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THE OPTIMAL CHOICE OF A SUBSET OF A POPULATION*

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We check a ranked population in random order. At each step we either accept or reject the observed sample. As a result we get a partition of the population into two parts: accepted and rejected. We are interested only in the case when all accepted samples are better than all rejected. Under this condition, we maximize the probability to obtain a fixed size k of the accepted part. An arbitrary gain function $\varphi(k)$ is also considered.

1. Introduction. There is extensive literature on the problem of optimal choice. In the simplest case, known as the secretary problem, the object is to choose one appropriate sample from a population of n, e.g., select the best [3], one of the few best [5], or maximize the expected rank [5]. We shall investigate the problem of choosing not just one sample but a subset of the population. We are only interested in subsets D with the property that each element of D is better than all elements of the rest of the population. We will call these *excellent* sets. We observe sequentially samples from the population and decide to accept or reject each sample as we go. After sampling the entire population we receive a pay-off depending on the excellent set obtained (the pay-off is zero for nonexcellent sets). As a special case we consider in §4 the problem of selecting the best k of a population of $n = 2k$ with maximal probability. In this case we describe explicitly the optimal strategy and we prove that the corresponding probability of success is $1/(k+1)$. We do not have an explicit solution if $n \neq 2k$. However, we get certain asymptotic results as $n \to \infty$ for fixed k and also as $k \to \infty, n \to \infty$ in such a way that $(2k - n)/\sqrt{n}$ remains finite. In the latter case we get as a limit a diffusion process with a constant diffusion and variable drift.

Some results identical to ours are contained in a technical report [4] which appeared after this paper had been submitted for publication. In particular, a simple induction proof is given for equation (4.2). The problem of optimal selection of the two best samples has been considered earlier in [6].

Clearly a strategy will produce an excellent subset if and only if it satisfies the following condition: if a sample is better (worse) than some previously accepted (rejected) then accept (reject) it. Such strategies will be called *admissible*. This condition tells how to handle all samples except ones which are worse than any previously accepted but better than all rejected. We will call these *marginal* samples. An admissible strategy is determined by indicating what to do with marginal samples.

Suppose at time t (i.e., after t samples have been observed and decided on) we observe sample number $t+1$ and it is marginal. We must decide either to accept or reject it based on the information we have at this time. Namely, the relative ranks of the samples already checked and which ones were accepted. However, not all this information is needed for our problem. It is sufficient to know only the number accepted or the number rejected. For symmetry, we prefer to introduce the difference

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and to describe the state X_t at time t as the number of accepted minus the number of rejected samples. In this way we get a sequence $X_0 = 0, X_1, X_2, \ldots, X_n$ where $X_{t+1} - X_t$ is equal to 1 if we accept and -1 if we reject sample number $t+1$. If this sample is not marginal our action is determined by the admissibility condition. This gives certain probabilities, $p_t(x)$ and $q_t(x)$, of transition from x to $x+1$ and $x-1$ respectively at each time t. A marginal sample can be treated as we like and a given decision increases one or both of these probabilities.

2. Controlled Markov Chain. We assume that all $n!$ possible orders in which we check our population have equal probability. This implies that all relative rankings of the first $t+1$ samples are equally likely. To be at state x at time t means that we have accepted $\frac{1}{2}(t+x)$, hence, sample number $t+1$ will be better than a previously accepted sample with probability $\frac{1}{2}(t+x)/(t+1)$ which gives us $p_t(x)$. Analogously, $q_t(x) = \frac{1}{2}(t-x)/(t+1)$. The difference

$$r_t = 1 - p_t(x) - q_t(x) = 1/(t+1)$$

does not depend on x. This amount of probability can be allocated to $p_t(x)$ and $q_t(x)$ by our own choice. In this way X_t is a controlled Markov chain with transition probabilities

$$P^\gamma\{X_{t+1} = x+1 \mid X_t = x\} = p_t(x) + \gamma_t(x)r_t,$$

$$P^\gamma\{X_{t+1} = x-1 \mid X_t = x\} = q_t(x) + (1-\gamma_t(x))r_t$$

where $\gamma_t(x)$, $0 \leqslant \gamma_t(x) \leqslant 1$, represents our strategy at state x and time t.

An excellent set obtained at the end is uniquely determined by X_n, and our objective is to maximize the mathematical expectation of $\varphi(X_n)$ for a given pay-off function φ. We put

$$F_t(x) = \sup_\gamma E^\gamma_{t,x}\varphi(X_n)$$

where $E^\gamma_{t,x}$ is the mathematical expectation if we start at time t in state x and use the strategy γ. According to the general theory of dynamic programming [1],

$$F_t(x) = p_t(x)F_{t+1}(x+1) + q_t(x)F_{t+1}(x-1) + r_t[F_{t+1}(x+1) \vee F_{t+1}(x-1)], \tag{2.1}$$

with

$$F_n(x) = \varphi(x).$$

3. Optimal strategy. In this section we derive the general form of the optimal strategy for arbitrary φ. Since $X_0 = 0$, the process visits even (odd) states at even (odd) times. Thus at time t, $X_t \in E_t = \{x : |x| \leqslant t,\ x+t \text{ is even}\}$ and it suffices to define φ only on E_n. If we let

$$A_t = \{x \in E_t : F_{t+1}(x+1) > F_{t+1}(x-1)\},$$

$$R_t = \{x \in E_t : F_{t+1}(x+1) < F_{t+1}(x-1)\}$$

and

$$C_t = \{x \in E_t : F_{t+1}(x+1) = F_{t+1}(x-1)\},$$

then by (2.1) the optimal strategy acquires the following simple form.

Optimal strategy. If sample number $t+1$ is marginal then accept (reject, or do either to) it if $X_t \in A_t(R_t, C_t$, respectively). Now we establish some relations between A_t, R_t, C_t and $A_{t+1}, R_{t+1}, C_{t+1}$.

LEMMA 3.1. *If* $x+1$ *and* $x-1$ *are in* A_{t+1} (R_{t+1}, C_{t+1}) *then* $x \in A_t$ $(R_t, C_t,$ *respectively*).

PROOF. Let Δ be the symmetric difference operator ($\Delta u(x) = u(x+1) - u(x-1)$). Suppose $x+1, x-1 \in A_{t+1}$. Then $\Delta F_{t+2}(x+1) > 0$ and $\Delta F_{t+2}(x-1) > 0$. It follows from (2.1) that $F_{t+1}(x+1) > F_{t+2}(x) > F_{t+1}(x-1)$ and so $x \in A_t$. The proof is the same for R_t and C_t.

LEMMA 3.2. *If* $x+1 \in C_{t+1}$ *and* $x-1 \in A_{t+1}(R_{t+1})$ *then* $x \in A_t(R_t)$. *The same result holds when* $x+1$ *and* $x-1$ *are interchanged.*

PROOF. It suffices to consider the case where $x+1 \in C_{t+1}$ and $x-1 \in A_{t+1}$. Then we have $F_{t+1}(x+1) = F_{t+2}(x) > F_{t+1}(x-1)$ and so $x \in A_t$.

REMARK. Lemmas 3.1 and 3.2 give an inductive procedure for determining A_t, R_t and C_t given A_{t+1}, R_{t+1} and C_{t+1} except at points $x \in E_t$ for which $x+1 \in A_{t+1}(R_{t+1})$ and $x-1 \in R_{t+1}(A_{t+1})$, i.e., points where F_{t+2} has a local minimum (maximum, resp.). Hence, if φ has say N critical points then the strategy at time t is determined by the strategy at time $t+1$ except at N points (or less).

4. Selection of the best half. Now consider the problem of selecting the best half from a population of n.

THEOREM 4.1. *For the problem of selecting exactly the best half, an admissible strategy is optimal if and only if marginal samples are accepted (rejected) when the total number of samples accepted so far is less (greater) than the number rejected. The optimal probability of success is* $1/(k+1)$, *where* $k = n/2$.

PROOF. For this problem $\varphi(x)$ is one at $x = 0$ and zero elsewhere. Hence $A_{n-1} = \{-1\}$, $R_{n-1} = \{1\}$ and $C_{n-1} = \{x : |x| > 1\}$. By Lemma 3.1 we see that $\{x \in E_t : |x| > n-t\} \subset C_t$. Since $\varphi(-x) = \varphi(x)$ and $p_t(-x) = q_t(x)$, it follows that $F_t(-x) = F_t(x)$. In particular, for t even $F_{t+1}(-1) = F_{t+1}(1)$ which implies that $0 \in C_t$. From Lemmas 3.1 and 3.2 we see that $A_t = \{x \in E_t : x > 0\}$, $R_t = \{x \in E_t : x > 0\}$ and $C_t = \{x \in E_t : x = 0\} \cup \{x \in E_t : |x| > n-t\}$.

To solve (2.1) first change the independent variables to the number rejected, $r = \frac{1}{2}(t-x)$, and the number accepted, $a = \frac{1}{2}(t+x)$. Then we have

$$(a+r+1)F(r,a) = rF(r+1,a) + aF(r,a+1) + F(r+1,a) \vee E(r,a+1). \quad (4.1)$$

Now letting $w(r,a) = F(r,a)/a(a+1)\ldots(r+a)$ we transform (4.1) into a difference equation. Solving it we express $w(r,a)$ in terms of $w(r+1,m)$, $m \geqslant a$. Then we show by induction that

$$F(r,r) = \frac{2r+1}{k+1} \frac{\binom{2r}{r}}{\binom{2k}{k-1}} \sum_{l=1}^{k-r} a_l^r \binom{2k-2r-l}{k-r-l}$$

where $a_1^r = 1$ and a_l^r is defined recursively by

$$a_l^r = \sum_{m=1}^{l-1} a_m^r \left\{ \binom{2l-m-2}{l-m-1}\left(2 + \frac{1}{r+l-1}\right) - \binom{2l-m-1}{l-m} \right\}, \qquad l > 1.$$

For $r = 0$, we have $a_l^0 = 1$. Hence the probability of success using an optimal strategy is

$$F(0,0) = 1/(k+1).$$

Letting γ^* denote the optimal strategy we can express the above result in terms of the controlled Markov chain X_n as

$$P^{\gamma^*}_{0,0}(X_n = 0) = 2/(n+2).$$

In fact, a simple modification of the above calculation gives us

$$P^{\gamma^*}_{0,0}(X_n = x) = \begin{cases} 4\dfrac{n+1}{(n+2+|x|)(n+|x|)}, & x \neq 0, \\ \dfrac{2}{n+2}, & x = 0, \end{cases} \tag{4.2}$$

for even n. Using this we can show that for any $\beta < 1$

$$\lim_n \sup_\gamma P^{\gamma}_{0,0}(X_n \in [-an^\beta, an^\beta]) = 0. \tag{4.3}$$

5. Selection of the k best out of n. In this section we consider the problem of selecting the excellent set containing exactly k elements. We will call this the (k, n) problem. The (k, n) and $(n-k, n)$ problems are dual in the sense that by interchanging our notions of accept/reject and better/worse we see that these two problems are interchanged. As in §4, denote the number of samples previously accepted by a and those rejected by r.

THEOREM 5.1. *There exists a partition of $\{0, 1, \ldots, n-1\}$ into two disjoint sets $T = \{t_0 < t_1 < \cdots < t_{k-1}\}$ and $S = \{s_0 < s_1 < \cdots < s_{n-k-1}\}$ which determine an optimal strategy as follows: accept a marginal sample if and only if $t \geqslant t_a$; that is, if and only if the number sampled so far is greater than t_a, where a is the number accepted so far. An equivalent formulation is: reject a marginal sample if and only if $t \geqslant s_r$. For the dual problem, S and T are interchanged.*

PROOF. First note that for $a > k$ or $r > n-k$ there is no chance for success. Also, for $a = k$ the optimal strategy is to reject marginal samples. Similarly, for $r = n-k$ the optimal strategy is to accept marginal samples. Hence it suffices to specify the strategy only for (r, a) in the rectangle $0 \leqslant a < k, 0 \leqslant r < n-k$. If there are any points (r, a) in this rectangle which are in C_t, assign them either to the acceptance region A or the rejection region R. By Lemmas 3.1 and 3.2, if $(r, a) \in R$ then $(r-1, a)$ and $(r, a+1)$ are in R. Also for $(r, a) \in A$, $(r+1, a)$ and $(r, a-1)$ are in A. Let $r^*(a)$ be the smallest value of r for which $(r, a) \in A$ (set $r^*(a)$ equal to $n-k$ if there are no such r's). Clearly, $0 \leqslant r^*(0) \leqslant r^*(1) \leqslant \cdots \leqslant r^*(k-1) \leqslant n-k$. Now if we let $t_a = r^*(a) + a$ then $T = \{t_a\}$ determines the optimal strategy stated in the theorem.

Let $a^*(r)$ be the smallest value of a for which $(r, a) \in R$ (set $a^*(r)$ equal to k if there are no such a's). Then $S = \{a^*(r) + r : 0 \leqslant r < n-k\}$ gives the alternate form of the optimal strategy. Suppose that $t_a = s_r$ for some a, r, then

$$r^*(a) + a = a^*(r) + r. \tag{5.1}$$

If $a^* > a$ then by the definition of a^*, $(a, r) \in A$. However, by (5.1) $r^* > r$ and so by the definition of r^*, $(a, r) \in R$. This contradiction forces us to consider $a^* \leqslant a$ but similar arguments again lead to a contradiction. Hence S and T are disjoint.

Now we investigate the (k, n) problem for a fixed k as n tends to infinity. We prove that the limits $\tau(k) = \lim t_{k-1}(k, n)/n$, $\sigma(k) = \lim t_{k-2}(k, n)/n$ and $p_k = \lim p_{k,n}$ exist for all k. It is known [4] that $p_1 = e^{-1}$ and $\tau(1) = e^{-1}$. We will show that $\tau(k) = e^{-1/k}$ and

$$1 - \frac{2k^2 - 1}{k(k-1)}\tau(k) = \tau(k)\log\sigma(k) - \sigma(k). \tag{5.2}$$

In particular, $\sigma(2) \approx 0.2291$ and $p_2 = 2\tau\sigma - \sigma^2 \approx 0.2254$.

A simple calculation shows that

$$F(r,k) = \frac{\binom{k+r}{k}}{\binom{n}{k}} \quad \text{and} \quad F(n-k,a) = \frac{\binom{n-k-a}{n-k}}{\binom{n}{k}}.$$

From $F(r,k)$ we find that for $k < n/2$

$$(n-k)e^{-1/k} + k - (2 - e^{-1/k}) < t_{k-1} < (n-k)e^{-1/k} + k$$

(for $k > n/2, t_{k-1} = n - 1$). Taking limits we get $\tau(k) = e^{-1/k}$.

Let $f_a(x)$ be the limit of $F_{k,n}(r,a)$ as r and n tend to infinity in such a way that r/n tends to x. We replace the system of difference equations (4.1) with a system of differential equations in two different ways. These give upper and lower bounds on $F(r,a)$ and passing to the limit we get

$$xf_a'(x) = \begin{cases} a(f_a(x) - f_{a+1}(x)) & \text{for } f_a(x) > f_{a+1}(x), \\ (a+1)(f_a(x) - f_{a+1}(x)) & \text{for } f_a(x) \leqslant f_{a+1}(x) \end{cases} \tag{5.3}$$

with $f_k(x) = x^k$ and $f_a(1) = 0$ for $a < k$. Then τ and σ are the solutions of the equations $f_{k-1}(\tau) = f_k(\tau)$ and $f_{k-2}(\sigma) = f_{k-1}(\sigma)$, respectively. Also $p_k = f_0(0)$. Equation (5.2), p_1, and p_2 now follow by solving (5.3) for $a = k - 1$ and $k - 2$.

It is interesting to compare these results to those obtained by Gusein-Zade [5] for the problem of selecting one of the k best out of n. In this case there exists a sequence $1 \leqslant \pi_1 \leqslant \pi_2 \leqslant \cdots \leqslant \pi_k \leqslant n$ which determines the optimal strategy as follows: accept the first sample for which $t \geqslant \pi_{x_t}$, where x_t is the rank of sample number t among those preceding it. The asymptotic results for this problem are:

$$\lim_{n\to\infty} \pi_k(k,n)/n = \begin{cases} e^{-1} & \text{for } k = 1, \\ \left(\dfrac{k-1}{2k-1}\right)^{1/(k-1)} & \text{for } k > 1. \end{cases}$$

$\sigma = \lim_{n\to\infty} \pi_1(2,n)/n \approx 0.347$ is the solution of the equation $1 - \log\frac{2}{3} = \sigma - \log\sigma$, $p_1 = e^{-1}$, and $p_2 = 2\sigma - \sigma^2 \approx 0.574$.

Consider the following suboptimal strategy: reject the first m samples and after that accept all marginal samples. The probability of success p^m is given by

$$p^m = \binom{n-k-1}{m-1} \Big/ \binom{n}{m}$$

which is maximized by taking $m = [(n+1)/(k+1)]$ ($[x]$ stands for the greatest integer less than x). From this we find that

$$p_k \geqslant \frac{1}{k+1}\left(1 - \frac{1}{k+1}\right)^k \sim \frac{1}{e}\,\frac{1}{k+1}.$$

6. Diffusion equation. In §2 we introduced a controlled Markov chain X_t. Now let

$$\xi_t^{(n)} = \frac{1}{\sqrt{n}} X_{tn}, \qquad t = 0, 1/n, 2/n, \ldots, 1,$$

$$u_\gamma^{(n)}(x,t) = E_{x,t}^\gamma \varphi(\xi_1^{(n)})$$

where γ is any admissible strategy and φ is a bounded, piecewise continuous, integrable function. Then according to the general theory [2], $u(x,t) = \lim_{n\to\infty} u_\gamma^{(n)}(x,t)$

exists and is the solution to

$$-u_t = \tfrac{1}{2} u_{xx} + \frac{x}{t} u_x, \qquad 0 < t \leq 1, \tag{6.1}$$

with $u(x, 1) = \varphi(x)$. The transition function $p(x, t; \cdot, s)$ corresponding to (6.1) is Gaussian with mean xs/t and variance $s(1-t)/t$. Taking φ to be the indicator function for the interval $[-a, a]$ we get

$$u(x, t) \leq a\sqrt{\frac{2}{\pi}\,\frac{t}{1-t}}\;. \tag{6.2}$$

Using (6.2) it is easy to get (4.3) for $\beta = \frac{1}{2}$.

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