Simplex Method

An Example.

maximize \(-x_1 + 3x_2 - 3x_3\)
subject to 
\[
\begin{align*}
3x_1 &- x_2 - 2x_3 \leq 7 \\
-2x_1 &- 4x_2 + 4x_3 \leq 3 \\
x_1 &- 2x_3 \leq 4 \\
-2x_1 &+ 2x_2 + x_3 \leq 8 \\
3x_1 &\leq 5 \\
\end{align*}
\]
\[x_1, x_2, x_3 \geq 0.\]
Rewrite with slack variables

maximize \[ \zeta = -x_1 + 3x_2 - 3x_3 \]
subject to \[ \begin{align*} w_1 &= 7 - 3x_1 + x_2 + 2x_3 \\ w_2 &= 3 + 2x_1 + 4x_2 - 4x_3 \\ w_3 &= 4 - x_1 + 2x_3 \\ w_4 &= 8 + 2x_1 - 2x_2 - x_3 \\ w_5 &= 5 - 3x_1 \end{align*} \]
\[ x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0. \]

Notes:
• This layout is called a dictionary.
• Setting \( x_1, x_2, \) and \( x_3 \) to 0, we can read off the values for the other variables: \( w_1 = 7, \ w_2 = 3, \) etc. This specific solution is called a dictionary solution.
• Dependent variables, on the left, are called basic variables.
• Independent variables, on the right, are called nonbasic variables.
Dictionary Solution is Feasible

\[
\begin{align*}
\text{maximize} & \quad \zeta = -x_1 + 3x_2 - 3x_3 \\
\text{subject to} & \quad w_1 = 7 - 3x_1 + x_2 + 2x_3 \\
& \quad w_2 = 3 + 2x_1 + 4x_2 - 4x_3 \\
& \quad w_3 = 4 - x_1 + 2x_3 \\
& \quad w_4 = 8 + 2x_1 - 2x_2 - x_3 \\
& \quad w_5 = 5 - 3x_1 \\
& \quad x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0.
\end{align*}
\]

Notes:

- All the variables in the current dictionary solution are nonnegative.
- Such a solution is called \textit{feasible}.
- The initial dictionary solution need not be feasible—we were just lucky above.
Simplex Method—First Iteration

- If $x_2$ increases, obj goes up.
- How much can $x_2$ increase? Until $w_4$ decreases to zero.
- Do it. End result: $x_2 > 0$ whereas $w_4 = 0$.
- That is, $x_2$ must become basic and $w_4$ must become nonbasic.
- Algebraically rearrange equations to, in the words of Jean-Luc Picard, ”Make it so.”
- This is a pivot.
A Pivot: $x_2 \leftrightarrow w_4$

\[
\begin{array}{c|cccc|cccc}
\text{obj} & \text{Current Dictionary} & & & \text{obj} & \text{Current Dictionary} \\
\hline
w_1 & 7.0 & - & 3.0 & x_1 & - & -1.0 & x_1 & - & 3.0 & x_2 & + & -3.0 & x_3 \\
w_2 & 3.0 & - & -2.0 & x_1 & - & -4.0 & x_2 & - & 4.0 & x_3 \\
w_3 & 4.0 & - & 1.0 & x_1 & - & 0.0 & x_2 & - & -2.0 & x_3 \\
w_4 & 8.0 & - & -2.0 & x_1 & - & 2.0 & x_2 & - & 1.0 & x_3 \\
w_5 & 5.0 & - & 3.0 & x_1 & - & 0.0 & x_2 & - & 0.0 & x_3 \\
\end{array}
\]

becomes

\[
\begin{array}{c|cccc|cccc}
\text{obj} & \text{Current Dictionary} & & & \text{obj} & \text{Current Dictionary} \\
\hline
w_1 & 11.0 & - & 2.0 & x_1 & - & -1.5 & w_4 & + & -4.5 & x_3 \\
w_2 & 19.0 & - & -6.0 & x_1 & - & 2.0 & w_4 & - & 6.0 & x_3 \\
w_3 & 4.0 & - & 1.0 & x_1 & - & 0.0 & w_4 & - & -2.0 & x_3 \\
w_4 & 4.0 & - & -1.0 & x_1 & - & 0.5 & w_4 & - & 0.5 & x_3 \\
w_5 & 5.0 & - & 3.0 & x_1 & - & 0.0 & w_4 & - & 0.0 & x_3 \\
\end{array}
\]
Here’s the dictionary after the first pivot:

<table>
<thead>
<tr>
<th></th>
<th>obj =</th>
<th>w1 =</th>
<th>w2 =</th>
<th>w3 =</th>
<th>x2 =</th>
<th>w5 =</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>12.0</td>
<td>11.0</td>
<td>19.0</td>
<td>4.0</td>
<td>4.0</td>
<td>5.0</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Current Dictionary</td>
<td>Current Dictionary</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>-2.0</td>
<td>-6.0</td>
<td>-1.0</td>
<td>-3.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>x1</td>
<td>x1</td>
<td>x1</td>
<td>x1</td>
<td>x1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.5</td>
<td>0.5</td>
<td>2.0</td>
<td>0.5</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-4.5</td>
<td>-1.5</td>
<td>6.0</td>
<td>-2.0</td>
<td>0.5</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>x3</td>
<td>x3</td>
<td>x3</td>
<td>x3</td>
<td>x3</td>
<td>x3</td>
</tr>
</tbody>
</table>

- Now, let $x_1$ increase.
- Of the basic variables, $w_5$ hits zero first.
- So, $x_1$ enters and $w_5$ leaves the basis.
- New dictionary is...
### Simplex Method—Final Dictionary

<table>
<thead>
<tr>
<th></th>
<th>Current Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>obj</strong></td>
<td>$\frac{46}{3}$</td>
</tr>
<tr>
<td><strong>$w_1$</strong></td>
<td>$\frac{23}{3}$</td>
</tr>
<tr>
<td><strong>$w_2$</strong></td>
<td>29</td>
</tr>
<tr>
<td><strong>$w_3$</strong></td>
<td>$\frac{7}{3}$</td>
</tr>
<tr>
<td><strong>$x_2$</strong></td>
<td>$\frac{17}{3}$</td>
</tr>
<tr>
<td><strong>$x_1$</strong></td>
<td>$\frac{5}{3}$</td>
</tr>
</tbody>
</table>

- It’s optimal (no pink)!
- Click [here](#) to practice the simplex method.
- For instructions, click [here](#).
Agenda

- Discuss *unboundedness*; (today)

- Discuss initialization/*infeasibility*; i.e., what if initial dictionary is not feasible. (today)

- Discuss *degeneracy*. (next lecture)
### Unboundedness

Consider the following dictionary:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>w1 =</th>
<th>w2 =</th>
<th>w3 =</th>
<th>w4 =</th>
<th>w5 =</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>4.0</td>
<td>10.0</td>
<td>7.0</td>
<td>6.0</td>
<td>6.0</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>-5.0</td>
<td>-1.0</td>
<td>0.0</td>
<td>-2.0</td>
<td>-3.0</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>3.0</td>
<td>-5.0</td>
<td>-4.0</td>
<td>-2.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>-1.0</td>
<td>2.0</td>
<td>3.0</td>
<td>4.0</td>
<td>-3.0</td>
</tr>
</tbody>
</table>

- Could increase either $x_1$ or $x_3$ to increase obj.
- Consider increasing $x_1$.
- Which basic variable decreases to zero first?
- Answer: none of them, $x_1$ can grow without bound, and obj along with it.
- This is how we detect *unboundedness* with the simplex method.
Initialization

Consider the following problem:

maximize \(-3x_1 + 4x_2\)
subject to
\(-4x_1 - 2x_2 \leq -8\)
\(-2x_1 \leq -2\)
\(3x_1 + 2x_2 \leq 10\)
\(-x_1 + 3x_2 \leq 1\)
\(-3x_2 \leq -2\)
\(x_1, x_2 \geq 0.\)

Phase-I Problem

- Modify problem by subtracting a new variable, \(x_0\), from each constraint and
- replacing objective function with \(-x_0\)
Phase-I Problem

\[
\begin{align*}
\text{maximize} & \quad -x_0 \\
\text{subject to} & \quad -x_0 - 4x_1 - 2x_2 \leq -8 \\
& \quad -x_0 - 2x_1 \leq -2 \\
& \quad -x_0 + 3x_1 + 2x_2 \leq 10 \\
& \quad -x_0 - x_1 + 3x_2 \leq 1 \\
& \quad -x_0 - 3x_2 \leq -2 \\
& \quad x_0, x_1, x_2 \geq 0.
\end{align*}
\]

- Clearly feasible: pick \( x_0 \) large, \( x_1 = 0 \) and \( x_2 = 0 \).
- If optimal solution has \( \text{obj} = 0 \), then original problem is feasible.
- Final phase-I basis can be used as initial phase-II basis (ignoring \( x_0 \) thereafter).
- If optimal solution has \( \text{obj} < 0 \), then original problem is infeasible.
Initialization—First Pivot

Applet depiction shows both the Phase-I and the Phase-II objectives:

- Dictionary is infeasible even for Phase-I.
- One pivot needed to get feasible.
- Entering variable is $x_0$.
- Leaving variable is one whose current value is most negative, i.e. $w_1$.
- After first pivot...
Going into second pivot:

- Feasible!
- Focus on the yellow highlights.
- Let $x_1$ enter.
- Then $w_5$ must leave.
- After second pivot...
**Initialization—Third Pivot**

Going into third pivot:

<table>
<thead>
<tr>
<th></th>
<th>Current Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>obj</strong></td>
<td>-4.5 + -0.75 w1 + 0.75 w5 + 3.25 x2</td>
</tr>
<tr>
<td><strong>x0</strong></td>
<td>-2.0 + 0.0 w1 - 1.0 w5 - 3.0 x2</td>
</tr>
<tr>
<td><strong>w2</strong></td>
<td>2.0 - 0.0 w1 - 1.0 w5 - 3.0 x2</td>
</tr>
<tr>
<td><strong>w3</strong></td>
<td>3.0 - -0.5 w1 - 0.5 w5 - 2.5 x2</td>
</tr>
<tr>
<td><strong>w4</strong></td>
<td>7.5 - 0.75 w1 - 1.75 w5 - 5.75 x2</td>
</tr>
<tr>
<td><strong>x1</strong></td>
<td>4.5 - -0.25 w1 - 0.75 w5 - 5.75 x2</td>
</tr>
</tbody>
</table>

- $x_2$ must enter.
- $x_0$ must leave.
- After third pivot...
End of Phase-I

Current dictionary:

<table>
<thead>
<tr>
<th></th>
<th>Current Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>obj</strong></td>
<td>$-7/3$ + $-3/4$ w1 + $11/6$ w5 + $0$ x0</td>
</tr>
<tr>
<td><strong>x2</strong></td>
<td>$2/3$ - $0$ w1 - $-1/3$ w5 - $0$ x0</td>
</tr>
<tr>
<td><strong>w2</strong></td>
<td>$4/3$ - $-1/2$ w1 - $1/3$ w5 - $0$ x0</td>
</tr>
<tr>
<td><strong>w3</strong></td>
<td>$11/3$ - $3/4$ w1 - $1/6$ w5 - $0$ x0</td>
</tr>
<tr>
<td><strong>w4</strong></td>
<td>$2/3$ - $-1/4$ w1 - $7/6$ w5 - $0$ x0</td>
</tr>
<tr>
<td><strong>x1</strong></td>
<td>$5/3$ - $-1/4$ w1 - $1/6$ w5 - $0$ x0</td>
</tr>
</tbody>
</table>

- Optimal for Phase-I (no yellow highlights).
- $\text{obj} = 0$, therefore original problem is feasible.
## Phase-II

Current dictionary:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>-7/3</th>
<th>-3/4</th>
<th>11/6</th>
<th>w1</th>
<th>w5</th>
<th>x0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x2</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>-1/3</td>
<td>w1</td>
<td>w5</td>
<td>0</td>
</tr>
<tr>
<td>w2</td>
<td>4/3</td>
<td>-1/2</td>
<td>w1</td>
<td>1/3</td>
<td>w5</td>
<td>0</td>
<td>x0</td>
</tr>
<tr>
<td>w3</td>
<td>11/3</td>
<td>3/4</td>
<td>w1</td>
<td>1/6</td>
<td>w5</td>
<td>0</td>
<td>x0</td>
</tr>
<tr>
<td>w4</td>
<td>2/3</td>
<td>-1/4</td>
<td>w1</td>
<td>7/6</td>
<td>w5</td>
<td>0</td>
<td>x0</td>
</tr>
<tr>
<td>x1</td>
<td>5/3</td>
<td>-1/4</td>
<td>w1</td>
<td>1/6</td>
<td>w5</td>
<td>0</td>
<td>x0</td>
</tr>
</tbody>
</table>

For Phase-II:
- Ignore column with $x_0$ in Phase-II.
- Ignore Phase-I objective row.

$w_5$ must enter. $w_4$ must leave...
Optimal Solution

Optimal!

Click here to practice the simplex method on problems that may have infeasible first dictionaries.

For instructions, click here.