Restarting

Consider an optimal dictionary:

\[ \zeta = \zeta^* - z_N^T x_N \]
\[ x_B = x_B^* - B^{-1} N x_N. \]

Now, suppose objective coefficients change from \( c \) to \( \tilde{c} \).

To adjust current dictionary,

- recompute \( z_N^* \), and
- recompute \( \zeta^* \).

Note that \( x_B^* \) remains unchanged. Therefore,

- Adjusted dictionary is \textit{primal feasible}.
- Apply primal simplex method.
- Likely to reach optimality quickly.

Had it been the right-hand sides \( b \) that changed, then

- Adjusted dictionary would be \textit{dual feasible}.
- Could apply dual simplex method.
Ranging

Given an optimal dictionary:

\[ \zeta = \zeta^* - z_N^T x_N \]
\[ x_B = x_B^* - B^{-1} N x_N. \]

Question: If \( c \) were to change to

\[ \tilde{c} = c + \mu \Delta c, \]

for what range of \( \mu \)'s does the current basis remain optimal?

Recall that:

\[ z_N^* = (B^{-1} N)^T c_B - c_N \]

Therefore, dual variables change as follows by \( \mu \Delta z_N \) where

\[ \Delta z_N = (B^{-1} N)^T \Delta c_B - \Delta c_N \]

We want:

\[ z_N^* + \mu \Delta z_N \geq 0 \]

From familiar ratio tests, we get

\[ \left( \min_{j \in N} \frac{\Delta z_j}{z_j^*} \right)^{-1} \leq \mu \leq \left( \max_{j \in N} \frac{\Delta z_j}{z_j^*} \right)^{-1}. \]

Comments:

- A similar analysis works for changes to the right-hand side.
- An example is worked out in the text.
Ranging with the Pivot Tool.

An initial dictionary:

The optimal dictionary:

Question: *If the coefficient on $x_2$ in original problem were changed to $1 + \mu$ (and everything remains unchanged), for what range of $\mu$'s does the current basis remain optimal?*
Ranging with the Pivot Tool—Continued.

Set artificial rhs column to zeros.

Set artificial objective row to \( x_2 \):

\[
\begin{align*}
\text{obj} &= \begin{array}{c}
8.0
\end{array} \\
\text{w1} &= \begin{array}{c}
4.0
\end{array} \quad + \quad \begin{array}{c}
0.0
\end{array} \\
\text{w2} &= \begin{array}{c}
3.0
\end{array} \quad + \quad \begin{array}{c}
0.0
\end{array} \\
\text{w3} &= \begin{array}{c}
1.0
\end{array} \quad + \quad \begin{array}{c}
0.0
\end{array} \\
\text{w6} &= \begin{array}{c}
1.0
\end{array} \quad + \quad \begin{array}{c}
0.0
\end{array} \\
\text{x2} &= \begin{array}{c}
2.0
\end{array} \quad + \quad \begin{array}{c}
0.0
\end{array} \\
\text{x1} &= \begin{array}{c}
3.0
\end{array} \quad + \quad \begin{array}{c}
0.0
\end{array}
\end{align*}
\]

\[\begin{array}{cccc}
-1.0 & \leq \mu & \leq & 1.0
\end{array}\]

The range of \( \mu \) values is shown at the bottom of the pivot tool.
The Primal-Dual Simplex Method.

An Example

\[
\begin{align*}
\text{maximize} & \quad -3x_1 + 11x_2 + 2x_3 \\
\text{subj. to} & \quad -x_1 + 3x_2 \leq 5 \\
& \quad 3x_1 + 3x_2 \leq 4 \\
& \quad 3x_2 + 2x_3 \leq 6 \\
& \quad -3x_1 - 5x_3 \leq -4 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Initial Dictionary:

\[
\begin{align*}
\zeta &= -3x_1 + 11x_2 + 2x_3 \\
w_1 &= 5 + x_1 - 3x_2 \\
w_2 &= 4 - 3x_1 - 3x_2 \\
w_3 &= 6 - 3x_2 - 2x_3 \\
w_4 &= -4 + 3x_1 + 5x_3
\end{align*}
\]

Note: neither primal nor dual feasible.
Perturb

Introduce a parameter $\mu$ and perturb:

\[
\zeta = \begin{array}{c}
-3x_1 + 11x_2 + 2x_3 \\
-\mu x_1 - \mu x_2 - \mu x_3
\end{array}
\]

\[
\begin{align*}
w_1 &= 5 + \mu + x_1 - 3x_2 \\
w_2 &= 4 + \mu - 3x_1 - 3x_2 \\
w_3 &= 6 + \mu - 3x_2 - 2x_3 \\
w_4 &= -4 + \mu + 3x_1 + 5x_3
\end{align*}
\]

For $\mu$ large, dictionary is optimal.

Question: For which $\mu$ values is dictionary optimal? Answer:

\[
\begin{align*}
-3 - \mu &\leq 0 \\
11 - \mu &\leq 0 \quad (*) \\
2 - \mu &\leq 0 \quad (*) \\
5 + \mu &\geq 0 \\
4 + \mu &\geq 0 \\
6 + \mu &\geq 0 \\
-4 + \mu &\geq 0 \quad (*)
\end{align*}
\]

Note: only those marked with (*) give inequalities that omit $\mu = 0$. Tightest:

\[\mu \geq 11\]

Achieved by: objective row perturbation on $x_2$. Let $x_2$ enter.
Who Leaves?

Do ratio test using current lowest $\mu$ value, i.e. $\mu = 11$:

\[
\begin{align*}
5 + 11 - 3x_2 & \geq 0 \\
4 + 11 - 3x_2 & \geq 0 \\
6 + 11 - 3x_2 & \geq 0 \\
-4 + 11 & \geq 0
\end{align*}
\]

Tightest:

\[
4 + 11 - 3x_2 \geq 0.
\]

Achieved by: constraint containing basic variable $w_2$.

Let $w_2$ leave.

After the pivot:

\[
\begin{align*}
\zeta &= 14.67 & - 14x_1 & - 3.67w_2 & + 2x_3 \\
& & + 0.33\mu w_2 & - \mu x_3 \\
\mu &= 1 & + 4x_1 & + w_2 \\
x_2 &= 1.33 & + 0.33\mu & - x_1 & - 0.33w_2 \\
w_3 &= 2 & + 3x_1 & + w_2 & - 2x_3 \\
w_4 &= -4 & + \mu & + 3x_1 & + 5x_3
\end{align*}
\]
Second Pivot

Using the advanced pivot tool, the current dictionary is:

Note: the parameter $\mu$ is not shown. **But it is there!** Question: For which $\mu$ values is dictionary optimal? Answer:

$$
\begin{align*}
-14 & \leq 0 \\
-3.67 + 0.33\mu & \leq 0 \\
2 - \mu & \leq 0 \\
1 & \geq 0 \\
1.33 + 0.33\mu & \geq 0 \\
2 & \geq 0 \\
-4 + \mu & \geq 0 \\
\end{align*}
$$

Tightest lower bound:

$$
\mu \geq 4
$$

Achieved by: constraint containing basic variable $w_4$. Let $w_4$ leave.
Who shall enter?
Recall the current dictionary:

Do *dual-type* ratio test using current lowest μ value, i.e. μ = 4:

\[
\begin{align*}
14 & + 0 \cdot 4 - 3y_4 \geq 0 \\
3.67 & - 0.33 \cdot 4 \geq 0 \\
-2 & + 1 \cdot 4 - 5y_4 \geq 0
\end{align*}
\]

Tightest:

\[-2 + 1 \cdot 4 - 5y_4 \geq 0.\]

Achieved by: objective term containing nonbasic variable \(x_3\).
Let \(x_3\) enter.
Question: For which \( \mu \) values is dictionary optimal? Answer:

\[
\begin{align*}
-15.2 + 0.6\mu & \leq 0 \\
-3.67 + 0.33\mu & \leq 0 \\
0.4 - 0.2\mu & \leq 0 \quad * \\
1 & \geq 0 \\
1.33 + 0.33\mu & \geq 0 \\
0.4 + 0.4\mu & \geq 0 \\
0.8 - 0.2\mu & \geq 0
\end{align*}
\]

Tightest lower bound:

\[ \mu \geq 2 \]

Achieved by: objective term containing nonbasic variable \( w_4 \). Let \( w_4 \) enter.
Third Pivot–Continued

Who shall leave?
Recall the current dictionary:

\[
\begin{align*}
\text{obj} &= 16.2667 + 0.0 \\
v_1 &= 1.0 + 0.0 - 4.0 x_1 - 4.0 x_2 - 4.0 w_4 \\
x_2 &= 1.3333 + 0.3333 - 1.0 x_1 - 1.0 x_2 - 1.0 x_3 - 0.4 w_4 \\
v_3 &= 0.4 + 0.4 - 4.2 x_1 - 4.2 x_2 - 4.2 x_3 - 0.4 w_4 \\
x_3 &= 0.8 + -0.2 - 0.6 x_1 - 0.6 x_2 - 0.6 x_3 - 0.2 w_4
\end{align*}
\]

Do *primal-type* ratio test using current lowest \( \mu \) value, i.e. \( \mu = 2 \):

\[
\begin{align*}
1 &+ 0 \times 2 &\geq 0 \\
1.33 &+ 0.33 \times 2 &\geq 0 \\
0.4 &+ 0.4 \times 2 &- 0.4w_4 &\geq 0 \\
0.8 &- 0.2 \times 2 &+ 0.2w_4 &\geq 0
\end{align*}
\]

Tightest:

\[
0.4 + 0.4 \times 2 - 0.4w_4 \geq 0
\]

Achieved by: constraint containing basic variable \( w_3 \).
Let \( w_3 \) leave.
Fourth Pivot

The current dictionary is:

\[
\begin{array}{cccccc}
\text{obj} & = & 16.6667 & + & -11.0 & x_1 & + & -2.6667 & v_2 & + & -1.0 & v_3 \\
\text{w}_1 & = & 1.0 & + & 0.0 & - & -4.0 & x_1 & - & -1.0 & v_2 & - & 0.0 & v_3 \\
x_2 & = & 1.3333 & + & 0.3333 & - & 1.0 & x_1 & - & 0.3333 & v_2 & - & 0.0 & v_3 \\
w_4 & = & 1.0 & + & 1.0 & - & -10.5 & x_1 & - & -2.5 & v_2 & - & 2.5 & v_3 \\
x_3 & = & 1.0 & + & 0.0 & - & -1.5 & x_1 & - & -0.5 & v_2 & - & 0.5 & v_3 \\
\end{array}
\]

It’s **optimal**!

Also, the range of \( \mu \) values includes \( \mu = 0 \):

\[
\begin{align*}
-11 & - 1.5 \mu & \leq & 0 \\
-2.67 & - 0.167 \mu & \leq & 0 \\
-1 & + 0.5 \mu & \leq & 0 \\
\hline
1 & \geq & 0 \\
1.33 & + 0.33 \mu & \geq & 0 \\
1 & + \mu & \geq & 0 \\
1 & \geq & 0 \\
\end{align*}
\]

That is,

\[-1 \leq \mu \leq 2\]

Range of \( \mu \) values is shown at bottom of pivot tool. Invalid ranges are highlighted in yellow.