Resource Allocation

Recall the resource allocation problem \((m = 2, n = 3)\):

\[
\begin{align*}
\text{maximize} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to} & \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \leq b_1 \\
& \quad a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \leq b_2 \\
& \quad x_1, x_2, x_3 \geq 0,
\end{align*}
\]

where

\[
\begin{align*}
c_j & = \text{profit per unit of product } j \text{ produced} \\
b_i & = \text{units of raw material } i \text{ on hand} \\
a_{ij} & = \text{units raw material } i \text{ required to produce 1 unit of prod } j.
\end{align*}
\]
Closing Up Shop

If we produce one unit less of product $j$, then we free up:

- $a_{1j}$ units of raw material 1 and
- $a_{2j}$ units of raw material 2.

Selling these unused raw materials for $y_1$ and $y_2$ dollars/unit yields $a_{1j}y_1 + a_{2j}y_2$ dollars.

Only interested if this exceeds lost profit on each product $j$:

$$a_{1j}y_1 + a_{2j}y_2 \geq c_j, \quad j = 1, 2, 3.$$ 

Consider a buyer offering to purchase our entire inventory.

Subject to above constraints, buyer wants to minimize cost:

$$\begin{align*}
\text{minimize} & \quad b_1y_1 + b_2y_2 \\
\text{subject to} & \quad a_{11}y_1 + a_{21}y_2 \geq c_1 \\
& \quad a_{12}y_1 + a_{22}y_2 \geq c_2 \\
& \quad a_{13}y_1 + a_{23}y_2 \geq c_3 \\
& \quad y_1, y_2 \geq 0.
\end{align*}$$
Duality

Every Problem:

\[
\text{maximize } \sum_{j=1}^{n} c_j x_j \\
\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, m \\
x_j \geq 0 \quad j = 1, 2, \ldots, n,
\]

Has a Dual:

\[
\text{minimize } \sum_{i=1}^{m} b_i y_i \\
\text{subject to } \sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1, 2, \ldots, n \\
y_i \geq 0 \quad i = 1, 2, \ldots, m.
\]
Dual of Dual

Primal Problem:

\[
\text{maximize } \sum_{j=1}^{n} c_j x_j \\
\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \\
x_j \geq 0 \quad j = 1, \ldots, n,
\]

Original problem is called the \textit{primal problem}.

A problem is defined by its data (notation used for the variables is arbitrary).

Dual in “Standard” Form:

\[
-\text{maximize } \sum_{i=1}^{m} -b_i y_i \\
\text{subject to } -\sum_{i=1}^{m} a_{ij} y_i \leq -c_j \quad j = 1, \ldots, n \\
y_i \geq 0 \quad i = 1, \ldots, m.
\]

Dual is “negative transpose” of primal.

\textbf{Theorem} Dual of dual is primal.
Weak Duality Theorem

If \((x_1, x_2, \ldots, x_n)\) is feasible for the primal and \((y_1, y_2, \ldots, y_m)\) is feasible for the dual, then

\[
\sum_j c_j x_j \leq \sum_i b_i y_i.
\]

Proof.

\[
\sum_j c_j x_j \leq \sum_j \left( \sum_i y_i a_{ij} \right) x_j \\
= \sum_{ij} y_i a_{ij} x_j \\
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i \\
\leq \sum_i b_i y_i.
\]
Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?

Answer is provided by the Strong Duality Theorem (coming later).
Simplex Method and Duality

A Primal Problem:

Its Dual:

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: $x_2$ enters, $w_2$ leaves.
Make analogous pivot in dual: $z_2$ leaves, $y_2$ enters.
Second Iteration

After First Pivot:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>w1</th>
<th>x2</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>2</td>
<td>-3/2</td>
<td>-3/4</td>
<td>-3/4</td>
<td>-3/4</td>
</tr>
<tr>
<td>3</td>
<td>3/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>4</td>
<td>3/4</td>
<td>9/4</td>
<td>1/4</td>
<td>9/4</td>
</tr>
</tbody>
</table>

Note: negative transpose property intact.

Again, use primal to pick pivot: $x_3$ enters, $w_1$ leaves.

Make analogous pivot in dual: $z_3$ leaves, $y_1$ enters.
After Second Iteration

Primal:

• Is optimal.

Dual:

• Negative transpose property remains intact.
• Is optimal.

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.
Strong Duality Theorem

Conclusion on previous slide is the essence of the strong duality theorem which we now state:

**Theorem.**  *If the primal problem has an optimal solution,*

\[ x^* = (x_1^*, x_2^*, \ldots, x_n^*) , \]

*then the dual also has an optimal solution,*

\[ y^* = (y_1^*, y_2^*, \ldots, y_m^*) , \]

*and*

\[ \sum_j c_j x_j^* = \sum_i b_i y_i^* . \]

**Paraphrase:**

If primal has an optimal solution, then there is no duality gap.
Duality Gap

Four possibilities:

• Primal optimal, dual optimal (no gap).
• Primal unbounded, dual infeasible (no gap).
• Primal infeasible, dual unbounded (no gap).
• Primal infeasible, dual infeasible (infinite gap).

Example of infinite gap:

maximize $2x_1 - x_2$
subject to $x_1 - x_2 \leq 1$
$-x_1 + x_2 \leq -2$
$x_1, x_2 \geq 0.$
Complementary Slackness

**Theorem.** At optimality, we have

\[
x_j z_j = 0, \quad \text{for } j = 1, 2, \ldots, n,
\]
\[
w_i y_i = 0, \quad \text{for } i = 1, 2, \ldots, m.
\]
Proof

Recall the proof of the Weak Duality Theorem:

\[
\sum_j c_j x_j \leq \sum_j (c_j + z_j) x_j = \sum_j \left( \sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j
\]

\[
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \leq \sum_i b_i y_i,
\]

The inequalities come from the fact that

\[
x_j z_j \geq 0, \quad \text{for all } j,
\]
\[
w_i y_i \geq 0, \quad \text{for all } i.
\]

By Strong Duality Theorem, the inequalities are equalities at optimality.
Dual Simplex Method

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

An Example. Showing both primal and dual dictionaries:

Looking at dual dictionary: \( y_2 \) enters, \( z_2 \) leaves.

On the primal dictionary: \( w_2 \) leaves, \( x_2 \) enters.

After pivot...
Going in, we have:

Looking at dual: $y_1$ enters, $z_4$ leaves.

Looking at primal: $w_1$ leaves, $x_4$ enters.
Dual Simplex Method Pivot Rule

Refering to the primal dictionary:

- Pick leaving variable from those rows that are *infeasible*.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...
Dual Simplex Method: Third Pivot

Going in, we have:

Which variable must leave and which must enter?

See next page...
Answer is: $x_2$ leaves, $x_1$ enters.

Resulting dictionary is OPTIMAL:
Dual-Based Phase I Method

Example:

<table>
<thead>
<tr>
<th>obj</th>
<th>2.0</th>
<th>x1</th>
<th>3.0</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-1.0</td>
<td>x1</td>
<td>-1.0</td>
<td>x2</td>
</tr>
<tr>
<td>+</td>
<td>-2.0</td>
<td>x1</td>
<td>-3.0</td>
<td>x2</td>
</tr>
<tr>
<td>+</td>
<td>-1.0</td>
<td>x1</td>
<td>-1.0</td>
<td>x2</td>
</tr>
<tr>
<td>+</td>
<td>-4.0</td>
<td>x1</td>
<td>-4.0</td>
<td>x3</td>
</tr>
</tbody>
</table>

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For *Phase I*, use the fake objective—it’s dual feasible.
- Two right-hand sides: the real one (on the left) and a fake (on the right).
- Ignore the fake right-hand side—we’ll use it in another algorithm later.

*Phase I—First Pivot:* $w_3$ leaves, $x_1$ enters.

After first pivot...
Recall current dictionary:

Dual pivot: \( w_2 \) leaves, \( x_2 \) enters.

After pivot:
Dual-Based Phase I Method—Third Pivot

Current dictionary:

Dual pivot:

\( w_1 \) leaves,

\( w_2 \) enters.

After pivot:

It’s feasible!
Current dictionary:

It’s feasible.

Ignore fake objective.

Use the real thing (top row).

Primal pivot: $x_3$ enters, $w_4$ leaves.
Final Dictionary

After pivot:

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>obj</td>
<td>6.75</td>
<td>+</td>
<td>4.5</td>
<td>w3</td>
<td>+</td>
<td>-2.0</td>
<td>w3</td>
<td>+</td>
</tr>
<tr>
<td>w2</td>
<td>11.25</td>
<td>+</td>
<td>6.25</td>
<td>-</td>
<td>-6.5</td>
<td>v3</td>
<td>-</td>
<td>3.5</td>
</tr>
<tr>
<td>x2</td>
<td>3.25</td>
<td>+</td>
<td>0.25</td>
<td>-</td>
<td>-1.5</td>
<td>v3</td>
<td>-</td>
<td>0.5</td>
</tr>
<tr>
<td>x1</td>
<td>0.5</td>
<td>+</td>
<td>-0.5</td>
<td>-</td>
<td>0.0</td>
<td>v3</td>
<td>-</td>
<td>0.0</td>
</tr>
<tr>
<td>x3</td>
<td>0.75</td>
<td>+</td>
<td>0.75</td>
<td>-</td>
<td>-0.5</td>
<td>v3</td>
<td>-</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Problem is unbounded!