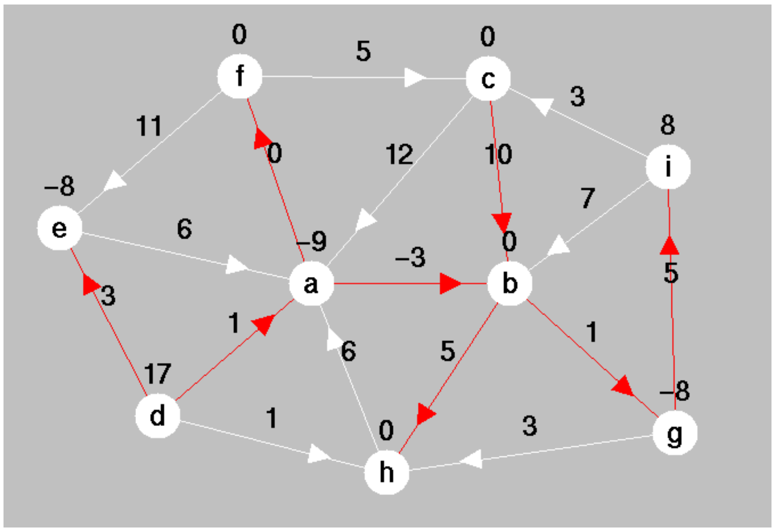

FIGURE 1. Data for a network flow problem. As usual the numbers above the nodes are supplies (negative values represent demands) and numbers shown above the arcs are unit shipping costs. The darkened arcs form a spanning tree.

(1) Consider the network flow problem shown Figure 1.
   (a) Compute primal flows for each tree arc.
   (b) Compute dual variables for each node.
   (c) Compute dual slacks for each nontree arc.
(2) Consider the tree solution for a minimum cost network flow problem shown in Figure 2.
   (a) Using the largest–coefficient rule in the dual network simplex method, what is the leaving arc?
   (b) What is the entering arc?
   (c) After *one* pivot, what is the new tree solution?
(3) One may assume without loss of generality that every node in a minimum cost network flow problem has at least two arcs associated with it. Why?
(4) The sum of the dual slacks around any cycle is a constant. What is that constant?

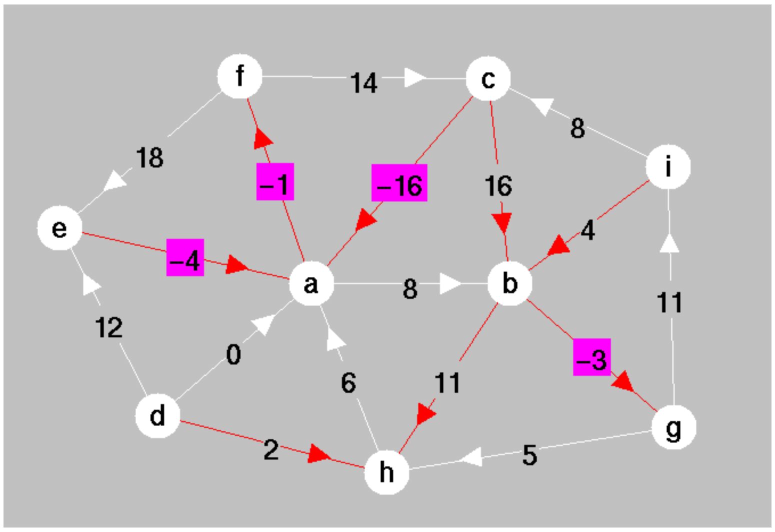

FIGURE 2. A tree solution for a network flow problem. As usual the numbers on the tree arcs represent primal flows while number on the nontree arcs are dual slacks.

(5) *Planar Networks.* A network is called planar if the nodes and arcs can be layed out on the two-dimensional plane in such a manner that no two arcs cross each other. For planar networks, the nodes are often refered to as *vertices* and the arcs are called *edges*. Consider a specific connected planar network. If one were to delete the vertices and the edges from the plane, one would be left with a disjoint collection of subsets of the plane. These subsets are called *faces*. It is important to note that there is one unbounded face. It is a face just like the other bounded ones. Figure 3 shows a connected planar network with its faces labeled.

Associated with each connected planar network is a *dual network* defined by interchanging vertices and faces. That is, place a dual vertex in the center of each primal face and connect, with a dual edge, any two dual vertices whose corresponding primal faces share an edge. Note: the dual vertex corresponding to the unbounded primal face could be placed anywhere in the unbounded face but we choose to put it *at infinity*. In this way, dual edges that have a head or a tail at this node can run off to infinity in any direction. (It would have been better to start by saying that a planar network is a network that can be embedded onto the two-dimensional sphere—in that case, the unbounded dual node could be put at the opposite pole.)

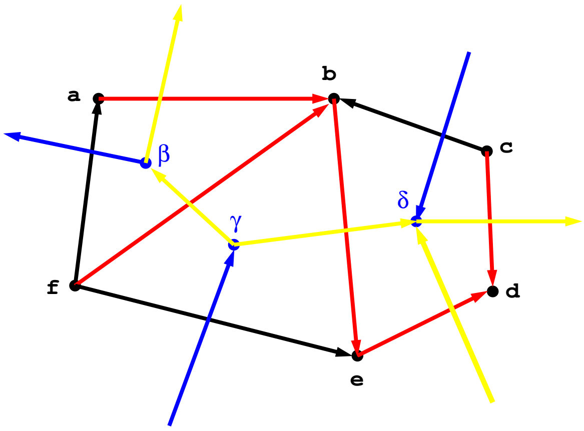

FIGURE 3. The primal network has nodes a through f and arcs shown in red and black. The red arcs form a spanning tree. The corresponding dual network has nodes α through δ (node α is "at infinity") and arcs shown in blue and yellow. The blue arcs form a spanning tree for the dual network.

Each dual edge crosses exactly one primal edge. Therefore, there is a one-to-one correspondence between primal edges and dual edges. The directionality of the dual edge is determined by rotating a tangent vector pointing along the direction of the primal edge counterclockwise until it is tangent to the dual edge.

Consider a spanning tree on the primal network and suppose that a primal-dual tree solution is given. We define a *spanning tree* on the dual network as follows. A dual edge is on the dual network's spanning tree if and only if the corresponding primal edge is not on the primal network's spanning tree.

(a) As always, let m denote the number of nodes and let n denote the number of arcs in a network. Let f denote the number of faces in a planar network. Show by induction on $m+n$ that $m = n - f + 2$.
(b) Show that the dual spanning tree defined above is in fact a spanning tree.
(c) Show that a dual pivot for a minimum cost network flow problem defined on the primal network is precisely the same as a primal pivot for the corresponding network flow problem on the dual network.