1. What is the dual of the following linear programming problem:

$$\begin{array}{lrcrcrcrcl}
\text{maximize} & x_1 & - & 2x_2 & & & & & & \\
\text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & \geq & 0 \\
& 4x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & \leq & 3 \\
& -x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 1 \\
& & & x_2, & x_3 & \geq & & 0 & . &
\end{array}$$

2. For x and y in $\mathbb{R}$, compute

$$\max_{x \geq 0} \min_{y \geq 0} (x - y) \quad \text{and} \quad \min_{y \geq 0} \max_{x \geq 0} (x - y)$$

and note whether or not they are equal.

3. Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & 2x_1 & + & 8x_2 & - & x_3 & - & 2x_4 & & \\
\text{subject to} & 2x_1 & + & 3 & & & + & 6x_4 & \leq & 6 \\
& -2x_1 & + & 4x_2 & + & 3x_3 & & & \leq & 1.5 \\
& 3x_1 & + & 2x_2 & - & 2x_3 & - & x_4 & \leq & 4 \\
& & & & & x_1, \; x_2, & x_3, & x_4 & \geq & 0.
\end{array}$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$\begin{array}{rcrcrcrcrcr}
\zeta & = & 3.5 & - & 0.25w_1 & + & 6.25x_2 & - & 0.5w_3 & - & 1.5x_4 \\
\hline
x_1 & = & 3.0 & - & 0.5w_1 & - & 1.5x_2 & & & - & 3.0x_4 \\
w_2 & = & 0.0 & + & 1.25w_1 & - & 3.25x_2 & - & 1.5w_3 & + & 13.5x_4 \\
x_3 & = & 2.5 & - & 0.75w_1 & - & 1.25x_2 & + & 0.5w_3 & - & 6.5x_4.
\end{array}$$

(a) Write down the dual problem.
(b) In the dictionary shown above, which variables are basic? Which are nonbasic?
(c) Write down the primal solution corresponding to the given dictionary. Is it feasible? Is it degenerate?
(d) Write down the corresponding dual dictionary.
(e) Write down the dual solution. Is it feasible?
(f) Do the primal/dual solutions you wrote above satisfy the complementary slackness property?
(g) Is the current primal solution optimal?
(h) For the next (primal) pivot, which variable will enter if the largest coefficient rule is used? Which will leave? Will the pivot be degenerate?

4. Consider the following variant of the resource allocation problem:

$$
\begin{aligned}
\text{maximize} \quad & c^T x \\
\text{subject to} \quad & Ax \le b \\
& 0 \le x \le u.
\end{aligned}
\tag{1}
$$

Here, c denotes the vector of unit prices for the products, b denotes the vector containing the number of units on hand of each raw material, and u denotes a vector of upper bounds on the number of units of each product that can be sold at the set price. Now, let's assume that the raw materials have not been purchased yet and it is part of the problem to determine b. Let p denote the vector of prices for each raw material. The problem then becomes an optimization over both x and b:

$$
\begin{aligned}
\text{maximize} \quad & c^T x - p^T b \\
\text{subject to} \quad & Ax - b \le 0 \\
& 0 \le x \le u \\
& b \ge 0.
\end{aligned}
$$

(a) Show that this problem always has an optimal solution.

(b) Let $y^*(b)$ denote optimal dual variables for the original resource allocation problem (1) (note that we've explicitly indicated that these dual variables depend on the vector b). Show that the optimal value of b, call it b^*, satisfies

$$y^*(b^*) = p.$$