

THE BORELL-EHRHARD GAME

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ABSTRACT. A precise description of the convexity of Gaussian measures is provided by sharp Brunn-Minkowski type inequalities due to Ehrhard and Borell. We show that these are manifestations of a game-theoretic mechanism: a minimax variational principle for Brownian motion. As an application, we obtain a Gaussian improvement of Barthe's reverse Brascamp-Lieb inequality.

1. INTRODUCTION

The convexity properties of probability measures play an important role in various areas of probability theory, analysis, and geometry. They arise in a fundamental manner, for example, in the study of concentration phenomena [29, 2] and in functional analysis and convex geometry [12, 1]. Among the most delicate results in this area are the remarkable convexity properties of Gaussian measures [14, 32, 28, 8, 10]. The aim of this paper is to shed some new light on the latter topic.

Let γ_n be the standard Gaussian measure on \mathbb{R}^n . The simplest expression of the convexity of Gaussian measures is given by the log-concavity property:

$$\lambda \log(\gamma_n(A)) + (1 - \lambda) \log(\gamma_n(B)) \leq \log(\gamma_n(\lambda A + (1 - \lambda)B))$$

for all $\lambda \in [0, 1]$ and Borel sets $A, B \subseteq \mathbb{R}^n$, where $A + B := \{x + y : x \in A, y \in B\}$ denotes Minkowski addition. This inequality is easily deduced from the classical Brunn-Minkowski inequality, which is the analogous statement for Lebesgue measure. However, while the importance of log-concavity can hardly be overstated, we expect in the case of Gaussian measures that convexity should appear in a much stronger form than can be explained by log-concavity alone. For example, the classical isoperimetric inequality for Euclidean volume is an easy and fundamental consequence of the Brunn-Minkowski inequality [21], but log-concavity fails to explain the analogous isoperimetric property of Gaussian measures [28].

A precise description of the convexity of Gaussian measures was developed in a remarkable paper by Ehrhard [14], who introduced the following sharp analogue of the Brunn-Minkowski inequality for Gaussian measures:

$$\lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)) \leq \Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)),$$

where $\Phi(x) := \gamma_1((-\infty, x])$. This inequality becomes equality when A, B are parallel halfspaces, and is a strict improvement over log-concavity as the function $\log \Phi$ is concave. It has numerous interesting and important implications, including the isoperimetric property of Gaussian measures that arises as a special case [28, 32].

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Given the fundamental nature of Ehrhard's inequality, it is natural to seek other Gaussian analogues of the rich family of results that appear in the classical Brunn-Minkowski theory (cf. [21, 4] and the references therein). Progress in this direction has remained relatively limited, however. Unlike the classical Brunn-Minkowski inequality, which is well understood from many different perspectives, only two approaches to Ehrhard's inequality are known.¹ Ehrhard's original proof [14], using a Gaussian analogue of Steiner symmetrization, is limited to the case where the sets A, B are convex; it was later extended by Latała [27] to eliminate the convexity assumption on one of the two sets. The long-standing problem of proving Ehrhard's inequality for arbitrary Borel sets was finally settled by Borell [8], who also introduced a number of significant generalizations of this inequality [9, 10]. Borell's elegant approach, using a nonlinear heat equation and the parabolic maximum principle, relies on some surprising cancellations (as will be explained below), complicating efforts to identify how it can be applied in other settings. A more abstract variant of Borell's approach is given in [5, 25, 24], but the mechanism that makes this approach work remains somewhat mysterious. Let us note, in addition, that unlike many other geometric inequalities (including the Gaussian isoperimetric inequality) that extend to more general settings, Ehrhard's inequality appears to be uniquely Gaussian; see [26, §4.3] for some discussion on this point.

The aim of this paper is to develop a new interpretation of Ehrhard's inequality: we will show that both Ehrhard's inequality and its generalizations due to Borell arise as manifestations of a stochastic game that appears to lie at the heart of these phenomena. This unexpected game-theoretic mechanism provides new insight into the success of earlier proofs, and allows us to identify new convexity results for Gaussian measures. In particular, we will develop a Gaussian improvement of Barthe's reverse Brascamp-Lieb inequality, addressing a question posed in [5].

1.1. Borell's stochastic method. To motivate the ideas that will be introduced in the sequel, let us begin by recalling a powerful approach, also due to Borell [7], for proving log-concavity of Gaussian measures.

In order to show that γ_n (or any other measure) is log-concave, it is natural to seek a representation formula for $\log(\gamma_n(A))$ from which the concavity property becomes evident. A fundamental representation of this type, the Gibbs variational principle, dates back to the earliest work on statistical mechanics [22]:

$$\log \left(\int e^f d\gamma_n \right) = \sup_{\mu} \left\{ \int f d\mu - H(\mu|\gamma_n) \right\},$$

where $H(\mu|\gamma_n)$ denotes relative entropy and the supremum is taken over all probability measures μ . The log-concavity property could be read off directly from this formulation using displacement convexity of relative entropy as developed in the theory of optimal transportation [37]. However, in the case of Gaussian measures, a simpler approach becomes available by identifying γ_n with the distribution of the value of a Brownian motion $\{W_t\}$ at time one. The advantage gained by this approach is that absolutely continuous changes of measure of Brownian motion admit an explicit characterization by Girsanov's theorem [33], which gives rise to the

¹ After this paper was completed, the author learned of recent work [34] where another rather delicate proof of Ehrhard's inequality is provided using the Ornstein-Uhlenbeck semigroup.

following reformulation of the Gibbs variational principle for Gaussian measures:

$$\log \left(\int e^f d\gamma_n \right) = \sup_{\alpha} \mathbf{E} \left[f \left(W_1 + \int_0^1 \alpha_t dt \right) - \frac{1}{2} \int_0^1 \|\alpha_t\|^2 dt \right],$$

where the supremum is taken over all progressively measurable processes α . This formula was originally obtained using PDE methods by Fleming [18]; the connection with the Gibbs variational principle was developed by Boué and Dupuis [11].

It was observed by Borell in [7] that log-concavity of the Gaussian measure is an almost immediate consequence of this identity. Let us illustrate this idea in its functional (Prékopa-Leindler) form. Let f, g, h be functions such that

$$\lambda \log(f(x)) + (1 - \lambda) \log(g(y)) \leq \log(h(\lambda x + (1 - \lambda)y))$$

for all x, y , and denote by α^f and α^g the maximizing processes when the above representation is applied to $\log f$ and $\log g$, respectively. Then we have

$$\begin{aligned} & \lambda \log \left(\int f d\gamma_n \right) + (1 - \lambda) \log \left(\int g d\gamma_n \right) \\ &= \lambda \mathbf{E} \left[\log f \left(W_1 + \int_0^1 \alpha_t^f dt \right) - \frac{1}{2} \int_0^1 \|\alpha_t^f\|^2 dt \right] \\ & \quad + (1 - \lambda) \mathbf{E} \left[\log g \left(W_1 + \int_0^1 \alpha_t^g dt \right) - \frac{1}{2} \int_0^1 \|\alpha_t^g\|^2 dt \right] \\ & \leq \mathbf{E} \left[\log h \left(W_1 + \int_0^1 (\lambda \alpha_t^f + (1 - \lambda) \alpha_t^g) dt \right) - \frac{1}{2} \int_0^1 \|\lambda \alpha_t^f + (1 - \lambda) \alpha_t^g\|^2 dt \right] \\ & \leq \log \left(\int h d\gamma_n \right). \end{aligned}$$

Log-concavity follows readily by choosing $f = \mathbf{1}_A$, $g = \mathbf{1}_B$, and $h = \mathbf{1}_{\lambda A + (1 - \lambda)B}$. The beauty of this stochastic approach is that it reduces log-concavity of Gaussian measures to a trivial fact, viz. convexity of the function $x \mapsto \|x\|^2$. This idea has been further developed in [30, 31, 13] to prove various other inequalities, some of which do not seem to be readily accessible by other methods.

It is tempting to approach Ehrhard's inequality by seeking a Gaussian improvement of the Gibbs variational principle. It is far from clear, however, why this should be possible. The Gibbs variational principle is not a mysterious result: it simply expresses Fenchel duality for the convex functional $f \mapsto \log(\int e^f d\gamma_n)$. On the other hand, classical results of Hardy, Littlewood, and Pólya [23, §3.16] imply that the functional $f \mapsto \Phi^{-1}(\int \Phi(f) d\gamma_n)$ cannot be convex.

In his proof of Ehrhard's inequality [8], Borell circumvents the lack of a representation formula by using partial differential equation methods. As a first step, he obtains a PDE for the transformation $v_f(t, x) := \Phi^{-1}(u_f(t, x))$ of the solution $u_f(t, x)$ of the heat equation with initial condition f (the latter arises naturally in this setting as the Markov semigroup of Brownian motion). It is not immediately obvious that the resulting nonlinear PDE, given in section 2.2 below, possesses any useful convexity properties. Instead, Borell considers directly the desired combination $C(t, x, y) := \lambda v_f(t, x) + (1 - \lambda)v_g(t, y) - v_h(t, \lambda x + (1 - \lambda)y)$, and observes that a fortuitous cancellation occurs: one can arrange the terms in the combined PDEs for v_f, v_g, v_h to obtain a parabolic PDE for C alone. This makes it possible to apply the parabolic maximum principle to deduce nonpositivity of C , which is essentially the statement of Ehrhard's inequality in its functional form.

1.2. The Borell-Ehrhard game. The main result of the present paper is a new stochastic representation formula that lies at the heart of Ehrhard’s inequality, in direct analogy with Borell’s stochastic approach to log-concavity. This principle provides significant insight into the mechanism behind the convexity properties of Gaussian measures, as well as a new tool to study such properties.

As was explained above, the lack of convexity of $f \mapsto \Phi^{-1}(\int \Phi(f) d\gamma_n)$ prohibits us from obtaining a representation formula by a convex duality argument. Instead, our main result shows that this functional can be represented by a minimax variational principle. An informal statement of our main result is as follows.

Theorem (informal statement). *For bounded and uniformly continuous f*

$$\Phi^{-1}\left(\int \Phi(f) d\gamma_n\right) = \sup_{\alpha} \inf_{\beta} \mathbf{E} \left[\int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle \alpha_t, \beta_t \rangle dt + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} f\left(W_1 + \int_0^1 \alpha_t dt\right) \right].$$

This expression can be interpreted as the value of a zero-sum stochastic game between two players. The first player can apply a force α_t at time t to the underlying Brownian motion. The second player cannot affect the dynamics of the Brownian motion, but can instead choose to end the game prematurely: her control β_t is the rate of termination of the game at time t (that is, the game ends prematurely in the interval $[t, t + dt)$ with probability $\|\beta_t\|^2 dt$). The remarkable feature of this game is that the running cost $\langle \alpha_t, \beta_t \rangle$ is not quadratic, as in the stochastic representation used to prove log-concavity, but rather *linear* in α . This reflects the fact that the Φ^{-1} transformation lies precisely at the border of where we can expect convexity to appear: it “linearizes” the quadratic cost that arises from the Gibbs variational principle. (It is pointed out in [13] that log-concavity of γ_n can be strengthened in a different sense by exploiting uniform convexity of the quadratic cost.)

As is typical in the theory of continuous-time games, it is essential to carefully define the information structure available to each player in order for the above stochastic representation to be valid. In section 2, we provide a precise formulation and proof of our main result. The essential observation behind the proof is that our stochastic game is closely connected to Borell’s PDE approach to Ehrhard’s inequality: the nonlinear heat equation of Borell can be identified as the Bellman-Isaacs equation [36, 19] for the value of our stochastic game. This observation leads not only to the above representation, but also reveals the reason behind the hidden convexity that appears somewhat mysteriously in Borell’s proof.

With the above stochastic representation in hand, it is a simple exercise to deduce Ehrhard’s inequality, and its generalizations due to Borell, in complete analogy to the stochastic proof of log-concavity. This exercise is carried out in section 3.

1.3. A Gaussian reverse Brascamp-Lieb inequality. As an illustration of the power of the stochastic approach, we will use it to obtain a Gaussian improvement of the reverse Brascamp-Lieb inequality of Barthe [3, 5]. Let us first recall Barthe’s inequality in its Brunn-Minkowski form (see section 4 for the functional form). Let E_1, \dots, E_k be linear subspaces of \mathbb{R}^n with $\dim(E_i) = n_i$. Denote by P_i the orthogonal projection on E_i , and let $\lambda_1, \dots, \lambda_k \geq 0$ be such that $\lambda_1 P_1 + \dots + \lambda_k P_k = I_n$. Then Barthe’s inequality states that for any Borel sets $A_i \subseteq E_i$, we have

$$\lambda_1 \log(\gamma_{n_1}(A_1)) + \dots + \lambda_k \log(\gamma_{n_k}(A_k)) \leq \log(\gamma_n(\lambda_1 A_1 + \dots + \lambda_k A_k)),$$

where we identify γ_{n_i} with the standard Gaussian measure on E_i . This is an extension of the log-concavity property where the sets A_i may lie in lower-dimensional subspaces of the ambient space (in which case log-concavity is a trivial statement).

In view of Ehrhard's inequality, one might hope that it is possible to replace the logarithm by Φ^{-1} in Barthe's inequality to obtain a Gaussian improvement. However, this is certainly impossible in general: if E_1, \dots, E_k are orthogonal subspaces that span \mathbb{R}^n , then Barthe's inequality is in fact equality for all choices of A_i and no Gaussian improvement is possible (see section 4.1). Nonetheless, it is possible to systematically improve Barthe's inequality in the Gaussian setting, as we will do in section 4. To this end, define for every $c \in (0, 1)$ the function

$$\Phi_c^{-1}(x) := \Phi^{-1}(cx) - \Phi^{-1}(c).$$

We will show in section 4 that, under the same assumptions as in Barthe's inequality,

$$\lambda_1 \Phi_c^{-1}(\gamma_{n_1}(A_1)) + \dots + \lambda_k \Phi_c^{-1}(\gamma_{n_k}(A_k)) \leq \Phi_c^{-1}(\gamma_n(\lambda_1 A_1 + \dots + \lambda_k A_k))$$

for every $c \in (0, 1)$. From this inequality, one can recover both Barthe's inequality ($c \downarrow 0$) and Ehrhard's inequality ($c \uparrow 1$) as special cases. The stochastic game approach was essential to discovering the correct formulation of this inequality.

1.4. Generalized means. While our main result sheds new light on the mechanism behind Ehrhard's inequality, there remains some residual mystery regarding the origin of this stochastic game. The stochastic representation used to prove log-concavity is entirely natural, as it arises simply as a specialization of the Gibbs variational principle to the Brownian setting. It is unclear, however, whether there exists a natural minimax generalization of the Gibbs variational principle that can provide an analogous explanation for the Borell-Ehrhard game.

From another perspective, however, there is nothing particularly surprising about the stochastic representations that we encountered so far. To place these results in a broader context, we can consider the more general functional $f \mapsto F^{-1}(\int F(f) d\gamma_n)$ for any strictly increasing function F . Such functionals, called generalized means, were studied by Hardy, Littlewood, and Pólya [23, chapter 3], who provide in particular necessary and sufficient conditions for such functionals to be convex. In section 5, we will show that any convex generalized mean admits a stochastic representation that is very similar to the special case $F(x) = e^x$, from which convexity can be immediately read off. Moreover, we will argue that essentially arbitrary choice of F will admit a stochastic game representation, so the appearance of a game in the case $F(x) = \Phi(x)$ is just one specific example. Of course, it is a special feature of this example that gave rise to Ehrhard's inequality; the potential utility of such representations in other contexts will depend on the problem at hand.

2. THE BORELL-EHRHARD GAME

2.1. Setting and main result. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a probability space with a complete and right-continuous filtration, and let $\{W_t\}$ be a standard n -dimensional \mathcal{F}_t -Brownian motion. We denote by γ_n the standard Gaussian measure on \mathbb{R}^n and by $\Phi(x) := \gamma_1((-\infty, x])$. Our main result is a variational principle for Gaussian measures that will be expressed as a stochastic game for the Brownian motion W .

As is often the case in continuous time games, it is important to carefully define what information is available to each player. Informally, we can view our game as the continuous time limit of a discrete time game where two players take turns

exercising some control on the underlying Brownian motion. We denote the controls of the first and second players at time t by α_t and β_t , respectively. As the second player comes after the first, her control may depend on the choice of control of the first player. Conversely, the control of the first player may depend on the choice of control of the second player in earlier turns. It is not entirely obvious how this information structure should be encoded when time is continuous.

For our purposes, it will be convenient to adopt an approach due to Elliott and Kalton [15, 19]. In this framework, the second player may choose any control.

Definition 2.1. A *control* is a progressively measurable n -dimensional process $\beta = \{\beta_t\}_{t \in [0,1]}$. Denote by \mathcal{C} the family of all controls such that $\mathbf{E}[\int_0^1 \|\beta_s\|^2 ds] < \infty$.

On the other hand, the action of the first player must explicitly account for the fact that she has access to the earlier choice of control of the second player. To this end, we introduce the notion of an (Elliott-Kalton) strategy.

Definition 2.2. A *strategy* is a map $\alpha : \mathcal{C} \rightarrow \mathcal{C}$ such that for every $t \in [0, 1]$ and $\beta, \beta' \in \mathcal{C}$ such that $\beta_s(\omega) = \beta'_s(\omega)$ for a.e. $(s, \omega) \in [0, t] \times \Omega$, we have $\alpha_s(\beta)(\omega) = \alpha_s(\beta')(\omega)$ for a.e. $(s, \omega) \in [0, t] \times \Omega$. Denote by \mathcal{S} the family of all strategies such that $\sup\{\mathbf{E}[\int_0^1 \|\alpha_s(\beta)\|^2 ds] : \mathbf{E}[\int_0^1 \|\beta_s\|^2 ds] \leq R\} < \infty$ for all $R < \infty$.

In the Elliott-Kalton approach, the second player chooses any control $\beta \in \mathcal{C}$, while the first player's control $\alpha(\beta)$ is defined by a strategy $\alpha \in \mathcal{S}$. The definition of a strategy ensures that the control of the first player depends causally on the control of the second player, thereby encoding the desired information structure.

With these formalities out of the way, we can now formulate our main result.

Theorem 2.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded and uniformly continuous, and define*

$$J_f[\alpha, \beta] := \mathbf{E} \left[\int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle \alpha_t, \beta_t \rangle dt + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} f \left(W_1 + \int_0^1 \alpha_t dt \right) \right]$$

for $\alpha, \beta \in \mathcal{C}$. Then

$$\Phi^{-1} \left(\int \Phi(f) d\gamma_n \right) = \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_f[\alpha(\beta), \beta] = \inf_{\alpha \in \mathcal{S}} \sup_{\beta \in \mathcal{C}} J_f[\alpha(\beta), \beta].$$

The remainder of this section is devoted to the proof of Theorem 2.3. The connection with geometric inequalities will be developed in sections 3 and 4 below.

Remark 2.4. Let us emphasize that we defined a strategy $\alpha \in \mathcal{S}$ as a map that assigns to every random process $\beta \in \mathcal{C}$ another random process $\alpha(\beta) \in \mathcal{C}$, not as a mapping of the sample paths of β . In particular, the process $\alpha(\beta)$ is progressively measurable by construction, so that no measurability issues arise in Theorem 2.3.

2.2. The Borell PDE. Throughout the proof, we will assume without loss of generality that f is bounded, smooth, and has bounded derivatives of all orders. Once the result is proved in this case, the conclusion is readily extended to functions f that are only bounded and uniformly continuous (as the latter can be approximated in the uniform topology by smooth functions with bounded derivatives by convolution with a smooth compactly supported kernel, cf. [20, §8.2]).

Define for $(t, x) \in [0, 1] \times \mathbb{R}^n$ the function

$$u(t, x) := \mathbf{E}[\Phi(f(W_1 - W_t + x))],$$

so that u solves the heat equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 0, \quad u(1, x) = \Phi(f(x)).$$

Define

$$v(t, x) := \Phi^{-1}(u(t, x)).$$

By the smoothness assumption on f and elementary properties of the heat equation, u and therefore v are bounded, smooth, and have bounded derivatives of all orders on $[0, 1] \times \mathbb{R}^n$. Moreover, it is readily verified that v satisfies

$$\frac{\partial v}{\partial t} + \frac{1}{2}\Delta v - \frac{1}{2}v\|\nabla v\|^2 = 0, \quad v(1, x) = f(x).$$

This equation was introduced by Borell [8] in his study of the Ehrhard inequality.

The following simple observation contains the main idea behind the proof of Theorem 2.3: the nonlinear term in Borell's PDE admits a variational interpretation.

Lemma 2.5. *Let c be a constant such that $2c \geq \sup_x f(x)$. Then*

$$-\frac{1}{2}v\|\nabla v\|^2 = \sup_{a \in \mathbb{R}^n} \inf_{b \in \mathbb{R}^n} \left\{ \langle a + cb, \nabla v + b \rangle - \frac{1}{2}v\|b\|^2 \right\},$$

where the optimizer $a^* = (c - v)\nabla v$, $b^* = -\nabla v$ is a saddle point.

Proof. Define for $a, b \in \mathbb{R}^n$ the objective

$$H(a, b) := \langle a + cb, \nabla v + b \rangle - \frac{1}{2}v\|b\|^2.$$

Then it is readily verified that

$$H(a, b^*) = -\frac{1}{2}v\|\nabla v\|^2, \quad H(a^*, b) = \frac{1}{2}(2c - v)\|b + \nabla v\|^2 - \frac{1}{2}v\|\nabla v\|^2.$$

But note that as $2c \geq f$, we have $2c - v \geq 0$ by the definition of v . Therefore

$$\sup_a \inf_b H(a, b) \leq \sup_a H(a, b^*) = -\frac{1}{2}v\|\nabla v\|^2 = \inf_b H(a^*, b) \leq \sup_a \inf_b H(a, b),$$

and the proof is complete. \square

Lemma 2.5 reveals that the partial differential equation satisfied by v is none other than the Bellman-Isaacs equation for the value of a stochastic game [19, 36]. We can now proceed along mostly standard lines to formalize this idea.

2.3. Upper bound. Fix $2c \geq f$, and consider the stochastic differential equation

$$dX_t^\beta = (c - v(t, X_t^\beta))\nabla v(t, X_t^\beta) dt + c\beta_t dt + dW_t, \quad X_0^\beta = 0$$

for $\beta \in \mathcal{C}$. As the function $(c - v)\nabla v$ is smooth with bounded derivatives, this equation has a unique strong solution X^β [33, Theorem 4.8]: that is, there is a progressively measurable map $F : \mathbb{W} \rightarrow \mathbb{W}$ such that $X^\beta = F[\{\int_0^t c\beta_s ds + W_t\}]$, where \mathbb{W} denotes the space of continuous paths with its canonical filtration. Define

$$\alpha_t^*(\beta) := (c - v(t, X_t^\beta))\nabla v(t, X_t^\beta).$$

Then evidently $\alpha^*(\beta) \in \mathcal{C}$ (in fact, it is uniformly bounded) and α^* depends causally on β by the strong solution property. Thus we have shown that $\alpha^* \in \mathcal{S}$ defines an Elliott-Kalton strategy in the sense of Definition 2.2.

Applying Itô's formula to the process $e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} v(t, X_t^\beta)$ gives

$$\begin{aligned} & \int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle \alpha_t^*(\beta) + c\beta_t, \beta_t \rangle dt + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} f(X_1^\beta) = \\ & v(0, 0) + \int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle \nabla v(t, X_t^\beta), dW_t \rangle \\ & + \int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \left\{ \frac{\partial v}{\partial t}(t, X_t^\beta) + \frac{1}{2} \Delta v(t, X_t^\beta) \right. \\ & \quad \left. + \langle \alpha_t^*(\beta) + c\beta_t, \nabla v(t, X_t^\beta) + \beta_t \rangle - \frac{1}{2} v(t, X_t^\beta) \|\beta_t\|^2 \right\} dt. \end{aligned}$$

We now observe that the last integral in this expression is nonnegative by Borell's PDE and Lemma 2.5. Moreover, the Brownian integral is a martingale as ∇v is bounded. Therefore, taking the expectation of this expression, we obtain

$$\begin{aligned} v(0, 0) & \leq \mathbf{E} \left[\int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle \alpha_t^*(\beta) + c\beta_t, \beta_t \rangle dt + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} f(X_1^\beta) \right] \\ & = J_f[\alpha^*(\beta) + c\beta, \beta] \end{aligned}$$

for every $\beta \in \mathcal{C}$. But evidently $\tilde{\alpha}^*(\beta) := \alpha^*(\beta) + c\beta$ defines another Elliott-Kalton strategy $\tilde{\alpha}^* \in \mathcal{S}$. We therefore readily obtain the upper bound in Theorem 2.3

$$\Phi^{-1} \left(\int \Phi(f) d\gamma_n \right) = v(0, 0) \leq \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_f[\alpha(\beta), \beta].$$

2.4. Lower bound. For the proof of the lower bound, fix any $\alpha \in \mathcal{S}$. Given this strategy, our aim is to construct a control $\beta \in \mathcal{C}$ that nearly minimizes $J_f[\alpha(\beta), \beta]$. We will do this by imitating the idea that our continuous game is the limit of discrete-time games, as was explained informally at the beginning of this section.

To this end, fix a time step $\delta = N^{-1}$ ($N \geq 1$). For $t \in [0, \delta)$, let $\beta_t := -\nabla v(0, 0)$. We now iteratively extend the definition of β as follows. Suppose that β has been defined on the interval $[0, k\delta)$. Let $\beta_t^k := \beta_t \mathbf{1}_{[0, k\delta)}(t)$, so that $\beta^k \in \mathcal{C}$. We define

$$\beta_t := -\nabla v \left(k\delta, W_{k\delta} + \int_0^{k\delta} \alpha_s(\beta^k) ds \right)$$

for $t \in [k\delta, (k+1)\delta)$. Iterating this process $N-1$ times results in a control $\beta \in \mathcal{C}$ that is defined for all times $t \in [0, 1]$ (we may arbitrarily choose $\beta_1 := 0$).

Now define the process

$$X_t := W_t + \int_0^t \alpha_s(\beta) ds.$$

As $\beta_t = \beta_t^k$ for all $t \in [0, k\delta)$ by construction, we have $\alpha_s(\beta)(\omega) = \alpha_s(\beta^k)(\omega)$ for a.e. $(s, \omega) \in [0, k\delta) \times \Omega$ by the definition of an Elliott-Kalton strategy. In particular, we have $\beta_t = -\nabla v(k\delta, X_{k\delta})$ for every $t \in [k\delta, (k+1)\delta)$ a.s.

Applying Itô's formula to $e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} v(t, X_t)$ as in the upper bound gives

$$J_f[\alpha(\beta), \beta] = v(0, 0) + \mathbf{E}[\Gamma]$$

where

$$\Gamma := \int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \left\{ \frac{1}{2} v(t, X_t) (\|\nabla v(t, X_t)\|^2 - \|\beta_t\|^2) + \langle \alpha_t(\beta), \nabla v(t, X_t) + \beta_t \rangle \right\} dt.$$

As v is bounded and has bounded derivatives of all orders, we can estimate

$$\begin{aligned} \Gamma &\leq C_1 \int_0^1 (1 + \|\alpha_t(\beta)\|) \|\nabla v(t, X_t) + \beta_t\| dt \\ &= C_1 \sum_{k=0}^{N-1} \int_{k\delta}^{(k+1)\delta} (1 + \|\alpha_t(\beta)\|) \|\nabla v(t, X_t) - \nabla v(k\delta, X_{k\delta})\| dt \\ &\leq C_2 \sum_{k=0}^{N-1} \int_{k\delta}^{(k+1)\delta} (1 + \|\alpha_t(\beta)\|) (\delta + \|X_t - X_{k\delta}\|) dt \end{aligned}$$

for constants C_1, C_2 that depend on f only. Note that for $t \leq (k+1)\delta$

$$\|X_t - X_{k\delta}\| \leq \|W_t - W_{k\delta}\| + \sqrt{\delta} \left[\int_{k\delta}^t \|\alpha_s(\beta)\|^2 ds \right]^{1/2}.$$

We can therefore estimate using Cauchy-Schwarz

$$\mathbf{E}[\Gamma] \leq C_3 \sqrt{\delta} \left(1 + \mathbf{E} \left[\int_0^1 \|\alpha_t(\beta)\|^2 dt \right] \right) \leq C_3 (K+1) \sqrt{\delta},$$

where $K := \sup\{\mathbf{E}[\int_0^1 \|\alpha_t(\beta')\|^2 dt] : \|\beta'\|_\infty \leq \|\nabla v\|_\infty\} < \infty$ by definition as $\alpha \in \mathcal{S}$ and where C_3 depends only on f . We have therefore shown that

$$\inf_{\beta' \in \mathcal{C}} J_f[\alpha(\beta'), \beta'] \leq J_f[\alpha(\beta), \beta] \leq v(0, 0) + C_3 (K+1) \sqrt{\delta}.$$

As $\delta > 0$ and $\alpha \in \mathcal{S}$ were arbitrary, we readily conclude that

$$\sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_f[\alpha(\beta), \beta] \leq v(0, 0) = \Phi^{-1} \left(\int \Phi(f) d\gamma_n \right).$$

2.5. End of proof. Combining the upper and lower bound, we have shown

$$\Phi^{-1} \left(\int \Phi(f) d\gamma_n \right) = \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_f[\alpha(\beta), \beta].$$

It remains to prove the second identity in Theorem 2.3. To this end, note that

$$\Phi^{-1} \left(\int \Phi(f) d\gamma_n \right) = -\Phi^{-1} \left(\int \Phi(-f) d\gamma_n \right) = -\sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_{-f}[\alpha(\beta), \beta]$$

as $\Phi(-x) = 1 - \Phi(x)$. But we can write

$$-\sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_{-f}[\alpha(\beta), \beta] = \inf_{\alpha \in \mathcal{S}} \sup_{\beta \in \mathcal{C}} (-J_{-f}[\alpha(\beta), \beta]) = \inf_{\alpha \in \mathcal{S}} \sup_{\beta \in \mathcal{C}} J_f[\alpha(\beta), -\beta].$$

As \mathcal{C} is invariant under the transformation $\beta \mapsto -\beta$ and \mathcal{S} is invariant under the transformation $\alpha(\beta) \mapsto \alpha(-\beta)$, the second identity in Theorem 2.3 follows.

3. THE EHRHARD AND BORELL INEQUALITIES

The aim of this short section is to show that the classical Gaussian Brunn-Minkowski inequality of Ehrhard [14, 8] and its generalizations due to Borell [9, 10] arise as immediate corollaries of Theorem 2.3. In section 4 below, we will extend this approach to derive new geometric inequalities for Gaussian measures.

3.1. Ehrhard's inequality. Ehrhard's inequality states that

$$\lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)) \leq \Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B))$$

for all Borel sets $A, B \subseteq \mathbb{R}^n$ and $\lambda \in [0, 1]$. By approximating the indicator functions of A and B by smooth functions, it is routine to deduce this inequality from the following functional form of the result (see [8] or section 4.4 below).

Corollary 3.1 ([14, 8]). *Let $\lambda \in [0, 1]$, and let f, g, h be uniformly continuous functions with values in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. Suppose that for all $x, y \in \mathbb{R}^n$*

$$\lambda \Phi^{-1}(f(x)) + (1 - \lambda) \Phi^{-1}(g(y)) \leq \Phi^{-1}(h(\lambda x + (1 - \lambda)y)).$$

Then

$$\lambda \Phi^{-1}\left(\int f d\gamma_n\right) + (1 - \lambda) \Phi^{-1}\left(\int g d\gamma_n\right) \leq \Phi^{-1}\left(\int h d\gamma_n\right).$$

Proof. Fix $\delta > 0$, and choose near-optimal $\alpha_f, \alpha_g \in \mathcal{S}$ and $\beta_h \in \mathcal{C}$ such that

$$\begin{aligned} \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_{\Phi^{-1}(f)}[\alpha(\beta), \beta] &\leq \inf_{\beta \in \mathcal{C}} J_{\Phi^{-1}(f)}[\alpha_f(\beta), \beta] + \delta, \\ \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_{\Phi^{-1}(g)}[\alpha(\beta), \beta] &\leq \inf_{\beta \in \mathcal{C}} J_{\Phi^{-1}(g)}[\alpha_g(\beta), \beta] + \delta, \end{aligned}$$

$$J_{\Phi^{-1}(h)}[\lambda \alpha_f(\beta_h) + (1 - \lambda) \alpha_g(\beta_h), \beta_h] \leq \inf_{\beta \in \mathcal{C}} J_{\Phi^{-1}(h)}[\lambda \alpha_f(\beta) + (1 - \lambda) \alpha_g(\beta), \beta] + \delta.$$

Then by Theorem 2.3

$$\begin{aligned} &\lambda \Phi^{-1}\left(\int f d\gamma_n\right) + (1 - \lambda) \Phi^{-1}\left(\int g d\gamma_n\right) \\ &\leq \lambda J_{\Phi^{-1}(f)}[\alpha_f(\beta_h), \beta_h] + (1 - \lambda) J_{\Phi^{-1}(g)}[\alpha_g(\beta_h), \beta_h] + 2\delta \\ &\leq J_{\Phi^{-1}(h)}[\lambda \alpha_f(\beta_h) + (1 - \lambda) \alpha_g(\beta_h), \beta_h] + 2\delta \\ &\leq \Phi^{-1}\left(\int h d\gamma_n\right) + 3\delta, \end{aligned}$$

and the proof is completed by letting $\delta \downarrow 0$. \square

3.2. Borell's Gaussian Brunn-Minkowski inequalities. In [9], Borell proves a substantial generalization of Ehrhard's inequality: he shows that

$$\lambda \Phi^{-1}(\gamma_n(A)) + \mu \Phi^{-1}(\gamma_n(B)) \leq \Phi^{-1}(\gamma_n(\lambda A + \mu B))$$

holds for all Borel sets $A, B \subseteq \mathbb{R}^n$ if and only if $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ (the necessity of the latter conditions is easily verified by explicit examples, see [9]). The deduction of this result from Theorem 2.3 requires only a minor modification of the proof of Corollary 3.1: it suffices to note that we do not need to choose the same Brownian motion W in the variational problems for f and g . By choosing instead two correlated Brownian motions, we immediately recover Borell's result.

Corollary 3.2 ([9]). *Let $\lambda, \mu \geq 0$, and let f, g, h be uniformly continuous functions with values in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. Suppose that for all $x, y \in \mathbb{R}^n$*

$$\lambda \Phi^{-1}(f(x)) + \mu \Phi^{-1}(g(y)) \leq \Phi^{-1}(h(\lambda x + \mu y)).$$

If $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$, then

$$\lambda \Phi^{-1}\left(\int f d\gamma_n\right) + \mu \Phi^{-1}\left(\int g d\gamma_n\right) \leq \Phi^{-1}\left(\int h d\gamma_n\right).$$

Proof. Let $\rho = (1 - \lambda^2 - \mu^2)/2\lambda\mu$. The assumptions $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ guarantee that $\rho \in [-1, 1]$. We can therefore define two standard n -dimensional Brownian motions $\{W_t\}$ and $\{\tilde{W}_t\}$ with quadratic covariation $\langle W^i, \tilde{W}^j \rangle_t = \rho t \delta_{ij}$. The point of this construction is that the process $\{\bar{W}_t\}$ defined as $\bar{W}_t := \lambda W_t + \mu \tilde{W}_t$ is again a standard n -dimensional Brownian motion.

Let \tilde{J}_f, \bar{J}_f be defined analogously to J_f in Theorem 2.3 where $\{W_t\}$ is replaced by $\{\tilde{W}_t\}$ and $\{W_t\}$, respectively. The remainder of the proof is identical to that of Corollary 3.1, where $J_{\Phi^{-1}(g)}$ is replaced by $\tilde{J}_{\Phi^{-1}(g)}$ and $J_{\Phi^{-1}(h)}$ by $\bar{J}_{\Phi^{-1}(h)}$. \square

Remark 3.3. We observe that it was essential for the success of the proof of Corollary 3.2 that the game described by Theorem 2.3 is defined on a general probability space: while the objective function $J_f[\alpha, \beta]$ depends only on a single Brownian motion $\{W_t\}$, we allowed the controls $\alpha, \beta \in \mathcal{C}$ to be adapted to a larger filtration $\{\mathcal{F}_t\}$ that is not necessarily generated by the underlying Brownian motion alone. This freedom was used crucially in the proof of Corollary 3.2; here we can take \mathcal{F}_t to be (the augmentation of) $\sigma\{W_s, \tilde{W}_s : s \leq t\}$, but we cannot ensure that the control $\lambda\alpha_f(\beta_h) + \mu\alpha_g(\beta_h)$ will depend only on $\{\bar{W}_t\}$.

The assumptions $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ in Corollary 3.2 are precisely the conditions required for the existence of correlated standard Brownian motions $\{W_{1,t}\}$ and $\{W_{2,t}\}$ such that $\lambda W_1 + \mu W_2$ is also a standard Brownian motion. Along identical lines, we immediately see that the inequality

$$\lambda_1 \Phi^{-1}(\gamma_n(A_1)) + \cdots + \lambda_k \Phi^{-1}(\gamma_n(A_k)) \leq \Phi^{-1}(\gamma_n(\lambda_1 A_1 + \cdots + \lambda_k A_k))$$

holds for all Borel sets $A_1, \dots, A_k \subseteq \mathbb{R}^n$ whenever there exist correlated standard Brownian motions $\{W_{i,t}\}$, $i = 1, \dots, k$ such that $\lambda_1 W_1 + \cdots + \lambda_k W_k$ is again a standard Brownian motion. The family of coefficients $\lambda_1, \dots, \lambda_k \geq 0$ for which this is the case is characterized by [5, Lemma 3], and we recover in this manner the general Gaussian Brunn-Minkowski inequality of Borell [10].

Remark 3.4. We have stated Corollaries 3.1 and 3.2 for simplicity under the assumption that the functions f, g, h are uniformly continuous and bounded away from zero and one. This case contains the main difficulty of the problem: it is routine to derive from this the corresponding results for sets [8], and one can subsequently derive versions of Corollaries 3.1 and 3.2 where the functions f, g, h are just Borel measurable with values in $[0, 1]$ as is explained in [28]. As these are standard results, we omit the details. However, in section 4.4 below, we will work out in detail a direct approximation argument in the setting of Theorem 4.2 that could also be applied here to deduce the measurable versions of Corollaries 3.1 and 3.2.

4. A GAUSSIAN REVERSE BRASCAMP-LIEB INEQUALITY

4.1. Barthe's inequality. Both the classical Brunn-Minkowski inequality and Ehrhard's inequality bound the measure of the Minkowski sum $\lambda A + (1 - \lambda)B$ from below in terms of the measures of A and B . Therefore, when either A or B has measure zero, these inequalities necessarily become trivial. Nonetheless, it is perfectly possible for $\lambda A + (1 - \lambda)B$ to have positive measure even when A and B are, for example, contained in lower-dimensional subspaces of \mathbb{R}^n . This phenomenon is captured quantitatively by a significant generalization of the classical Brunn-Minkowski inequality due to Barthe [3], which we presently recall.

Fix $\lambda_1, \dots, \lambda_k \geq 0$, and let B_1, \dots, B_k be linear maps $B_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ such that

$$\sum_{i=1}^k \lambda_i B_i^* B_i = I_n, \quad B_i B_i^* = I_{n_i} \text{ for all } i.$$

Note that B_i^* isometrically embeds \mathbb{R}^{n_i} in the linear subspace $E_i = \text{Im}(B_i^*)$ of \mathbb{R}^n . Let $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be functions such that

$$\lambda_1 \log(f_1(x_1)) + \dots + \lambda_k \log(f_k(x_k)) \leq \log(h(\lambda_1 B_1^* x_1 + \dots + \lambda_k B_k^* x_k))$$

for all $x_i \in \mathbb{R}^{n_i}$. Then Barthe's inequality states that

$$\lambda_1 \log \left(\int f_1 d\gamma_{n_1} \right) + \dots + \lambda_k \log \left(\int f_k d\gamma_{n_k} \right) \leq \log \left(\int h d\gamma_n \right)$$

(see [5, 31] for the formulation in terms of Gaussian rather than Lebesgue measure). When f_i are taken to be indicator functions of sets, this reduces to the following generalization of the Brunn-Minkowski inequality: for any Borel sets $A_i \subseteq E_i$

$$\lambda_1 \log(\gamma_{n_1}(A_1)) + \dots + \lambda_k \log(\gamma_{n_k}(A_k)) \leq \log(\gamma_n(\lambda_1 A_1 + \dots + \lambda_k A_k)),$$

where we implicitly identify γ_{n_i} with the standard Gaussian measure on E_i .

Remark 4.1. Barthe's inequality is also called the reverse Brascamp-Lieb inequality. The classical Brascamp-Lieb inequality is an analogous multilinear generalization of Hölder's inequality. Just as the Prékopa-Leindler inequality could formally be viewed as a reverse form of Hölder's inequality, Barthe's inequality can be viewed as a reverse form of the Brascamp-Lieb inequality. Let us note that we have stated the inequality in its "geometric" form, which is most natural for our purposes. The general form of the reverse Brascamp-Lieb inequality (for general matrices B_i) can be deduced from the geometric form, see [6] and [31] for details.

When $n_i = n$ and $B_i = I_n$ for all i , Barthe's inequality reduces to the Prékopa-Leindler inequality. However, we know that the latter is far from optimal for Gaussian measures: the sharp form of the Prékopa-Leindler inequality in the Gaussian case is precisely Ehrhard's inequality (Corollary 3.1), where the logarithm is replaced by Φ^{-1} . It is therefore natural to ask whether there exists an analogous Gaussian improvement of Barthe's inequality. This question was raised in [5, §4.2]. We will show in section 4.2 that there does in fact exist an interesting family of inequalities of this form, but the correct formulation of such inequalities is not entirely obvious. Before we develop these inequalities, let us briefly discuss what sort of improvement could reasonably be expected.

One might optimistically hope that as in the case of Ehrhard's inequality, we may simply replace \log by Φ^{-1} in Barthe's inequality to obtain the analogous Gaussian form. However, not only is this impossible, but in fact no improvement of Barthe's inequality is possible in general. To see why, consider the case where E_1 and E_2 are two orthogonal subspaces of \mathbb{R}^2 , which forces $\lambda_1 = \lambda_2 = 1$. Suppose the inequality

$$L(\gamma_1(A_1)) + L(\gamma_1(A_2)) \leq L(\gamma_2(A_1 + A_2))$$

holds for a function L . As $\gamma_2(A_1 + A_2) = \gamma_1(A_1)\gamma_1(A_2)$ in this case, we must have

$$L(x) + L(y) \leq L(xy)$$

for all $x, y \in [0, 1]$, which is clearly violated when $L(x) = \Phi^{-1}(x)$ (let $x = y = \frac{1}{2}$). On the other hand, the above inequality holds with equality when $L(x) = \log x$.

It follows that Barthe's inequality is already optimal in the orthogonal setting and cannot be improved by any alternative choice of function L .

We have now considered two extreme cases. When $E_1 = \dots = E_k = \mathbb{R}^n$, Ehrhard's inequality is sharp and the optimal choice of function is $L = \Phi^{-1}$. On the other hand, when E_1, \dots, E_k are orthogonal subspaces, Barthe's inequality is sharp and the optimal choice of function is $L = \log$. One can therefore not expect that any single choice of function L can provide a systematic Gaussian refinement of Barthe's inequality: any general improvement requires the choice of L to depend at least on the parameters λ_i and B_i . This feature is integral to the formulation of the Gaussian reverse Brascamp-Lieb inequalities that we will prove presently: we will introduce a family of inequalities that interpolate, in some sense, between the Ehrhard and Barthe inequalities; the best choice of inequality within this family must depend on the parameters to which it is applied.

4.2. A Gaussian refinement. In the remainder of this section, we place ourselves in the same setting as in the above formulation of Barthe's inequality: that is, we fix $\lambda_1, \dots, \lambda_k \geq 0$ and let B_1, \dots, B_k be linear maps $B_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ such that

$$\sum_{i=1}^k \lambda_i B_i^* B_i = I_n, \quad B_i B_i^* = I_{n_i} \text{ for all } i.$$

As before, we define the subspaces $E_i = \text{Im}(B_i^*)$. We also define the function

$$\Phi_c^{-1}(x) := \Phi^{-1}(cx) - \Phi^{-1}(c), \quad x \in [0, 1]$$

for $c \in (0, 1)$. We will prove the following Gaussian form of Barthe's inequality.

Theorem 4.2. *Let $c \in (0, 1)$, and let f_1, \dots, f_k, h be Borel measurable functions $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$, $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with values in $[0, 1]$. Suppose that*

$$\lambda_1 \Phi_c^{-1}(f_1(x_1)) + \dots + \lambda_k \Phi_c^{-1}(f_k(x_k)) \leq \Phi_c^{-1}(h(\lambda_1 B_1^* x_1 + \dots + \lambda_k B_k^* x_k))$$

for all $x_i \in \mathbb{R}^{n_i}$. Then

$$\lambda_1 \Phi_c^{-1} \left(\int f_1 d\gamma_{n_1} \right) + \dots + \lambda_k \Phi_c^{-1} \left(\int f_k d\gamma_{n_k} \right) \leq \Phi_c^{-1} \left(\int h d\gamma_n \right).$$

We immediately deduce the following generalization of Ehrhard's inequality.

Corollary 4.3. *For any $c \in (0, 1)$ and Borel sets $A_i \subseteq E_i$, $i = 1, \dots, k$, we have*

$$\lambda_1 \Phi_c^{-1}(\gamma_{n_1}(A_1)) + \dots + \lambda_k \Phi_c^{-1}(\gamma_{n_k}(A_k)) \leq \Phi_c^{-1}(\gamma_n(\lambda_1 A_1 + \dots + \lambda_k A_k)).$$

Proof. Choose $f_i(x) = \mathbf{1}_{A_i}(B_i^* x)$ and $h(x) = \mathbf{1}_{\lambda_1 A_1 + \dots + \lambda_k A_k}(x)$. \square

It is instructive to note that both Ehrhard's inequality and Barthe's generalized Brunn-Minkowski inequality arise as limiting cases of Corollary 4.3.

Let us first recover Barthe's inequality. To this end, recall that

$$\Phi(-y) = \int_y^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = (1 + o(1)) \frac{e^{-y^2/2}}{y\sqrt{2\pi}} \quad \text{as } y \rightarrow \infty.$$

A simple computation shows that

$$\Phi^{-1}(x)^2 = -2 \log x - \log \log(1/x) - \log 4\pi + o(1) \quad \text{as } x \downarrow 0,$$

so that

$$\Phi_c^{-1}(x) = \frac{\Phi^{-1}(cx)^2 - \Phi^{-1}(c)^2}{\Phi^{-1}(c) + \Phi^{-1}(cx)} = \frac{-2 \log x + o(1)}{\Phi^{-1}(c) + \Phi^{-1}(cx)} \quad \text{as } c \downarrow 0.$$

This implies, in particular, that

$$\lim_{c \downarrow 0} \Phi_c^{-1}(x) \sqrt{-2 \log c} = \log x.$$

Thus Barthe's Brunn-Minkowski inequality is recovered as $c \downarrow 0$ in Corollary 4.3.

On the other hand, to recover Ehrhard's inequality, set $n_i = n$ and $B_i = I_n$ for all i . This forces $\lambda_1 + \dots + \lambda_k = 1$, so that Corollary 4.3 reduces to

$$\lambda_1 \Phi^{-1}(c\gamma_n(A_1)) + \dots + \lambda_k \Phi^{-1}(c\gamma_n(A_k)) \leq \Phi^{-1}(c\gamma_n(\lambda_1 A_1 + \dots + \lambda_k A_k)).$$

Thus Ehrhard's inequality is recovered as $c \uparrow 1$ in Corollary 4.3.

We have therefore seen that Corollary 4.3 is never worse than Barthe's Brunn-Minkowski inequality, and can be substantially better. For general parameters, one has the freedom to optimize over c to obtain the best inequality in this family.

4.3. Proof of Theorem 4.2: smooth case. The proof of Theorem 4.2 is based on a minor but crucial variation on Theorem 2.3. Before we turn to the proof, let us briefly discuss how this extension and the function Φ_c arise naturally when attempting to extend Ehrhard's inequality to the setting of Theorem 4.2.

We begin by noting that the Gaussian measure γ_{n_i} on \mathbb{R}^{n_i} can be obtained as a projection of the Gaussian measure γ_n . In particular, as $B_i B_i^* = I_{n_i}$ by assumption, $B_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ is a projection, and we can write

$$\int f_i d\gamma_{n_i} = \int (f_i \circ B_i) d\gamma_n.$$

It is therefore natural to attempt to obtain a Gaussian analogue of Barthe's inequality by applying Theorem 2.3 to the functions $\Phi^{-1}(f_i \circ B_i)$. If we had

$$\begin{aligned} \Phi^{-1} \left(\int (f_i \circ B_i) d\gamma_n \right) &\stackrel{?}{=} \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} \mathbf{E} \left[\int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle B_i^* B_i \alpha_t(\beta), \beta_t \rangle dt \right. \\ &\quad \left. + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} \Phi^{-1} \left(f_i \left(B_i W_1 + \int_0^1 B_i \alpha_t(\beta) dt \right) \right) \right], \end{aligned}$$

then we could directly repeat the proof of Corollary 3.1 to show that the analogue of Barthe's inequality holds even for the function Φ^{-1} rather than Φ_c^{-1} (which is false as discussed above). However, Theorem 2.3 only gives $\langle \alpha_t(\beta), \beta_t \rangle$ in the running cost, rather than $\langle B_i^* B_i \alpha_t(\beta), \beta_t \rangle$ as written in the above formula.

This issue nearly resolves itself by inspection of the proof of Theorem 2.3. We observe that the optimal strategy α^* that is defined in the proof of Theorem 2.3 is of the form $\alpha_t^*(\beta) \propto \nabla v(t, X_t^\beta)$. As $\nabla(f_i \circ B_i) = B_i^*(\nabla f_i \circ B_i)$, it follows readily that $\nabla(f_i \circ B_i)$, and therefore $\nabla v(t, x)$, take values in the range of B_i^* . In particular,

$$B_i^* B_i \alpha_t^*(\beta) = \alpha_t^*(\beta)$$

as B_i is a projection. It therefore follows that we may essentially insert the missing projection $B_i^* B_i$ "for free" in the statement of Theorem 2.3.

The major problem that we have overlooked in the above discussion is that if $f_i \leq c$, then it is not $\alpha^*(\beta)$ but rather $\tilde{\alpha}^*(\beta) := \alpha^*(\beta) + \Phi^{-1}(c)\beta$ that appears as the saddle point in the proof of Theorem 2.3. In the proof of Ehrhard's inequality, this regularization (which arises in an essential manner in Lemma 2.5 in order for the game to be well-posed) was completely irrelevant as it could be absorbed into the definition of the optimal strategy $\tilde{\alpha}^*$. In the present case, however, the regularizing term is precisely the obstruction that prevents Theorem 4.2 from being valid for the function Φ^{-1} , as we cannot assume that β takes values in the range of B_i^* .

Instead, we will see in the proof of Proposition 4.4 below that a simple application of the fundamental theorem of calculus allows us to transform the additional regularization term in the running cost into an additive constant that depends only on the uniform upper bound on the function f_i . The definition of the function Φ_c^{-1} simply arises by absorbing the uniform bound and additive constant (note that applying Φ_c^{-1} to functions f_i with values in $[0, 1]$, as is done in Theorem 4.2, corresponds to applying Φ^{-1} to functions with values in $[0, c]$; in particular, Theorem 4.2 would fail if we allowed the functions f_i to take values larger than one, even though the function Φ_c^{-1} could naturally be extended to such values.)

Having described the main ideas behind the proof of Theorem 4.2, we are ready to state the key extension of Theorem 2.3 that will be needed.

Proposition 4.4. *Let $f : \mathbb{R}^m \rightarrow (-\infty, 0]$ be bounded and uniformly continuous, let $c \in (0, 1)$, and let $B : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map such that $BB^* = I_m$. Define*

$$J_f^{B,c}[\alpha, \beta] := \mathbf{E} \left[\int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \langle B^* B \alpha_t, \beta_t \rangle dt \right. \\ \left. + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} f \left(BW_1 + \int_0^1 B \alpha_t dt + \frac{\Phi^{-1}(c)}{2} \int_0^1 B \beta_t dt \right) \right]$$

for $\alpha, \beta \in \mathcal{C}$. Then we have

$$\Phi_c^{-1} \left(\int \Phi_c(f) d\gamma_m \right) = \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_f^{B,c}[\alpha(\beta), \beta].$$

Proof. We begin by noting that, by the definition of Φ_c^{-1} , we can write

$$\Phi_c^{-1} \left(\int \Phi_c(f) d\gamma_m \right) = \Phi^{-1} \left(\int \Phi(\Phi^{-1}(c) + f \circ B) d\gamma_n \right) - \Phi^{-1}(c),$$

where we used that B is a projection (so that $\gamma_m = \gamma_n B^{-1}$).

Define $g := \Phi^{-1}(c) + f \circ B$. As $f \leq 0$, we have $g \leq \Phi^{-1}(c)$. Following *verbatim* the proof of the upper bound of Theorem 2.3, we have

$$\Phi^{-1} \left(\int \Phi(g) d\gamma_n \right) \leq J_g[\alpha^*(\beta) + \frac{1}{2} \Phi^{-1}(c)\beta, \beta]$$

for all $\beta \in \mathcal{C}$, where $\alpha^* \in \mathcal{S}$ is a strategy of the form

$$\alpha_t^*(\beta) = \left(\frac{1}{2} \Phi^{-1}(c) - v(t, X_t^\beta) \right) \nabla v(t, X_t^\beta), \quad v(t, x) = \Phi^{-1}(\mathbf{E}[\Phi(g)(W_1 - W_t + x)])$$

for a suitably defined random process X^β . The crucial observation at this point is that as $\nabla g(x) = B^* \nabla f(Bx)$, we have $\nabla v(t, x) \in \text{Im}(B^*)$ for all t, x . In particular, the optimal strategy α^* satisfies $B^* B \alpha^*(\beta) = \alpha^*(\beta)$ for every $\beta \in \mathcal{C}$. Therefore

$$\Phi^{-1} \left(\int \Phi(g) d\gamma_n \right) \leq \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_g[B^* B \alpha(\beta) + \frac{1}{2} \Phi^{-1}(c)\beta, \beta].$$

On the other hand, the corresponding lower bound follows immediately from Theorem 2.3 (as strategies of the form $B^* B \alpha(\beta) + \frac{1}{2} \Phi^{-1}(c)\beta$ form a subset of all possible strategies \mathcal{S}). Putting everything together, we have now shown that

$$\Phi_c^{-1} \left(\int \Phi_c(f) d\gamma_m \right) = \sup_{\alpha \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} J_g[B^* B \alpha(\beta) + \frac{1}{2} \Phi^{-1}(c)\beta, \beta] - \Phi^{-1}(c).$$

To complete the proof, it suffices to note that

$$\begin{aligned} & J_g[B^*B\alpha + \frac{1}{2}\Phi^{-1}(c)\beta, \beta] - \Phi^{-1}(c) \\ &= J_f^{B,c}[\alpha, \beta] + \Phi^{-1}(c) \mathbf{E} \left[\frac{1}{2} \int_0^1 e^{-\frac{1}{2} \int_0^t \|\beta_s\|^2 ds} \|\beta_t\|^2 dt + e^{-\frac{1}{2} \int_0^1 \|\beta_t\|^2 dt} - 1 \right] \\ &= J_f^{B,c}[\alpha, \beta], \end{aligned}$$

where we used the fundamental theorem of calculus. \square

With Proposition 4.4 in hand, we immediately obtain:

Corollary 4.5. *Theorem 4.2 is valid under the additional assumption that the functions f_1, \dots, f_k, h are uniformly continuous with values in $[\varepsilon, 1]$ for some $\varepsilon > 0$.*

The proof is identical to that of Corollary 3.1, and we omit the details.

Remark 4.6. Corollary 4.3 can be deduced directly from Corollary 4.5 by introducing smooth approximations of the indicator functions of the sets A_1, \dots, A_k and $\lambda_1 A_1 + \dots + \lambda_k A_k$. Such an argument is given in [8], and can be readily applied in the present setting. We therefore do not need the full strength of Theorem 4.2 to deduce Corollary 4.3. However, while the proof of Theorem 4.2 requires a bit more work, it yields a result that is potentially of broader utility.

4.4. Proof of Theorem 4.2: general case. The important part Theorem 4.2 is already contained in Corollary 4.5 above. The remaining arguments in the proof of Theorem 4.2 are technical: we must approximate the measurable functions f_1, \dots, f_k, h by uniformly continuous functions so that Corollary 4.5 can be applied. The requisite approximation arguments are worked out in this section. (Closely related approximation arguments can also be found in [13].)

We begin by proving Theorem 4.2 in the case that f_1, \dots, f_k and h are upper-semicontinuous. The following lemma makes it possible to approximate upper-semicontinuous functions by uniformly continuous functions without violating the assumption of Theorem 4.2, so that Corollary 4.5 can be applied.

Lemma 4.7. *Let f_1, \dots, f_k, h be upper-semicontinuous functions $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$, $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with values in $[\varepsilon, 1]$ for some $\varepsilon > 0$. Let $c \in (0, 1)$, and suppose that*

$$\lambda_1 \Phi_c^{-1}(f_1(x_1)) + \dots + \lambda_k \Phi_c^{-1}(f_k(x_k)) \leq \Phi_c^{-1}(h(\lambda_1 B_1^* x_1 + \dots + \lambda_k B_k^* x_k))$$

for all $x_i \in \mathbb{R}^{n_i}$. Then there exist for every $s > 0$ uniformly continuous functions f_1^s, \dots, f_k^s, h^s with values in $[\varepsilon, 1]$ such that

$$\lambda_1 \Phi_c^{-1}(f_1^s(x_1)) + \dots + \lambda_k \Phi_c^{-1}(f_k^s(x_k)) \leq \Phi_c^{-1}(h^s(\lambda_1 B_1^* x_1 + \dots + \lambda_k B_k^* x_k))$$

for all $x_i \in \mathbb{R}^{n_i}$, and such that $f_i^s \rightarrow f_i$ and $h^s \rightarrow h$ pointwise as $s \downarrow 0$.

Proof. Define f_i^s and h^s by the sup-convolutions

$$\Phi_c^{-1}(f_i^s(x)) := \sup_{y \in \mathbb{R}^{n_i}} \{ \Phi_c^{-1}(f_i(y)) - s^{-1} \|x - y\| \},$$

$$\Phi_c^{-1}(h^s(x)) := \sup_{y \in \mathbb{R}^n} \{ \Phi_c^{-1}(h(y)) - s^{-1} \|x - y\| \}.$$

It is easily seen that f_i^s, h^s take values in $[\varepsilon, 1]$, and that $\Phi_c^{-1}(f_i^s)$ and $\Phi_c^{-1}(h^s)$ are s^{-1} -Lipschitz; thus f_i^s and h^s are certainly uniformly continuous. We now claim that $h^s \rightarrow h$ as $s \downarrow 0$. To see this, choose for every $s > 0$ a point y_s such that

$$\Phi_c^{-1}(h^s(x)) \leq \Phi_c^{-1}(h(y_s)) - s^{-1} \|x - y_s\| + s.$$

As $h^s \geq \varepsilon$ and $h \leq 1$, this evidently implies $\|x - y_s\| \leq s^2 - s\Phi_c^{-1}(\varepsilon)$ for all s , so that $y_s \rightarrow x$ as $s \downarrow 0$. But we can now estimate

$$\begin{aligned} \Phi_c^{-1}(h(x)) &\leq \liminf_{s \downarrow 0} \Phi_c^{-1}(h^s(x)) \leq \limsup_{s \downarrow 0} \Phi_c^{-1}(h^s(x)) \\ &\leq \limsup_{s \downarrow 0} \Phi_c^{-1}(h(y_s)) \leq \Phi_c^{-1}(h(x)), \end{aligned}$$

where we have used that h is upper-semicontinuous in the last line. This shows that $h^s \rightarrow h$ pointwise as $s \downarrow 0$, and $f_i^s \rightarrow f_i$ follows identically. Finally, note that

$$\begin{aligned} &\lambda_1 \Phi_c^{-1}(f_1^s(x_1)) + \cdots + \lambda_k \Phi_c^{-1}(f_k^s(x_k)) \\ &= \sup_{y_1, \dots, y_k} \{ \lambda_1 \Phi_c^{-1}(f_1(y_1)) + \cdots + \lambda_k \Phi_c^{-1}(f_k(y_k)) \\ &\quad - s^{-1} \lambda_1 \|x_1 - y_1\| - \cdots - s^{-1} \lambda_k \|x_k - y_k\| \} \\ &\leq \sup_{y_1, \dots, y_k} \{ \Phi_c^{-1}(h(\lambda_1 B_1^* y_1 + \cdots + \lambda_k B_k^* y_k)) \\ &\quad - s^{-1} \| \lambda_1 B_1^*(x_1 - y_1) + \cdots + \lambda_k B_k^*(x_k - y_k) \| \} \\ &\leq \Phi_c^{-1}(h^s(\lambda_1 B_1^* x_1 + \cdots + \lambda_k B_k^* x_k)), \end{aligned}$$

where we have used that $\|B_i^* z\| = \|z\|$ for $z \in \mathbb{R}^{n_i}$ and the triangle inequality. \square

Using Lemma 4.7 and Corollary 4.5, we can now prove the following.

Corollary 4.8. *Theorem 4.2 is valid under the additional assumption that the functions f_1, \dots, f_k, h are upper-semicontinuous with values in $[0, 1]$.*

Proof. We first approximate f_1, \dots, f_k, h by functions that are bounded away from zero. To this end, fix $\varepsilon \in (0, 1)$ and let $\delta := \max_i \Phi_c(\lambda_i \Phi_c^{-1}(\varepsilon))$. Define the upper-semicontinuous functions $\bar{h} := h \vee \delta$ and $\bar{f}_i := f_i \vee \varepsilon$ for all i . We claim that

$$\lambda_1 \Phi_c^{-1}(\bar{f}_1(x_1)) + \cdots + \lambda_k \Phi_c^{-1}(\bar{f}_k(x_k)) \leq \Phi_c^{-1}(\bar{h}(\lambda_1 B_1^* x_1 + \cdots + \lambda_k B_k^* x_k)).$$

Indeed, if $f_i(x_i) > \varepsilon$ for all i this follows from the assumption of Theorem 4.2, while if $f_i(x_i) \leq \varepsilon$ for some i the left-hand side is at most $\Phi_c^{-1}(\delta)$.

Applying Lemma 4.7, we can find uniformly continuous functions $\bar{f}_1^s, \dots, \bar{f}_k^s, \bar{h}^s$ with values in $[\varepsilon, 1]$ such that $\bar{f}_i^s \rightarrow \bar{f}_i$ and $\bar{h}^s \rightarrow \bar{h}$ pointwise as $s \downarrow 0$ and

$$\lambda_1 \Phi_c^{-1}(\bar{f}_1^s(x_1)) + \cdots + \lambda_k \Phi_c^{-1}(\bar{f}_k^s(x_k)) \leq \Phi_c^{-1}(\bar{h}^s(\lambda_1 B_1^* x_1 + \cdots + \lambda_k B_k^* x_k))$$

for every $s > 0$. Corollary 4.5 implies

$$\lambda_1 \Phi_c^{-1} \left(\int \bar{f}_1^s d\gamma_{n_1} \right) + \cdots + \lambda_k \Phi_c^{-1} \left(\int \bar{f}_k^s d\gamma_{n_k} \right) \leq \Phi_c^{-1} \left(\int \bar{h}^s d\gamma_n \right).$$

The conclusion follows using dominated convergence as $s \downarrow 0$ and $\varepsilon \downarrow 0$. \square

We can now complete the proof of Theorem 4.2.

Proof of Theorem 4.2. Let $\tilde{f}_1, \dots, \tilde{f}_k$ be upper-semicontinuous functions with compact support and with values in $[0, 1]$ such that $\tilde{f}_i \leq f_i$ for all i . Define \tilde{h} by

$$\Phi_c^{-1}(\tilde{h}(x)) := \sup_{\lambda_1 B_1^* x_1 + \cdots + \lambda_k B_k^* x_k = x} \{ \lambda_1 \Phi_c^{-1}(\tilde{f}_1(x_1)) + \cdots + \lambda_k \Phi_c^{-1}(\tilde{f}_k(x_k)) \}.$$

Then $\tilde{h} \leq h$ by construction, and \tilde{h} is also upper-semicontinuous [35, Prop. 1.27]. Moreover, the upper-semicontinuous functions $\tilde{f}_1, \dots, \tilde{f}_k$ and \tilde{h} clearly satisfy the assumptions of Theorem 4.2. Therefore, Corollary 4.8 implies

$$\lambda_1 \Phi_c^{-1} \left(\int \tilde{f}_1 d\gamma_{n_1} \right) + \dots + \lambda_k \Phi_c^{-1} \left(\int \tilde{f}_k d\gamma_{n_k} \right) \leq \Phi_c^{-1} \left(\int h d\gamma_n \right).$$

The conclusion now follows by taking the supremum on the left-hand side over all compactly supported upper-semicontinuous functions $\tilde{f}_i \leq f_i$ [20, Prop. 7.14]. \square

5. GENERALIZED MEANS

Unlike the logarithmic functional $f \mapsto \log(\int e^f d\gamma_n)$, whose stochastic representation has a natural interpretation through the Gibbs variational principle, the emergence of a stochastic game representation for $f \mapsto \Phi^{-1}(\int \Phi(f) d\gamma_n)$ may appear rather unexpected. To provide some further insight into such representations, we aim in this section to place the result of Theorem 2.3 in a broader context.

Throughout this section, let $I \subset \mathbb{R}$ be a compact interval, and let $F : I \rightarrow \mathbb{R}$ be a smooth function that is strictly increasing $F' > 0$. Following Hardy, Littlewood, and Pólya [23, chapter 3], we define the generalized mean \mathfrak{M}_F as

$$\mathfrak{M}_F(f) := F^{-1} \left(\int F(f) d\gamma_n \right)$$

for any measurable function $f : \mathbb{R}^n \rightarrow I$. We will argue below that the generalized mean \mathfrak{M}_F admits a stochastic representation for any sufficiently regular function F : from this perspective, there is nothing particularly special about the specific cases $F(x) = e^x$ and $F(x) = \Phi(x)$ that we encountered so far. Of course, the potential utility of such stochastic representations in other settings depends on the problem at hand. For example, to establish Brunn-Minkowski type inequalities, we crucially exploited a special feature of the functions $F(x) = e^x$ and $F(x) = \Phi(x)$: in both cases, the running cost in the stochastic representation proves to be a concave function of the strategy that is being maximized over. While such structural features of the representation are specific to particular choices of F , the existence of a stochastic representation is not anything special in its own right.

In their study of generalized means, Hardy, Littlewood, and Pólya [23, §3.16] obtained necessary and sufficient conditions for $f \mapsto \mathfrak{M}_F(f)$ to be a convex functional. In section 5.1, we will show that stochastic representations provide an interesting perspective on this characterization: the conditions of Hardy, Littlewood, and Pólya are precisely those that are needed to obtain a stochastic representation for \mathfrak{M}_F involving only a supremum (as in the case $F(x) = e^x$). In particular, we can state a very general expression for the stochastic representation in this setting, despite that the Fenchel transform of \mathfrak{M}_F (and therefore the natural analogue of the Gibbs variational principle) rarely admits a tractable expression. For generalized means that are not convex, we will outline in section 5.2 how one can obtain in this case a stochastic game representation of \mathfrak{M}_F under essentially no assumptions on F . As the explicit expressions that define such games for general F do not provide much insight, we do not state a general theorem, but rather illustrate by means of an example how easily such representations can be obtained in practice.

5.1. The convex case. The following result due to Hardy, Littlewood, and Pólya characterizes precisely when the functional $f \mapsto \mathfrak{M}_F(f)$ is convex.

Theorem 5.1 ([23, §3.16]). *The generalized mean functional $f \mapsto \mathfrak{M}_F(f)$ is convex if and only if the function F is convex and the function F'/F'' is concave.*

Proof. The following facts are explicitly stated and proved in [23, §3.16]:

- Convexity of F is necessary for \mathfrak{M}_F to be convex.
- If F is strictly convex $F'' > 0$, then concavity of F'/F'' is necessary and sufficient for \mathfrak{M}_F to be convex.

For completeness, we spell out what happens when F is convex but fails to be strictly convex. We should consider two separate cases:

- If F'' vanishes everywhere in I , then F is linear and convexity of \mathfrak{M}_F is trivial (note that in this case $F'/F'' \equiv +\infty$ is clearly concave).
- If F'' vanishes at some point but not everywhere in I , then F'/F'' must blow up to $+\infty$ near that point as we assumed that $F' > 0$ and that F is smooth. This implies there is a subinterval $J \subset I$ on which $F'' > 0$ but where F'/F'' fails to be concave, so convexity of \mathfrak{M}_F must fail.

We have therefore established all possible cases of Theorem 5.1. \square

When $F(x) = e^x$, the condition of Theorem 5.1 is evidently satisfied; in this case, convexity of \mathfrak{M}_F is simply the statement of Hölder's inequality. On the other hand, when $F(x) = \Phi(x)$, the condition for convexity fails to be satisfied on any interval I . In particular, while log-concavity could formally be viewed as a “reverse” form of Hölder's inequality, there cannot exist a Gaussian improvement of Hölder's inequality that is analogous to Ehrhard's improvement of log-concavity. Similarly, in contrast to the Gaussian improvement of the reverse Brascamp-Lieb inequality of Theorem 4.2, there cannot exist an analogous Gaussian improvement of the direct Brascamp-Lieb inequality (of which Hölder's inequality is a special case).

Remark 5.2. The proof of Theorem 5.1 shows that unless F is linear, convexity of the functional \mathfrak{M}_F requires that F is strictly convex $F'' > 0$ everywhere in I . We will therefore assume the latter without loss of generality in our development of stochastic representations for convex generalized means.

The relevance of the conditions of Theorem 5.1 is far from obvious at first sight. We will presently see that these conditions arise in a very natural manner when we attempt to obtain a stochastic representation for \mathfrak{M}_F .

We begin by developing the argument of section 2.2 in the present setting. Let $f : \mathbb{R}^n \rightarrow I$ be a Lipschitz function and define for $(t, x) \in [0, 1] \times \mathbb{R}^n$

$$u(t, x) := \mathbf{E}[F(f(W_1 - W_t + x))],$$

so that u solves the heat equation. Define

$$v(t, x) := F^{-1}(u(t, x)).$$

Note that as F is smooth and $F' > 0$, the function F^{-1} is smooth by the inverse function theorem. Therefore, by elementary properties of the heat equation, v takes values in I , is smooth and has bounded derivatives of all orders on $[0, 1 - \varepsilon] \times \mathbb{R}^n$ for every $\varepsilon > 0$, and $v(t, x) \rightarrow f(x)$ uniformly in x as $t \rightarrow 1$. Using

$$\frac{\partial u}{\partial t} = F'(v) \frac{\partial v}{\partial t}, \quad \Delta u = F'(v) \Delta v + F''(v) \|\nabla v\|^2$$

and the heat equation for u shows that v satisfies the PDE

$$\frac{\partial v}{\partial t} + \frac{1}{2}\Delta v + \frac{1}{2}\frac{F''(v)}{F'(v)}\|\nabla v\|^2 = 0, \quad v(1, x) = f(x).$$

We now readily see the relevance of the conditions of Theorem 5.1: as the function $(x, y) \mapsto \|x\|^2/y$ is convex for $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+$, the conditions of Theorem 5.1 are precisely those that ensure that the nonlinear term in this PDE is a convex function of $(\nabla v, v)$. In particular, we can express this term as follows.

Lemma 5.3. *Suppose that $F'' > 0$ and that F'/F'' is concave. Denote by $R := (-F'/F'')^*$ the Fenchel transform of the convex function $-F'/F''$. Then*

$$\frac{1}{2}\frac{F''(v)}{F'(v)}\|\nabla v\|^2 = \sup_{a \in \mathbb{R}^n} \sup_{b \in \mathbb{R}} \left\{ \langle a, \nabla v \rangle + \frac{1}{2}vb\|a\|^2 - \frac{1}{2}R(b)\|a\|^2 \right\},$$

where the optimizer is $a^* = (F''(v)/F'(v))\nabla v$ and $b^* = F'(v)F'''(v)/F''(v)^2 - 1$.

Proof. The optimization $\sup_{b \in \mathbb{R}} \{vb - R(b)\} = -F'(v)/F''(v)$ is simply the definition of the Fenchel conjugate. Moreover, as F is assumed to be smooth, the optimizer is given by $b^* = (-F'/F'')'(v)$ [35, Prop. 11.3]. The optimization over a is trivial. \square

Lemma 5.3 reveals that the partial differential equation satisfied by v is none other than the Bellman equation for the value of a stochastic control problem [18].

Theorem 5.4. *Let $F : I \rightarrow \mathbb{R}$ be a nonlinear smooth and strictly increasing function such that \mathfrak{M}_F is convex. Then \mathfrak{M}_F admits the stochastic representation*

$$\mathfrak{M}_F(f) = \sup_{\alpha \in \mathcal{C}_b^k} \sup_{\beta \in \mathcal{C}_b^1} K_f[\alpha, \beta]$$

for every lower-semicontinuous function $f : \mathbb{R}^n \rightarrow I$, where

$$K_f[\alpha, \beta] := \mathbf{E} \left[e^{\frac{1}{2} \int_0^1 \beta_t \|\alpha_t\|^2 dt} f \left(W_1 + \int_0^1 \alpha_t dt \right) - \frac{1}{2} \int_0^1 e^{\frac{1}{2} \int_0^t \beta_s \|\alpha_s\|^2 ds} R(\beta_t) \|\alpha_t\|^2 dt \right]$$

with $R := (-F'/F'')^*$. Here \mathcal{C}_b^k denotes the family of all k -dimensional uniformly bounded and progressively measurable processes.

This result should be viewed as an explicit stochastic representation of Fenchel duality for the convex functional \mathfrak{M}_F . In particular, as $K_f[\alpha, \beta]$ is linear in f , the convexity of \mathfrak{M}_F is immediately obvious from the representation.

Proof. We first assume that the function f is Lipschitz. Define $W_t^\alpha := W_t + \int_0^t \alpha_s ds$. Applying Itô's formula to $e^{\frac{1}{2} \int_0^t \beta_s \|\alpha_s\|^2 ds} v(t, W_t^\alpha)$ yields

$$\begin{aligned} & e^{\frac{1}{2} \int_0^1 \beta_t \|\alpha_t\|^2 dt} f(W_1^\alpha) - \frac{1}{2} \int_0^1 e^{\frac{1}{2} \int_0^t \beta_s \|\alpha_s\|^2 ds} R(\beta_t) \|\alpha_t\|^2 dt = \\ & v(0, 0) + \int_0^1 e^{\frac{1}{2} \int_0^t \beta_s \|\alpha_s\|^2 ds} \langle \nabla v(t, W_t^\alpha), dW_t \rangle \\ & + \int_0^1 e^{\frac{1}{2} \int_0^t \beta_s \|\alpha_s\|^2 ds} \left\{ \frac{\partial v}{\partial t}(t, W_t^\alpha) + \frac{1}{2} \Delta v(t, W_t^\alpha) + \langle \alpha_t, \nabla v(t, W_t^\alpha) \rangle \right. \\ & \quad \left. + \frac{1}{2} \beta_t \|\alpha_t\|^2 v(t, W_t^\alpha) - \frac{1}{2} R(\beta_t) \|\alpha_t\|^2 \right\} dt. \end{aligned}$$

As f is Lipschitz, ∇v is uniformly bounded so the stochastic integral is a martingale. Taking the expectation, and using Lemma 5.3 and the partial differential equation for v yields $K_f[\alpha, \beta] \leq v(0, 0) = \mathfrak{M}_F(f)$ for every $\alpha \in \mathcal{C}_b^n$ and $\beta \in \mathcal{C}_b^1$. Thus

$$\sup_{\alpha \in \mathcal{C}_b^n} \sup_{\beta \in \mathcal{C}_b^1} K_f[\alpha, \beta] \leq \mathfrak{M}_F(f).$$

It remains to note that the inequality is equality if we choose the optimal controls

$$\begin{aligned} \alpha_t^* &= \frac{F''(v(t, X_t))}{F'(v(t, X_t))} \nabla v(t, X_t), \\ \beta_t^* &= \frac{F'(v(t, X_t)) F'''(v(t, X_t))}{F''(v(t, X_t))^2} - 1, \end{aligned}$$

where X_t is the solution of the stochastic differential equation

$$dX_t = \frac{F''(v(t, X_t))}{F'(v(t, X_t))} \nabla v(t, X_t) dt + dW_t, \quad X_0 = 0.$$

Here we note that by our assumptions, F has bounded derivatives of all orders and F' and F'' are uniformly bounded away from zero, v and ∇v are uniformly bounded, and $v(t, \cdot)$ has bounded derivatives of all orders for $t < 1$, so that this stochastic differential equation has a unique strong solution and $\alpha^* \in \mathcal{C}_b^n$, $\beta^* \in \mathcal{C}_b^1$.

Now assume f is only lower-semicontinuous. Let $f_k(x) = \inf_y \{f(y) + k\|x - y\|\}$. Then $f_k : \mathbb{R}^n \rightarrow I$ is Lipschitz for every k and $f_k \uparrow f$ pointwise as in the proof of Lemma 4.7. The result follows using monotone convergence by applying the stochastic representation of $\mathfrak{M}_F(f_k)$ and taking the supremum over k . \square

Let us illustrate Theorem 5.4 in some simple examples.

Example 5.5. Consider the case $F(x) = e^x$ that arises from the Gibbs variational principle. Then $F'/F'' \equiv 1$, so we readily compute $R = (-F'/F'')^*$ as

$$R(x) = \begin{cases} 1 & \text{for } x = 0, \\ +\infty & \text{for } x \neq 0. \end{cases}$$

Substituting this expression into Theorem 5.4, we immediately recover the stochastic representation discussed in the introduction.

Example 5.6. Consider the case $F(x) = x^p$ for $p > 1$, where we choose $I = [x_-, x_+]$ for some $0 < x_- < x_+ < \infty$. Then $F', F'' > 0$ and $F'(x)/F''(x) = x/(p-1)$ is certainly concave. We readily compute $R = (-F'/F'')^*$ as

$$R(x) = \begin{cases} 0 & \text{for } x = -1/(p-1), \\ +\infty & \text{for } x \neq -1/(p-1). \end{cases}$$

Substituting this expression into Theorem 5.4 yields

$$\left(\int f^p d\gamma_n \right)^{1/p} = \sup_{\alpha \in \mathcal{C}_b^n} \mathbf{E} \left[e^{-\frac{1}{2(p-1)} \int_0^1 \|\alpha_t\|^2 dt} f \left(W_1 + \int_0^1 \alpha_t dt \right) \right].$$

This result could also be obtained along the lines of [11] by applying Girsanov's theorem to the representation $(\int f^p d\gamma_n)^{1/p} = \sup_{g>0: \int g d\gamma_n=1} \int g^{1-1/p} f d\gamma_n$.

Example 5.7. Consider the case $F(x) = xe^x$ on $I = [0, C]$ for some $C < \infty$. Then $F', F'' > 0$ and $F'(x)/F''(x) = (1+x)/(2+x)$ is concave. We compute

$$R(x) = \begin{cases} -2\sqrt{-x} - 2x + 1 & \text{for } x \leq 0, \\ +\infty & \text{for } x > 0. \end{cases}$$

Substituting this expression into Theorem 5.4 yields

$$\begin{aligned} \mathbb{W}\left(\int f e^f d\gamma_n\right) &= \sup_{\alpha \in \mathcal{C}_b^n} \sup_{\gamma \in \mathcal{C}_b^1} \mathbf{E}\left[e^{-\frac{1}{2} \int_0^1 \gamma_t^2 \|\alpha_t\|^2 dt} f\left(W_1 + \int_0^1 \alpha_t dt\right)\right. \\ &\quad \left. - \frac{1}{2} \int_0^1 e^{-\frac{1}{2} \int_0^t \gamma_s^2 \|\alpha_s\|^2 ds} (2\gamma_t^2 - 2|\gamma_t| + 1) \|\alpha_t\|^2 dt\right], \end{aligned}$$

where $W = F^{-1}$ is the Lambert W -function and we defined $\beta_t := -\gamma_t^2$ to enforce nonpositivity. This expression can be simplified slightly by introducing the new control $\eta_t := |\gamma_t| \alpha_t$. Rearranging the above expression then yields

$$\begin{aligned} \mathbb{W}\left(\int f e^f d\gamma_n\right) &= \sup_{\alpha, \eta \in \mathcal{C}_b^n} \mathbf{E}\left[e^{-\frac{1}{2} \int_0^1 \|\eta_t\|^2 dt} f\left(W_1 + \int_0^1 \alpha_t dt\right)\right. \\ &\quad \left. - \frac{1}{2} \int_0^1 e^{-\frac{1}{2} \int_0^t \|\eta_s\|^2 ds} (\|\eta_t\|^2 + \|\alpha_t - \eta_t\|^2) dt\right]. \end{aligned}$$

We remark that the stochastic representation appears in surprisingly tractable form, while it is not clear whether it is possible to obtain a tractable analogue of the Gibbs variational principle $\mathfrak{M}_F(f) = \sup_{\mu} \{\int f d\mu - \mathfrak{M}_F^*(\mu)\}$ in this example.

5.2. Generalized means and stochastic games. The essential idea behind the proof of Theorem 5.4 was that when \mathfrak{M}_F is convex, the nonlinear equation

$$\frac{\partial v}{\partial t} + \frac{1}{2} \Delta v + \frac{1}{2} \frac{F''(v)}{F'(v)} \|\nabla v\|^2 = 0$$

could be expressed as a supremum of linear parabolic equations

$$\sup_a \left\{ \frac{\partial v}{\partial t} + \frac{1}{2} \Delta v + \langle c_1(a), \nabla v \rangle + c_2(a)v + c_3(a) \right\} = 0$$

for some functions c_1, c_2, c_3 . Such a representation cannot hold when \mathfrak{M}_F fails to be convex. Nonetheless, even in the absence of convexity, we can try to express the above nonlinear equation in the more complicated form

$$\sup_a \inf_b \left\{ \frac{\partial v}{\partial t} + \frac{1}{2} \Delta v + \langle c_1(a, b), \nabla v \rangle + c_2(a, b)v + c_3(a, b) \right\} = 0$$

for some functions c_1, c_2, c_3 . If this is possible, then the arguments of Theorem 2.3 could be adapted to obtain a stochastic game representation for \mathfrak{M}_F .

It has long been understood in the PDE literature that while convexity is a very special property, almost any reasonable nonlinearity can be expressed in the form of a game; see [16, 17] and the references therein. In the present context, this implies that it is possible to obtain stochastic game representations for generalized means \mathfrak{M}_F under essentially no assumptions on the function F . Let us outline one particular approach to obtaining such representations. In [16, §4.1], it is observed that any locally Lipschitz function $\Psi : \mathbb{R}^k \rightarrow \mathbb{R}$ can be represented as

$$\Psi(x) = \max_{a \in \mathbb{R}^k} \min_{b \in \mathbb{R}^k} \left\{ \int_0^1 \langle \nabla \Psi((1-t)a + tb), x - a \rangle dt + \Psi(a) \right\}$$

(indeed, it suffices to note that $a^* = b^* = x$ is a saddle point). Now assume, as in the general setting of this section, that $F : I \rightarrow \mathbb{R}$ is a smooth function on the

compact interval I that is strictly increasing $F' > 0$, and consider the function

$$\Psi(x, y) = \frac{1}{2} \frac{F''(x)}{F'(x)} \|y\|^2, \quad (x, y) \in I \times \mathbb{R}^n.$$

Then the gradient of Ψ is locally bounded, and thus the above maximin representation holds. It is a simple exercise to repeat the proof of Theorem 2.3 in the present setting to obtain a completely general stochastic game representation for \mathfrak{M}_F .

Remark 5.8. There are two minor issues that require care in extending Theorem 2.3 to the general setting. First, the representation will hold for functions f that are smooth with bounded derivatives, but one cannot trivially extend to bounded uniformly continuous functions (as in the present case the exponential factor in front of f in the representation need not have a universal upper bound). Second, for the same reason, one should work with the smaller classes of controls and strategies $\mathcal{C}_b := \{\beta \in \mathcal{C} : \|\beta\|_\infty < \infty\}$, $\mathcal{S}_b := \{\alpha \in \mathcal{S} : \sup_{\|\beta\|_\infty \leq R} \|\alpha(\beta)\|_\infty < \infty \forall R < \infty\}$.

Carrying out the approach outlined above would give rise to a very general representation of \mathfrak{M}_F as a stochastic game. However, this representation is not canonical. The usefulness of a stochastic game representation in a given situation will generally rely on some structural properties of the representation that may be far from evident in this particular formulation. For example, applying the above representation to the case $F(x) = \Phi(x)$ yields a rather ugly expression from which one would be hard-pressed to conclude the validity of Ehrhard's inequality. While the existence of stochastic game representations for general \mathfrak{M}_F sheds some light on the origin of the phenomenon observed in Theorem 2.3, a genuinely useful representation of this kind should be specifically chosen to possess the desired structural properties that are relevant to the problem under consideration. We have already seen an example of this in the previous section, where special representations were chosen for convex \mathfrak{M}_F from which the convexity property becomes evident, and in Theorem 2.3. As a further illustration we provide one additional example.

Example 5.9. Consider $F(x) = 1 - e^{-x^2/2}$ on $I = [\varepsilon, 2c]$ for $0 < \varepsilon < 2c < \infty$. This function behaves very similarly to $\Phi(x)$ as $x \rightarrow \infty$, at least to leading order in the exponent. We might therefore expect a stochastic game representation of \mathfrak{M}_F that is similar to that of Theorem 2.3. Let us see how this can be achieved.

We begin by computing

$$\frac{F''(x)}{F'(x)} = -x + \frac{1}{x}.$$

Note that the term $-x$ is precisely what arises for Φ , but we now have an additional term. To obtain a representation that is similar to that for Φ , we apply Lemma 2.5 to the first term and introduce an additional control for the second term:

$$\begin{aligned} \frac{1}{2} \frac{F''(v)}{F'(v)} \|\nabla v\|^2 &= -\frac{1}{2} v \|\nabla v\|^2 + \frac{1}{2v} \|\nabla v\|^2 \\ &= \sup_{a \in \mathbb{R}^n} \inf_{b \in \mathbb{R}^n} \left\{ \langle a + cb, \nabla v + b \rangle - \frac{1}{2} v \|b\|^2 \right\} + \sup_{\tilde{a} \in \mathbb{R}^n} \left\{ \langle \tilde{a}, \nabla v \rangle - \frac{1}{2} v \|\tilde{a}\|^2 \right\}. \end{aligned}$$

One can now repeat the proof of Theorem 2.3 to obtain the representation

$$\begin{aligned} \mathfrak{M}_F(f) &= \sqrt{-2 \log \left(\int e^{-f^2/2} d\gamma_n \right)} \\ &= \sup_{\alpha, \tilde{\alpha} \in \mathcal{S}} \inf_{\beta \in \mathcal{C}} \mathbf{E} \left[\int_0^1 e^{-\frac{1}{2} \int_0^t (\|\tilde{\alpha}_s(\beta)\|^2 + \|\beta_s\|^2) ds} \langle \alpha_t(\beta), \beta_t \rangle dt \right. \\ &\quad \left. + e^{-\frac{1}{2} \int_0^1 (\|\tilde{\alpha}_t(\beta)\|^2 + \|\beta_t\|^2) dt} f \left(W_1 + \int_0^1 (\alpha_t(\beta) + \tilde{\alpha}_t(\beta)) dt \right) \right] \end{aligned}$$

for any bounded, uniformly continuous, and nonnegative function f . Notice that, while this representation is quite close to that of Theorem 2.3 (in particular, we see that $\mathfrak{M}_F \geq \mathfrak{M}_\Phi$ by setting $\tilde{\alpha} = 0$), the present representation is not concave in $(\alpha, \tilde{\alpha})$ and we therefore do not obtain an Ehrhard-type inequality for \mathfrak{M}_F .

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