PHYS 203 **Princeton University** **Fall 2005**

Midterm Exam Solutions

1. (30 points) **Bouncing Ball**

A small ball of mass *m* is released with a zero velocity from a height *h* above the floor. In the absence of air resistance the ball will hit the floor in a time $t_0 = \sqrt{2h/g}$. Now suppose there is a small resistance force proportional to the velocity, $F = -kmv$. The time until the ball hits the floor will increase by a small amount $t = t_0 + t_1$. For small *k* the time increment t_1 is proportional to *k*.

a) Determine the equation describing the position of the ball as a function of time.

b) Calculate t_1 to first non-trivial order in *k*.

For your reference the Taylor series is given by $f(x_0 + x) = f(x_0) + \sum_n \left(\frac{\partial^n f}{\partial x^n}\right)_{x0} \frac{x^n}{n!}$

a) The equation of motion is (x axis is directed down)

$$m\ddot{x} = -mk\dot{x} + mg$$
$$\ddot{x} + k\dot{x} = g$$

This is an inhomogeneous linear equation. The characteristic equation for the homogeneous differential equation $\ddot{x} + k\dot{x} = 0$ *is* $r^2 + kr = 0;\ r = 0,\ r = -k$. *The particular solution can be guessed* $x = (g/k)t$. *Hence the complete solution to the equation of motion is* $x(t) = A + Be^{-kt} + (g/k)t$. *The initial conditions are* $x(0) = 0, \dot{x}(0) = 0$. *Therefore the constants A and B have to satisfy the following two equations*

$$A + B = 0$$
$$-kB + g/k = 0$$

So $B = g/k^2, A = -g/k^2$ *and the position as a function of time is given by*

$$x(t) = \frac{g}{k^2}(e^{-kt} - 1) + \frac{g}{k}t$$

b) *The time when the ball hits the floor is given by x(t) = h. This is a transcendental equation for t and cannot be solved exactly. But we can solve it easily for the first order correction due to air resistance. Set* $t = t_0 + t_1$ *where* t_0 *is the time you would get without air resistance. Since k is small, the exponential function can be expanded in a Taylor series around zero*

$$h = \frac{g}{k^2}\left(1 - k(t_0 + t_1) + \frac{k^2}{2}(t_0 + t_1)^2 - \frac{k^3}{6}(t_0 + t_1)^3 - 1\right) + \frac{g}{k}(t_0 + t_1)$$

Also since $t_1 << t_0$, *the powers of* $t_0 + t_1$ *can be expanded to first order in* t_1

$$h = \frac{g}{k^2}\left(1 - k(t_0 + t_1) + \frac{k^2}{2}(t_0^2 + 2t_1 t_0) - \frac{k^3}{6}(t_0^3 + 3t_1 t_0^2) - 1\right) + \frac{g}{k}(t_0 + t_1)$$

$$h = -\frac{g(t_0 + t_1)}{k} + \frac{g}{2}(t_0^2 + 2t_1 t_0) - \frac{gk}{6}(t_0^3 + 3t_1 t_0^2) + \frac{g}{k}(t_0 + t_1)$$

Using $t_0 = \sqrt{2h/g}$ *we get*

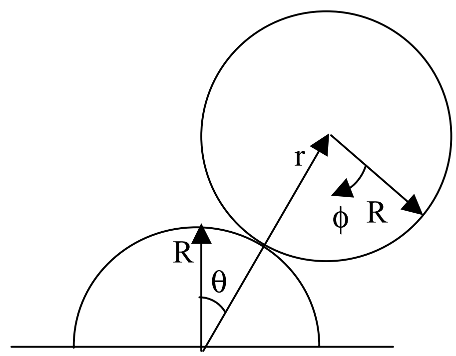
$$g t_1 t_0 = \frac{gk}{6}(t_0^3 + 3t_1 t_0^2)$$

$$t_1 = \frac{k}{6} t_0^2 = \frac{kh}{3g}$$

Note that we dropped the term proportional to t_1 *on the right hand side because it is small compared with the other term. Also it is clear after the fact that the exponent had to be expanded to third order. If the expansion were truncated sooner, one would get a trivial result.*

2. (40 points) **Slipping Cylinder**

A cylinder of radius *R* and mass *m* is placed on top of a fixed circular surface of the same radius and allowed to roll off. The cylinder begins to slip when the angle θ reaches a critical value θ_c. The goal of the problem is to determine the coefficient of static friction μ between the surfaces.



a) Write down the Lagrangian and the necessary equations of constraint.

Since we are concerned with friction in this problem, we will need to know both the normal force and the tangential friction force. Hence, we need to write the Lagrangian equations with two equations of constraint:
$g_1 = r - 2R = 0$ *and* $g_2 = \theta - \phi = 0$ *and two underdetermined constraints* λ_1 *and* λ_2

The Lagrangian in terms of variables θ, ϕ, *and r is*

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 - mgr\cos\theta$$

The equations of motion are

$$r: \quad mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda_1 = 0$$

$$\theta: \quad mgr\sin\theta - mr^2\ddot{\theta} + \lambda_2 = 0$$

$$\phi: \quad -I\ddot{\phi} - \lambda_2 = 0$$

b) What is the condition for the cylinder to begin to slip?

The usual equation for slipping is $F_f = \mu N$. *Here* λ_1 *gives the normal force of constraint, while* λ_2 *gives the torque that prevents slipping. Hence the cylinder begins to slip when* $\lambda_2 = \pm \mu R \lambda_1$. *The friction force can be either positive or negative, whatever is needed to prevent slipping.*

c) Obtain the Lagrangian equation of motion containing only the variable θ.

Now that we found the relationship between the forces, we can eliminate variables r and ϕ from the Lagrangian equations of motion using equations of constraint

$$\ddot{r}=0: \quad \lambda_1 = mg\cos\theta - 2mR\dot{\theta}^2$$
$$\theta=\phi: \quad 2mgR\sin\theta - 4mR^2\ddot{\theta} - I\ddot{\theta}=0$$
$$\phi: \; \lambda_2 = -I\ddot{\theta}$$

c) What quantity is conserved for $\theta < \theta_c$?

The energy is conserved before the cylinder begins to slip because friction does no work

$$E = H = T + U = 2mR^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 + 2mgR\cos\theta = E_0 = 2mgR$$

d) Obtain an equation for μ as a function of the angle θ_c

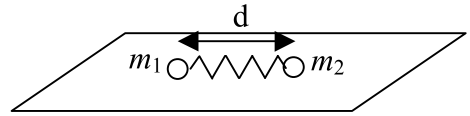
Now we substitute expressions for $\dot{\theta}^2$ *and* $\ddot{\theta}$ *into the condition for slipping (also* $I = mR^2/2$)

$$\ddot{\theta} = \frac{4g}{9R}\sin\theta$$
$$\dot{\theta}^2 = \frac{8g(1-\cos\theta)}{9R}$$
$$\lambda_2 = -\frac{mR^2}{2}\frac{4g}{9R}\sin\theta_c = \pm\mu R\lambda_1 = \mu R\left(mg\cos\theta_c - m\frac{16g(1-\cos\theta_c)}{9}\right)$$
$$-2\sin\theta_c = \mu(16 - 25\cos\theta_c)$$
$$\mu = \frac{2\sin\theta_c}{25\cos\theta_c - 16}$$

Pick the sign so μ *is positive. Also note that* $\cos\theta_c$ *has to be greater than 16/25 (the result obtained in the homework problem), otherwise the cylinder will simply fly off the surface. One needs an infinite* μ *to keep it from slipping just before it flies off as the normal force goes to zero.*

3. (30 points) **Twirling masses**

Two masses m_1 and m_2 are connected by a spring with a zero equilibrium distance and a spring constant k, so $F = -kd$, where d is the distance between the masses. The masses are free to slide on a frictionless horizontal table.



a) How can this problem be reduced to a problem of motion in a central potential? What is the effective potential in this case?

This configuration is similar to a homework problem. There are no external forces or torques. Hence the linear and angular momentum of the system is constant. One can go to a reference frame where the center of mass is at rest and describe the motion in terms of a *reduced mass $\mu = m_1 m_2/(m_1+m_2)$ and single variable d = r. Hence, the problem is equivalent to motion in a central force potential with an effective potential is given by*

$V_{eff} = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2}$ *where l is the angular momentum of the system.*

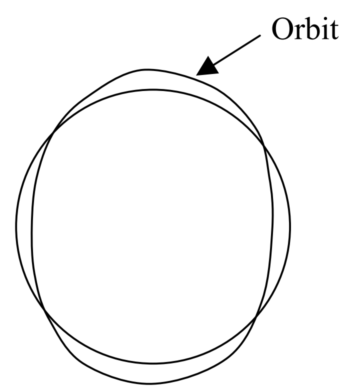
b) Calculate the frequency of small oscillations around a circular orbit.

The frequency of small oscillations is given by $\Omega = \sqrt{\frac{d^2V_{eff}/dr^2\big|_{r_{eq}}}{\mu}}$

$$V'(r) = kr - \frac{l^2}{\mu r^3}$$
$$V''(r) = k + \frac{3l^2}{\mu r^4}$$

At equilibrium $V'(r_{eq}) = 0,\ r_{eq}{}^4 = l^2/\mu k$ *and* $V''(r_{eq}) = k + 3k,\ \Omega = 2\sqrt{k/\mu}$

c) Will the orbit precess? Sketch an orbit with a small oscillation around the circular orbit.



The period of the orbit is given by

$$T = \frac{2\pi r_{eq}}{v} = \frac{2\pi\mu r^2{}_{eq}}{l} = \frac{2\pi\mu}{l}\sqrt{\frac{l^2}{\mu k}} = 2\pi\sqrt{\mu/k}$$

Hence TΩ = 4Pi, in one complete orbit the will be two complete oscillations. The orbit does not precess.