Linear Programming: Chapter 1
Introduction

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Resource Allocation

maximize \[ c_1x_1 + c_2x_2 + \cdots + c_nx_n \]
subject to \[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \]
\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \]
\[ x_1, x_2, \ldots, x_n \geq 0, \]

where

\[ c_j = \text{profit per unit of product } j \text{ produced} \]
\[ b_i = \text{units of raw material } i \text{ on hand} \]
\[ a_{ij} = \text{units of raw material } i \text{ required to produce one unit of product } j. \]
minimize \( c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \)
subject to
\[
\begin{align*}
l_1 & \leq a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq u_1 \\
l_2 & \leq a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq u_2 \\
\vdots \\
l_m & \leq a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq u_m \\
x_1, x_2, \ldots, x_n & \geq 0,
\end{align*}
\]
where
\[
\begin{align*}
c_j &= \text{cost per unit of food } j \\
l_i &= \text{minimum daily allowance of nutrient } i \\
u_i &= \text{maximum daily allowance of nutrient } i \\
a_{ij} &= \text{units of nutrient } i \text{ contained in one unit of food } j.
\end{align*}
\]
Shape Optimization (Telescope Design)

The problem is to design and build a space telescope that will be able to “see” planets around nearby stars (other than the Sun).

Consider a telescope. Light enters the front of the telescope. This is called the pupil plane. The telescope focuses all the light passing through the pupil plane from a given direction at a certain point on the focal plane, say \((0, 0)\).

However, the wave nature of light makes it impossible to concentrate all of the light at a point. Instead, a small disk, called the Airy disk, with diffraction rings around it appears.

These diffraction rings are bright relative to any planet that might be orbiting a nearby star and so would completely hide the planet. The Sun, for example, would appear \(10^{10}\) times brighter than the Earth to a distant observer.

By placing a mask over the pupil, one can design the shape and strength of the diffraction rings. The problem is to find an optimal shape so as to put a very deep null very close to the Airy disk.
Airy Disk and Diffraction Rings

A conventional telescope has a circular opening as depicted by the left side of the figure. Visually, a star then looks like a small disk with rings around it, as depicted on the right.

The rings grow progressively dimmer as this log-plot shows:
Airy Disk and Diffraction Rings—Log Scaling

Here’s the same Airy disk from the previous slide plotted using a logarithmic brightness scale with $10^{-11} = -110\text{dB}$ set to black:

The problem is to find an aperture mask, i.e. a pupil plane mask, that yields a $-100\text{ dB}$ null somewhere near the first diffraction ring. A hard problem! Such a null would appear almost black in this log-scaled image.
Electric Field

Consider “tinting” the front opening of the telescope using a nonuniform tint given by $A(r)$.

In such a situation, the image plane electric field of a star is a rotationally symmetric real function $E(\rho)$:

$$E(\rho) = 2\pi \int_0^{D/2} A(r) J_0(2\pi r \rho) r dr$$

The intensity of the light at radius $\rho$ from the center of the image plane is given by the square of the electric field.
Maximizing Throughput

We maximize the “area” under $A(r)$ (make the tinting as bright as possible) subject to very strict contrast constraints:

$$\text{maximize } \int_0^{D/2} A(r) rdr$$

subject to $-10^{-5}E(0) \leq E(\rho) \leq 10^{-5}E(0)$, for $\rho_{\text{min}} \leq \rho \leq \rho_{\text{max}}$

$0 \leq A(r) \leq 1$, for $0 \leq r \leq D/2$

The first constraint guarantees $10^{-10}$ light intensity throughout a desired annulus of the focal plane, and the remaining constraint ensures that the tinting is really a tinting.
Solution via Linear Programming

\[ \mathcal{O} = \{(\xi, 0) : \xi_0 \leq \xi \leq \xi_1\} \]

\[ \rho_{\text{min}} = 4 \]

\[ \rho_{\text{max}} = 60 \]

Thrput = 10%
Finite Impulse Response (FIR) Filter Design

- Audio is stored digitally in a computer as a stream of short integers: $u_k$, $k \in \mathbb{Z}$.

- When the music is played, these integers are used to drive the displacement of the speaker from its resting position.

- For CD quality sound, 44100 short integers get played per second per channel.

|     | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | -32768 | 8   | -23681 | 16  | 12111 | 1 | -32768 | 9   | -18449 | 17  | 17311 | 2 | -32768 | 10  | -11025 | 18  | 21311 | 3 | -30753 | 11  | -6913  | 19  | 23055 | 4 | -28865 | 12  | -4337  | 20  | 23519 | 5 | -29105 | 13  | -1329  | 21  | 25247 | 6 | -29201 | 14  | 1743   | 22  | 27535 | 7 | -26513 | 15  | 6223   | 23  | 29471 |
FIR Filter Design—Continued

- A finite impulse response (FIR) filter takes as input a digital signal and convolves this signal with a finite set of fixed numbers $h_0, \ldots, h_n$ to produce a filtered output signal:

$$y_k = \sum_{i=-n}^{n} h_i u_{k-i}.$$ 

- Sparing the details, the output power at frequency $\nu$ is given by

$$|H(\nu)|^2$$

where

$$H(\nu) = \sum_{k=-n}^{n} h_k e^{2\pi i k \nu} = h(0) + 2 \sum_{k=1}^{n} h_k \cos(2\pi k \nu),$$

- Similarly, the mean absolute deviation from a flat frequency response over a frequency range, say $\mathcal{L} \subset [0, 1]$, is given by

$$\frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} |H(\nu) - 1| \, d\nu$$
Filter Design: Woofer, Midrange, Tweeter

\[
\text{minimize } \int_0^1 |H_w(\nu) + H_m(\nu) + H_t(\nu) - 1| d\nu
\]

subject to
\[-\epsilon \leq H_w(\nu) \leq \epsilon, \quad \nu \in W = [.2, .8] \]
\[-\epsilon \leq H_m(\nu) \leq \epsilon, \quad \nu \in M = [.4, .6] \cup [.9, 1] \]
\[-\epsilon \leq H_t(\nu) \leq \epsilon, \quad \nu \in T = [.7, .3] \]

where

\[
H_i(\nu) = h_0^i + 2 \sum_{k=1}^{n} h_k^i \cos(2\pi k \nu), \quad i = W, M, T
\]

\[
h_k^i = \text{filter coefficients, i.e., decision variables}
\]
Conversion to a Linear Programming Problem

minimize \[ \int_{0}^{1} t(\nu) d\nu \]

subject to \[ t(\nu) \leq H_w(\nu) + H_m(\nu) + H_t(\nu) - 1 \leq t(\nu) \quad \nu \in [0, 1] \]
\[ -\epsilon \leq H_w(\nu) \leq \epsilon, \quad \nu \in W \]
\[ -\epsilon \leq H_m(\nu) \leq \epsilon, \quad \nu \in M \]
\[ -\epsilon \leq H_t(\nu) \leq \epsilon, \quad \nu \in T \]
Specific Example

filter length: $n = 14$

integral discretization: $N = 1000$

Demo: orig-clip woofer midrange tweeter reassembled

Ref: J.O. Coleman, U.S. Naval Research Laboratory,
CISS98 paper available: engr.umbc.edu/~jeffc/pubs/abstracts/ciss98.html
Portfolio Optimization

Markowitz Shares the 1990 Nobel Prize

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences
in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR’S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS
AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize
in Economic Sciences with one third each, to

Professor Harry Markowitz, City University of New York, USA,
Professor Merton Miller, University of Chicago, USA,
Professor William Sharpe, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called,
Capital Asset Pricing Model (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources
among various areas of production. It is to a large extent through financial markets that saving in
different sectors of the economy is transferred to firms for investments in buildings and machines.
Financial markets also reflect firms’ expected prospects and risks, which implies that risks can be spread
and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry
Markowitz who developed a theory for households’ and firms’ allocation of financial assets under
uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally
invested in assets which differ in regard to their expected return and risk, and thereby also how risks can
be reduced.

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**Notation:** $R_j(t) = \text{return on investment } j \text{ in time period } t.$
Risk vs. Reward

**Reward**—estimated using historical means:

\[ \text{reward}_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t). \]

**Risk**—Markowitz defined risk as the variability of the returns as measured by the historical variances:

\[ \text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} (R_j(t) - \text{reward}_j)^2. \]

However, to get a linear programming problem (and for other reasons) we use the sum of the absolute values instead of the sum of the squares:

\[ \text{risk}_j = \frac{1}{T} \sum_{t=1}^{T} |R_j(t) - \text{reward}_j|. \]
Hedging

*Investment A*: up 20%, down 10%, equally likely—a risky asset.

*Investment B*: up 20%, down 10%, equally likely—another risky asset.

*Correlation*: up years for A are down years for B and vice versa.

*Portfolio*—half in A, half in B: up 5% every year! No risk!
Portfolios

Fractions: \( x_j \) = fraction of portfolio to invest in \( j \).

Portfolio's Historical Returns:

\[
R(t) = \sum_j x_j R_j(t)
\]

Portfolio’s Reward:

\[
\text{reward}(x) = \frac{1}{T} \sum_{t=1}^{T} R(t) = \frac{1}{T} \sum_{t=1}^{T} \sum_j x_j R_j(t)
\]
Portfolio’s Risk:

\[
\text{risk}(x) = \frac{1}{T} \sum_{t=1}^{T} |R(t) - \text{reward}(x)|
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j} x_j R_j(t) - \frac{1}{T} \sum_{s=1}^{T} \sum_{j} x_j R_j(s) \right|
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j} x_j \left( R_j(t) - \frac{1}{T} \sum_{s=1}^{T} R_j(s) \right) \right|
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j} x_j (R_j(t) - \text{reward}_j) \right|
\]
A Markowitz-Type Model

**Decision Variables:** the fractions \( x_j \).

**Objective:** maximize return, minimize risk.

**Fundamental Lesson:** can’t simultaneously optimize two objectives.

**Compromise:** set an upper bound \( \mu \) for risk and maximize reward subject to this bound constraint:

- Parameter \( \mu \) is called risk aversion parameter.
- Large value for \( \mu \) puts emphasis on reward maximization.
- Small value for \( \mu \) puts emphasis on risk minimization.

**Constraints:**

\[
\frac{1}{T} \sum_{t=1}^{T} \left| \sum_j x_j (R_j(t) - \text{reward}_j) \right| \leq \mu
\]

\[
\sum_j x_j = 1
\]

\[
x_j \geq 0 \quad \text{for all } j
\]
Optimization Problem

maximize \( \frac{1}{T} \sum_{t=1}^{T} \sum_{j} x_j R_j(t) \)

subject to \( \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j} x_j (R_j(t) - \text{reward}_j) \right| \leq \mu \)

\[ \sum_{j} x_j = 1 \]

\[ x_j \geq 0 \quad \text{for all } j \]

Because of absolute values not a linear programming problem.

Easy to convert...
maximize \[ \frac{1}{T} \sum_{t=1}^{T} \sum_{j} x_j R_j(t) \]

subject to \[ -y_t \leq \sum_{j} x_j (R_j(t) - \text{reward}_j) \leq y_t \quad \text{for all } t \]

\[ \frac{1}{T} \sum_{t=1}^{T} y_t \leq \mu \]

\[ \sum_{j} x_j = 1 \]

\[ x_j \geq 0 \quad \text{for all } j \]
Efficient Frontier

Varying risk bound $\mu$ produces the so-called efficient frontier. Portfolios on the efficient frontier are reasonable. Portfolios not on the efficient frontier can be strictly improved.

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Efficient Frontier

- T-Bills
- Corp. Bonds
- Long Bonds
- S&P 500
- Wilshire 5000
- NASDAQ Composite
- Gold
- EAFE

Graph showing efficient frontier with various assets including T-Bills, Corp. Bonds, Long Bonds, S&P 500, Wilshire 5000, NASDAQ Composite, Gold, and EAFE.