1. *Piecewise Linear Approximation.* Given real numbers $b_1 < b_2 < \cdots < b_k$, let f be a continuous function on $\mathbb{R}$ that is linear on each interval $[b_i, b_{i+1}]$, $i = 0, 1, \ldots, k$ (for convenience we let $b_0 = -\infty$ and $b_{k+1} = \infty$). Such a function is called *piecewise linear* and the numbers b_i are called *breakpoints*. Piecewise linear functions are often used to approximate (continuous) nonlinear functions. The purpose of this exercise is to show how and why.
   (a) Every piecewise linear function can be written as a sum of a constant plus a linear term plus a sum of absolute value terms:

   $$f(x) = d + a_0 + \sum_{i=1}^{k} a_i |x - b_i|.$$

   Let c_i denote the slope of f on the interval $[b_i, b_{i+1}]$. Derive an explicit expression for each of the a_j's (including a_0) in terms of the c_i's.
   (b) In terms of the c_i's, give necessary and sufficient conditions for f to be convex.
   (c) In terms of the a_j's, give necessary and sufficient conditions for f to be convex.
   (d) Assuming that f is convex and is a term in the objective function for a linearly constrained optimization problem, derive an equivalent linear programming formulation involving at most k extra variables and constraints.
   (e) Repeat the first four parts of this problem using $\max(x - b_i, 0)$ in place of $|x - b_i|$.
2. Let f be the function of 2 real variables defined by

   $$f(x, y) = x^2 - 2xy + y^2.$$

   Show that f is convex.
3. A function f of 2 real variables is called a *monomial* if it has the form

   $$f(x, y) = x^m y^n$$

   for some nonnegative integers m and n. Which monomials are convex?
4. Let ϕ be a convex function of a single real variable. Let f be a function defined on $\mathbb{R}^n$ by the formula

   $$f(x) = \phi(a^T x + b),$$

   where a is an n-vector and b is a scalar. Show that f is convex.

5. Which of the following functions are convex (assume that the domain of the function is all of $\mathbb{R}^n$ unless specified otherwise)?
   (a) $4x^2 - 12xy + 9y^2$
   (b) $x^2 + 2xy + y^2$
   (c) x^2y^2
   (d) $x^2 - y^2$
   (e) e^{x-y}
   (f) $e^{x^2-y^2}$
   (g) $\frac{x^2}{y}$ on $\{(x, y) : y > 0\}$
6. Given a symmetric square matrix A, the quadratic form $x^T Ax = \sum_{i,j} a_{ij}x_ix_j$ generalizes the notion of the square of a variable. The generalization of the notion of the fourth power of a variable is an expression of the form

$$f(x) = \sum_{i,j,k,l} a_{ijkl}x_ix_jx_kx_l.$$

The four-dimensional array of numbers $A = \{a_{ijkl} : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n, 1 \leq l \leq n\}$ is called a *4-tensor*. As with quadratic expressions, we may assume that A is symmetric:

$$a_{ijkl} = a_{jkli} = \cdots = a_{lkij}$$

(i.e., given i, j, k, l, all $4! = 24$ premutations must give the same value for the tensor).
   (a) Give conditions on the 4-tensor A to guarantee that f is convex.
   (b) Suppose that some variables, say y_i's, are related to some other variables, say x_j's, in a linear fashion:

$$y_i = \sum_j f_{ij}x_j.$$

Express $\sum_i y_i^4$ in terms of the x_j's. In particular, give an explicit expression for the 4-tensor and show that it satisfies the conditions derived in part (a).

7. Consider the problem

$$\begin{array}{ll} \text{minimize} & ax_1 + x_2 \\ \text{subject to} & \sqrt{\epsilon^2 + x_1^2} \leq x_2. \end{array}$$

where $-1 < a < 1$.

(a) Graph the feasible set: $\left\{(x_1, x_2) : \sqrt{\epsilon^2 + x_1^2} \leq x_2\right\}$. Is the problem convex?

(b) Following the steps in the middle of p. 391 of the text, write down the first-order optimality conditions for the barrier problem associated with barrier parameter $\mu > 0$.

(c) Solve explicitly the first-order optimality conditions. Let $(x_1(\mu), x_2(\mu))$ denote the solution.

(d) Graph the central path, $(x_1(\mu), x_2(\mu))$, as μ varies from 0 to ∞.