Linear Programming: Chapter 2
The Simplex Method

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An Example.

maximize \(-x_1 + 3x_2 - 3x_3\)

subject to
\[
\begin{align*}
3x_1 - x_2 - 2x_3 & \leq 7 \\
-2x_1 - 4x_2 + 4x_3 & \leq 3 \\
x_1 & - 2x_3 \leq 4 \\
-2x_1 + 2x_2 + x_3 & \leq 8 \\
3x_1 & \leq 5
\end{align*}
\]

\(x_1, x_2, x_3 \geq 0.\)
Rewrite with slack variables

maximize \( \zeta = -x_1 + 3x_2 - 3x_3 \)
subject to \( w_1 = 7 - 3x_1 + x_2 + 2x_3 \)
\( w_2 = 3 + 2x_1 + 4x_2 - 4x_3 \)
\( w_3 = 4 - x_1 + 2x_3 \)
\( w_4 = 8 + 2x_1 - 2x_2 - x_3 \)
\( w_5 = 5 - 3x_1 \)

\( x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0. \)

Notes:
• This layout is called a dictionary.
• Setting \( x_1, x_2, \) and \( x_3 \) to 0, we can read off the values for the other variables: \( w_1 = 7, w_2 = 3, \) etc. This specific solution is called a dictionary solution.
• Dependent variables, on the left, are called basic variables.
• Independent variables, on the right, are called nonbasic variables.
Dictionary Solution is Feasible

maximize \( \zeta = -x_1 + 3x_2 - 3x_3 \)

subject to

\[ w_1 = 7 - 3x_1 + x_2 + 2x_3 \]
\[ w_2 = 3 + 2x_1 + 4x_2 - 4x_3 \]
\[ w_3 = 4 - x_1 + 2x_3 \]
\[ w_4 = 8 + 2x_1 - 2x_2 - x_3 \]
\[ w_5 = 5 - 3x_1 \]

\( x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0 \).

Notes:

- All the variables in the current dictionary solution are nonnegative.
- Such a solution is called **feasible**.
- The initial dictionary solution need not be feasible—we were just lucky above.
• If $x_2$ increases, obj goes \textit{up}.
• How much can $x_2$ increase? Until $w_4$ decreases to zero.
• Do it. End result: $x_2 > 0$ whereas $w_4 = 0$.
• That is, $x_2$ must become \textit{basic} and $w_4$ must become \textit{nonbasic}.
• Algebraically rearrange equations to, in the words of Jean-Luc Picard, "Make it so."
• This is a \textit{pivot}. 

\begin{center}
\begin{tabular}{|c|c|}
\hline
\textbf{Current Dictionary} & \\
\hline
$\text{obj} = 0.0$ & $+ \begin{array}{c}
-1.0 \\
3.0 \\
-2.0 \\
1.0 \\
-3.0 \\
\end{array}$ \text{\textit{x1}} + $\begin{array}{c}
3.0 \\
-1.0 \\
-4.0 \\
0.0 \\
2.0 \\
\end{array}$ \text{\textit{x2}} + $\begin{array}{c}
-3.0 \\
-2.0 \\
4.0 \\
-2.0 \\
1.0 \\
\end{array}$ \text{\textit{x3}} \\
\hline
$w_1 = 7.0$ & $- 3.0$ \text{\textit{x1}} - $-1.0$ \text{\textit{x2}} - $-2.0$ \text{\textit{x3}} \\
$w_2 = 3.0$ & $- 2.0$ \text{\textit{x1}} - $-4.0$ \text{\textit{x2}} - $4.0$ \text{\textit{x3}} \\
$w_3 = 4.0$ & $- 1.0$ \text{\textit{x1}} - $0.0$ \text{\textit{x2}} - $-2.0$ \text{\textit{x3}} \\
$w_4 = 8.0$ & $- 2.0$ \text{\textit{x1}} - $2.0$ \text{\textit{x2}} - $1.0$ \text{\textit{x3}} \\
$w_5 = 5.0$ & $- 3.0$ \text{\textit{x1}} - $0.0$ \text{\textit{x2}} - $0.0$ \text{\textit{x3}} \\
\hline
\end{tabular}
\end{center}
A Pivot: $x_2 \leftrightarrow w_4$

| obj  | 0.0  | + | -1.0 | x1 | + | 3.0 | x2 | + | -3.0 | x3 |
| w1   | 7.0  | - | 3.0  | x1 | - | -1.0 | x2 | - | -2.0 | x3 |
| w2   | 3.0  | - | -2.0 | x1 | - | -4.0 | x2 | - | 4.0  | x3 |
| w3   | 4.0  | - | 1.0  | x1 | - | 0.0  | x2 | - | -2.0 | x3 |
| w4   | 8.0  | - | -2.0 | x1 | - | 2.0  | x2 | - | 1.0  | x3 |
| w5   | 5.0  | - | 3.0  | x1 | - | 0.0  | x2 | - | 0.0  | x3 |

becomes

| obj  | 12.0 | + | 2.0  | x1 | + | -1.5 | w4 | + | -4.5 | x3 |
| w1   | 11.0 | - | 2.0  | x1 | - | 0.5  | w4 | - | -1.5 | x3 |
| w2   | 19.0 | - | -6.0 | x1 | - | 2.0  | w4 | - | 6.0  | x3 |
| w3   | 4.0  | - | 1.0  | x1 | - | 0.0  | w4 | - | -2.0 | x3 |
| x2   | 4.0  | - | -1.0 | x1 | - | 0.5  | w4 | - | 0.5  | x3 |
| w5   | 5.0  | - | 3.0  | x1 | - | 0.0  | w4 | - | 0.0  | x3 |
Here’s the dictionary after the first pivot:

<table>
<thead>
<tr>
<th></th>
<th>Current Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td>obj</td>
<td>12.0 + 2.0 x1 + -1.5 w4 + -4.5 x3</td>
</tr>
<tr>
<td>w1</td>
<td>11.0 - 2.0 x1 - 0.5 w4 - -1.5 x3</td>
</tr>
<tr>
<td>w2</td>
<td>19.0 - -6.0 x1 - 2.0 w4 - 6.0 x3</td>
</tr>
<tr>
<td>w3</td>
<td>4.0 - 1.0 x1 - 0.0 w4 - -2.0 x3</td>
</tr>
<tr>
<td>x2</td>
<td>4.0 - -1.0 x1 - 0.5 w4 - 0.5 x3</td>
</tr>
<tr>
<td>w5</td>
<td>5.0 - 3.0 x1 - 0.0 w4 - 0.0 x3</td>
</tr>
</tbody>
</table>

- Now, let $x_1$ increase.
- Of the basic variables, $w_5$ hits zero first.
- So, $x_1$ enters and $w_5$ leaves the basis.
- New dictionary is...
## Simplex Method—Final Dictionary

<table>
<thead>
<tr>
<th></th>
<th>Current Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>obj</strong></td>
<td>$\frac{46}{3}$ + $\frac{-2}{3}$ $w_5$ + $\frac{-3}{2}$ $w_4$ + $\frac{-9}{2}$ $x_3$</td>
</tr>
<tr>
<td><strong>w1</strong></td>
<td>$\frac{23}{3}$ - $\frac{-2}{3}$ $w_5$ - $\frac{1}{2}$ $w_4$ - $\frac{-3}{2}$ $x_3$</td>
</tr>
<tr>
<td><strong>w2</strong></td>
<td>$29$ - $2$ $w_5$ - $2$ $w_4$ - $6$ $x_3$</td>
</tr>
<tr>
<td><strong>w3</strong></td>
<td>$\frac{7}{3}$ - $\frac{-1}{3}$ $w_5$ - $0$ $w_4$ - $-2$ $x_3$</td>
</tr>
<tr>
<td><strong>x2</strong></td>
<td>$\frac{17}{3}$ - $\frac{1}{3}$ $w_5$ - $\frac{1}{2}$ $w_4$ - $\frac{1}{2}$ $x_3$</td>
</tr>
<tr>
<td><strong>x1</strong></td>
<td>$\frac{5}{3}$ - $\frac{1}{3}$ $w_5$ - $0$ $w_4$ - $0$ $x_3$</td>
</tr>
</tbody>
</table>

- It’s optimal (no pink)!
- Click [here](#) to practice the simplex method.
- For instructions, click [here](#).
Agenda

- Discuss *unboundedness*; (today)

- Discuss initialization/*infeasibility*; i.e., what if initial dictionary is not feasible. (today)

- Discuss *degeneracy*. (next lecture)
Unboundedness

Consider the following dictionary:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>Current Dictionary</th>
<th>Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>2.0 x1 +</td>
<td>-1.0 x2 +</td>
</tr>
<tr>
<td>w1 =</td>
<td>4.0</td>
<td>-5.0 x1 -</td>
<td>3.0 x2 -</td>
</tr>
<tr>
<td>w2 =</td>
<td>10.0</td>
<td>-1.0 x1 -</td>
<td>-5.0 x2 -</td>
</tr>
<tr>
<td>w3 =</td>
<td>7.0</td>
<td>0.0 x1 -</td>
<td>-4.0 x2 -</td>
</tr>
<tr>
<td>w4 =</td>
<td>6.0</td>
<td>-2.0 x1 -</td>
<td>-2.0 x2 -</td>
</tr>
<tr>
<td>w5 =</td>
<td>6.0</td>
<td>-3.0 x1 -</td>
<td>0.0 x2 -</td>
</tr>
</tbody>
</table>

- Could increase either $x_1$ or $x_3$ to increase obj.
- Consider increasing $x_1$.
- Which basic variable decreases to zero first?
- Answer: none of them, $x_1$ can grow without bound, and obj along with it.
- This is how we detect *unboundedness* with the simplex method.
Initialization

Consider the following problem:

\[
\begin{align*}
\text{maximize} & \quad -3x_1 + 4x_2 \\
\text{subject to} & \quad -4x_1 - 2x_2 \leq -8 \\
& \quad -2x_1 \leq -2 \\
& \quad 3x_1 + 2x_2 \leq 10 \\
& \quad -x_1 + 3x_2 \leq 1 \\
& \quad -3x_2 \leq -2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

Phase-I Problem

- Modify problem by subtracting a new variable, \( x_0 \), from each constraint and
- replacing objective function with \( -x_0 \)
Phase-I Problem

maximize $-x_0$
subject to
\[-x_0 - 4x_1 - 2x_2 \leq -8\]
\[-x_0 - 2x_1 \leq -2\]
\[-x_0 + 3x_1 + 2x_2 \leq 10\]
\[-x_0 + x_1 + 3x_2 \leq 1\]
\[-x_0 - 3x_2 \leq -2\]

$x_0, x_1, x_2 \geq 0$.

• Clearly feasible: pick $x_0$ large, $x_1 = 0$ and $x_2 = 0$.
• If optimal solution has obj = 0, then original problem is feasible.
• Final phase-I basis can be used as initial phase-II basis (ignoring $x_0$ thereafter).
• If optimal solution has obj < 0, then original problem is infeasible.
Applet depiction shows both the Phase-I and the Phase-II objectives:

<table>
<thead>
<tr>
<th>obj</th>
<th>Current Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0 x0 + 0.0 x1 + 4.0 x2</td>
</tr>
<tr>
<td>0.0</td>
<td>-1.0 x0 + 0.0 x1 + 0.0 x2</td>
</tr>
<tr>
<td>w1 = -8.0</td>
<td>-1.0 x0 - 4.0 x1 - 2.0 x2</td>
</tr>
<tr>
<td>w2 = -2.0</td>
<td>-1.0 x0 - 2.0 x1 - 0.0 x2</td>
</tr>
<tr>
<td>w3 = 10.0</td>
<td>-1.0 x0 - 3.0 x1 - 2.0 x2</td>
</tr>
<tr>
<td>w4 = 1.0</td>
<td>-1.0 x0 - 3.0 x1 - 3.0 x2</td>
</tr>
<tr>
<td>w5 = -2.0</td>
<td>-1.0 x0 - 0.0 x1 - 3.0 x2</td>
</tr>
</tbody>
</table>

- Dictionary is infeasible even for Phase-I.
- One pivot needed to get feasible.
- Entering variable is $x_0$.
- Leaving variable is one whose current value is most negative, i.e. $w_1$.
- After first pivot...
Initiaization—Second Pivot

Going into second pivot:

- Feasible!
- Focus on the yellow highlights.
- Let $x_1$ enter.
- Then $w_5$ must leave.
- After second pivot...
Going into third pivot:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>Current Dictionary</th>
<th>Dictionary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-4.5</td>
<td>-0.75</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>-2.0</td>
<td>0.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>x0</td>
<td>2.0</td>
<td>0.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>w2</td>
<td>3.0</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>w3</td>
<td>7.5</td>
<td>0.75</td>
<td>-1.75</td>
</tr>
<tr>
<td>w4</td>
<td>4.5</td>
<td>-0.25</td>
<td>-0.75</td>
</tr>
<tr>
<td>x1</td>
<td>1.5</td>
<td>-0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>

- $x_2$ must enter.
- $x_0$ must leave.
- After third pivot...
End of Phase-I

Current dictionary:

```
<table>
<thead>
<tr>
<th></th>
<th>-7/3</th>
<th>-3/4</th>
<th>w1 +</th>
<th>11/6</th>
<th>w5 +</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>x2</td>
<td>2/3</td>
<td>0</td>
<td>w1 -</td>
<td>-1/3</td>
<td>w5 -</td>
<td>0</td>
</tr>
<tr>
<td>w2</td>
<td>4/3</td>
<td>-1/2</td>
<td>w1 -</td>
<td>1/3</td>
<td>w5 -</td>
<td>0</td>
</tr>
<tr>
<td>w3</td>
<td>11/3</td>
<td>3/4</td>
<td>w1 -</td>
<td>1/6</td>
<td>w5 -</td>
<td>0</td>
</tr>
<tr>
<td>w4</td>
<td>2/3</td>
<td>-1/4</td>
<td>w1 -</td>
<td>7/6</td>
<td>w5 -</td>
<td>0</td>
</tr>
<tr>
<td>x1</td>
<td>5/3</td>
<td>-1/4</td>
<td>w1 -</td>
<td>1/6</td>
<td>w5 -</td>
<td>0</td>
</tr>
</tbody>
</table>
```

- Optimal for Phase-I (no yellow highlights).
- \( \text{obj} = 0 \), therefore original problem is feasible.
Phase-II

Current dictionary:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>x2</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>x1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-7/3</td>
<td>2/3</td>
<td>4/3</td>
<td>11/3</td>
<td>2/3</td>
<td>5/3</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>3/4</td>
<td>-1/4</td>
<td>-1/4</td>
</tr>
<tr>
<td></td>
<td>-3/4</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>11/6</td>
<td>w1</td>
<td>w1</td>
<td>w1</td>
<td>w1</td>
<td>w1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-1/3</td>
<td>1/3</td>
<td>1/6</td>
<td>7/6</td>
<td>1/6</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>w5</td>
<td>w5</td>
<td>w5</td>
<td>w5</td>
<td>w5</td>
</tr>
<tr>
<td></td>
<td>w5</td>
<td>x0</td>
<td>x0</td>
<td>x0</td>
<td>x0</td>
<td>x0</td>
</tr>
</tbody>
</table>

For Phase-II:

- Ignore column with $x_0$ in Phase-II.
- Ignore Phase-I objective row.

$w_5$ must enter. $w_4$ must leave...
Optimal Solution

• Optimal!

• Click [here](#) to practice the simplex method on problems that may have infeasible first dictionaries.

• For instructions, click [here](#).