ORF 522: Lecture 10

Linear Programming: Chapter 14
Network Flows: Theory

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Networks

Basic elements:

- $\mathcal{N}$ **Nodes** (let $m$ denote number of them).
- $\mathcal{A}$ **Directed Arcs**
  - subset of all possible arcs: $\{(i, j) : i, j \in \mathcal{N}, i \neq j\}$.
  - arcs are **directed**: $(i, j) \neq (j, i)$. 

![Network Diagram](image.png)
Network Flow Data

- \( b_i, \ i \in \mathcal{N} \), supply at node \( i \)
- \( c_{ij}, \ (i, j) \in \mathcal{A} \), cost of shipping 1 unit along arc \((i, j)\).

Note: demands are recorded as negative supplies.
Network Flow Problem

Decision Variables:

\[ x_{ij}, \ (i, j) \in A, \quad \text{quantity} \ to \ ship \ along \ arc \ (i, j). \]

Objective:

\[
\text{minimize} \ \sum_{(i,j) \in A} c_{ij}x_{ij}
\]
Constraints:

- Mass conservation (aka flow balance):
  \[ \text{inflow}(k) - \text{outflow}(k) = \text{demand}(k) = -\text{supply}(k), \quad k \in \mathcal{N} \]
  \[ \implies \sum_{i: (i, k) \in \mathcal{A}} x_{ik} - \sum_{j: (k, j) \in \mathcal{A}} x_{kj} = -b_k, \quad k \in \mathcal{N} \]

- Nonnegativity:
  \[ x_{ij} \geq 0, \quad (i, j) \in \mathcal{A} \]
Matrix Notation

minimize $c^T x$
subject to $Ax = -b$
$x \geq 0$

where

$$c^T = \begin{bmatrix} 2 & 4 & 9 & 11 & 4 & 3 & 8 & 7 & 0 & 15 & 16 & 18 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -19 \\ 16 \\ -33 \\ 0 \\ 36 \end{bmatrix}$$

Notes

- $A$ is called node-arc incidence matrix.
- $A$ is large and sparse and integer.
Dual Problem

maximize \(-b^T y\)
subject to \(A^T y + z = c\)
\[z \geq 0\]

In network notation:

maximize \(-\sum_{i \in N} b_i y_i\)
subject to \(y_j - y_i + z_{ij} = c_{ij}\) \((i, j) \in A\)
\[z_{ij} \geq 0\] \((i, j) \in A\)
Complementarity Relations

• The primal variables are nonnegative.

• Therefore the associated dual constraints are inequalities.

• The dual slack variables are complementary to the primal variables:

\[ x_{ij}z_{ij} = 0, \quad (i, j) \in A \]

• The primal constraints are equalities.

• Therefore they have no slack variables.

• The corresponding dual variables, the \( y_i \)'s, are free variables.

• No complementarity conditions apply to them.
Definition: Subnetwork

Network

Subnetwork
Connected vs. Disconnected

Connected

Disconnected
Cyclic vs. Acyclic

Cyclic

Acyclic
Trees

Tree = Connected + Acyclic

Not Trees
Spanning Trees

Spanning Tree—A tree touching every node

Tree Solution

\[ x_{ij} = 0 \quad \text{for } (i, j) \notin \text{Tree Arcs} \]

Note: Tree solutions are easy to compute—start at the leaves and work inward...
Online Pivot Tool–Notations

Data:
- **Costs** on arcs shown above arcs.
- **Supplies** at nodes shown above nodes.

Variables:
- **Primal flows** shown on tree arcs.
- **Dual slacks** shown on nontree arcs.
- **Dual variables** shown below nodes.
Tree Solutions–An Example

Data:

- Fix a root node, say a.
- **Primal flows** on tree arcs calculated recursively from leaves inward.
- **Dual variables** at nodes calculated recursively from root node outward along tree arcs using:
  \[ y_j - y_i = c_{ij} \]
- **Dual slacks** on nontree arcs calculated using:
  \[ z_{ij} = y_i - y_j + c_{ij} \]