Consistent procedures for cluster tree estimation and pruning

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Abstract—For a density $f$ on $\mathbb{R}^d$, a high-density cluster is any connected component of $\{x : f(x) \geq \lambda\}$, for some $\lambda > 0$. The set of all high-density clusters forms a hierarchy called the cluster tree of $f$. We present two procedures for estimating the cluster tree given samples from $f$. The first is a robust variant of the single linkage algorithm for hierarchical clustering. The second is based on the $k$-nearest neighbor graph of the samples. We give finite-sample convergence rates for these algorithms which also imply consistency, and we derive lower bounds on the sample complexity of cluster tree estimation. Finally, we study a tree pruning procedure that guarantees, under milder conditions than usual, to remove clusters that are spurious while recovering those that are salient.

Index Terms—Clustering algorithms, convergence.

I. INTRODUCTION

We consider the problem of hierarchical clustering in a “density-based” setting, where a cluster is formalized as a region of high density. Given data drawn i.i.d. from some unknown distribution with density $f$ in $\mathbb{R}^d$, the goal is to estimate the “hierarchical cluster structure” of the density, where a cluster is defined as a connected subset of an $f$-level set $\{x \in \mathcal{X} : f(x) \geq \lambda\}$. These subsets form an infinite tree structure as $\lambda$ varies, in the sense that each cluster at some level $\lambda$ is contained in a cluster at a lower level $\lambda' < \lambda$. This infinite tree is called the cluster tree of $f$ and is illustrated in Figure 1.

Our formalism of the cluster tree (Section II-B) and our notion of consistency follow early work on clustering, in particular that of Hartigan [1]. Much subsequent work has been devoted to estimating the connected components of a single level set; see, for example, [2]–[7]. In contrast to these results, the present work is concerned with the simultaneous estimation of all level sets of an unknown density: recovering the cluster tree as a whole.

Are there hierarchical clustering algorithms which converge to the cluster tree? Previous theory work, [1] and [8], has provided partial consistency results for the well-known single-linkage clustering algorithm, while other work [9] has suggested ways to overcome the deficiencies of this algorithm by making it more robust, but without proofs of convergence. In this paper, we propose a novel way to make single-linkage more robust, while retaining most of its elegance and simplicity (see Figure 3). We show that this algorithm implicitly creates a hierarchy of geometric graphs, and we relate connected components of these graphs to clusters of the underlying density. We establish finite-sample rates of convergence for the clustering procedure (Theorem III.3); the centerpiece of our argument is a result on continuum percolation (Theorem IV.7). This also implies consistency in the sense of Hartigan.

We then give an alternative procedure based on the $k$-nearest neighbor graph of the sample (see Figure 4). Such graphs are widely used in machine learning, and interestingly there is still much to understand about their expressiveness. We show that by successively removing points from this graph, we can create a hierarchical clustering that also converges to the cluster tree, at roughly the same rate as the linkage-based scheme (Theorem III.4).

Next, we use tools from information theory to give a lower bound on the problem of cluster tree estimation (Theorem VI.1), which matches our upper bounds in its dependence on most of the parameters of interest.

The convergence results for our two hierarchical clustering procedures nevertheless leave open the possibility that the trees they produce contain spurious branching. This is a well-studied problem in the cluster tree literature, and we address it with a pruning method (Figure 9) that preserves the consistency properties of the tree estimators while providing finite-sample
guarantees on the removal of false clusters (Theorem VII.5). This procedure is based on simple intuition that can carry over to other cluster tree estimators.

II. DEFINITIONS AND PREVIOUS WORK

Let \( X \) be a subset of \( \mathbb{R}^d \). We exclusively consider Euclidean distance on \( X \), denoted \( \| \cdot \| \). Let \( B(x, r) \) be the closed ball of radius \( r \) around \( x \).

A. Clustering

We start by considering the more general context of clustering. While clustering procedures abound in statistics and machine learning, it remains largely unclear whether clusters in finite data—for instance, the clusters returned by a particular procedure—reveal anything meaningful about the underlying distribution from which the data is sampled. Understanding what statistical estimation based on a finite data set reveals about the underlying distribution is a central preoccupation of statistics and machine learning; however this kind of analysis has proved elusive in the case of clustering, except perhaps in the case of density-based clustering.

Consider for instance \( k \)-means, possibly the most popular clustering procedure in use today. If this procedure returns \( k \) clusters on an \( n \)-sample from a distribution \( f \), what do these clusters reveal about \( f \)? Pollard [10] proved a basic consistency result: if the algorithm always finds the global minimum of the \( k \)-means cost function (which, incidentally, is NP-hard and thus computationally intractable in general; see [11, Theorem 3]), then as \( n \to \infty \), the clustering is the globally optimal \( k \)-means solution for \( f \), suitably defined. Even then, it is unclear whether the best \( k \)-means solution to \( f \) is an interesting or desirable quantity in settings outside of vector quantization.

Our work, and more generally work on density-based clustering, relies on meaningful formalisms of how a clustering of data generalizes to unambiguous structures of the underlying distribution. The main such formalism is that of the cluster tree.

B. The cluster tree

We start with notions of connectivity. A path \( P \) in \( S \subset \mathcal{X} \) is a continuous function \( P : [0,1] \to S \). If \( x = P(0) \) and \( y = P(1) \), we write \( x \leadsto y \) and we say that \( x \) and \( y \) are connected in \( S \). This relation – “connected in \( S \)” – is an equivalence relation that partitions \( S \) into its connected components. We say \( S \subset \mathcal{X} \) is connected if it has a single connected component.

The cluster tree is a hierarchy each of whose levels is a partition of a subset of \( \mathcal{X} \), which we will occasionally call a subpartition of \( \mathcal{X} \). Write \( \Pi(\mathcal{X}) = \{ \text{subpartitions of } \mathcal{X} \} \).

Definition II.1. For any \( f : \mathcal{X} \to \mathbb{R} \), the cluster tree of \( f \) is a function \( \mathcal{C}_f : \mathbb{R} \to \Pi(\mathcal{X}) \) given by \( \mathcal{C}_f(\lambda) = \text{connected components of } \{ x \in \mathcal{X} : f(x) \geq \lambda \} \). Any element of \( \mathcal{C}_f(\lambda) \), for any \( \lambda \), is called a cluster of \( f \).

For any \( \lambda \), \( \mathcal{C}_f(\lambda) \) is a set of disjoint clusters of \( \mathcal{X} \). They form a hierarchy in the following sense.

Lemma II.2. Pick any \( \lambda' \leq \lambda \). Then:
1) For any \( C \in \mathcal{C}_f(\lambda) \), there exists \( C' \in \mathcal{C}_f(\lambda') \) such that \( C \subseteq C' \).
2) For any \( C \in \mathcal{C}_f(\lambda) \) and \( C' \in \mathcal{C}_f(\lambda') \), either \( C \subseteq C' \) or \( C \cap C' = \emptyset \).

We will sometimes deal with the restriction of the cluster tree to a finite set of points \( x_1, \ldots, x_n \). Formally, the restriction of a subpartition \( C \subset \Pi(\mathcal{X}) \) to these points is defined to be \( \mathcal{C}[x_1, \ldots, x_n] = \{ C \cap \{ x_1, \ldots, x_n \} : C \in C \} \). Likewise, the restriction of the cluster tree is \( \mathcal{C}_f[x_1, \ldots, x_n] : \mathbb{R} \to \Pi(\{ x_1, \ldots, x_n \}) \), where \( \mathcal{C}_f[x_1, \ldots, x_n](\lambda) = \mathcal{C}_f(\lambda)[x_1, \ldots, x_n] \) (Figure 2).

C. Notion of convergence and previous work

Suppose a sample \( X_n \subset \mathcal{X} \) of size \( n \) is used to construct a tree \( C_n \) that is an estimate of \( C_f \). Hartigan [1] provided a sensible notion of consistency for this setting.

Definition II.3. For any sets \( A, A' \subset \mathcal{X} \), let \( A_n \) (resp, \( A'_n \)) denote the smallest cluster of \( C_n \) containing \( A \cap X_n \) (resp, \( A' \cap X_n \)). We say \( C_n \) is consistent if, whenever \( A \) and \( A' \) are different connected components of \( \{ x : f(x) \geq \lambda \} \) (for some \( \lambda > 0 \)), \( \Pr(A_n \neq \text{disjoint from } A'_n) \to 1 \) as \( n \to \infty \).

It is well known that if \( X_n \) is used to build a uniformly consistent density estimate \( f_n \) (that is, \( \sup_x |f_n(x) - f(x)| \to 0 \)), then the cluster tree \( \mathcal{C}_{f_n} \) is consistent; see Appendix A for details. The problem is that \( \mathcal{C}_{f_n} \) is not easy to compute for typical density estimates \( f_n \); imagine, for instance, how one might go about trying to find level sets of a mixture of Gaussians. Wong and Lane [12] have an efficient procedure that tries to approximate \( \mathcal{C}_{f_n} \) when \( f_n \) is a \( k \)-nearest neighbor density estimate, but they have not shown that it preserves the consistency of \( \mathcal{C}_{f_n} \).

On the other hand, there is a simple and elegant algorithm that is a plausible estimator of the cluster tree: single linkage (or Kruskal’s algorithm). Given a data set \( x_1, \ldots, x_n \in \mathbb{R}^d \), it operates as follows.

1) For each \( i \), set \( r_2(x_i) \) to the distance from \( x_i \) to its nearest neighbor.
2) As \( r \) grows from 0 to \( \infty \):
   a) Construct a graph \( G_r \) with nodes \( \{ x_i : r_2(x_i) \leq r \} \).
Include edge \((x_i, x_j)\) if \(\|x_i - x_j\| \leq r\).  

b) Let \(C_n(r)\) be the connected components of \(G_r\).

Hartigan [1] has shown that single linkage is consistent in one dimension (that is, for \(d = 1\)). But he also demonstrates, by a lovely reduction to continuum percolation, that this consistency fails in higher dimension \(d \geq 2\). The problem is the requirement that \(A \cap X_n \subseteq A_r\); by the time the clusters are large enough that one of them contains all of \(A\), there is a reasonable chance that this cluster will be so big as to also contain part of \(A'\).

With this insight, Hartigan defines a weaker notion of fractional consistency, under which \(A_n\) (resp., \(A'_n\)) need not contain all of \(A \cap X_n\) (resp., \(A' \cap X_n\)), but merely a positive fraction of it -- and ought to pass arbitrarily close (as \(n \to \infty\)) to the remainder. He then shows that single linkage achieves this weaker consistency for any pair \(A, A'\) for which the ratio
\[
\frac{\inf\{f(x) : x \in A \cup A'\}}{\sup\{\inf\{f(x) : x \in P\} : \text{paths } P \text{ from } A \to A'\}}
\]
is sufficiently large. More recent work by Penrose [8] closes the gap and shows fractional consistency whenever this ratio is > 1.

A more robust version of single linkage has been proposed by Wishart [9]: when connecting points at distance \(r\) from each other, only consider points that have at least \(k\) neighbors within distance \(r\) (for some \(k > 2\)). Thus initially, when \(r\) is small, only the regions of highest density are available for linkage, while the rest of the data set is ignored. As \(r\) gets larger, more and more of the data points become candidates for linkage. This scheme is intuitively sensible, but Wishart does not provide a proof of convergence. Thus it is unclear how to set \(k\), for instance.

Several papers [4]-[7] have recently considered the problem of recovering the connected components of \(\{x : f(x) \geq \lambda\}\) for a user-specified \(\lambda\): the flat version of our problem. Most similar to the work in this paper is the algorithm of [4], which uses the \(k\)-nearest neighbor graph of the data. These level set results invariably require niceness conditions on the specific level set being recovered, often stated in terms of the smoothness of the boundary of clusters, and/or regularity conditions on the density \(f\) on clusters of the given level set. It is unclear whether these conditions hold for all level sets of a general density, in other words how restrictive these conditions are in the context of recovering the entire cluster tree. In contrast, under mild requirements on the distribution, our conditions on the recovered level sets hold for any level set as the sample size \(n\) increases. The main distributional requirement for consistency is that of continuity of the density \(f\) on a compact support \(\Lambda'\).

A different approach is taken in a paper of Steinwart [13], which does not require the user to specify a density level, but rather automatically determines the smallest \(\lambda\) at which \(C_f(\lambda)\) has two components. In [13] the continuity requirements on the density are milder than for other results in the literature, including ours. However it does restrict attention to bimodal densities due to technical hurdles of the flat case: different levels of the cluster tree are collapsed together in the flat case making it difficult to recover a given level from data especially in the case of multimodal densities. Interestingly, the hierarchical setting resolves some of the technical hurdles of the flat case since levels of the cluster tree would generally appear at different levels of a sensible hierarchical estimator. This makes it possible in this paper to give particularly simple estimators, and to analyze them under quite modest assumptions on the data.

A related issue that has received quite a lot of attention is that of pruning a cluster tree estimate: removing spurious clusters. A recent result of Rinaldo et al [14] gives meaningful statistical guarantees, but is based on the cluster tree of an empirical density estimate, which is algorithmically problematic as discussed earlier. Stuetzle and Nugent [15] have an appealing top-down scheme for estimating the cluster tree, along with a post-processing step (called runt pruning) that helps identify modes of the distribution. The consistency of this method has not yet been established. We provide a consistent pruning procedure for both our procedures.

The present results are based in part on earlier conference versions, [16] and [17]. The result of [16] analyzes the consistency of our first cluster tree estimator (Figure 3) but provides no pruning method for the estimator. The result of [17] analyzes the second cluster tree estimator (Figure 4) and shows how to prune it. However the pruning method is tuned to this second estimator and works only under strict Hölder continuity requirements on the density. The present work first provides a unified analysis of both estimators using techniques developed in [16]. Second, building on insight from [17], we derive a new pruning method which provably works for either estimator without Hölder conditions on the distribution. In particular, the pruned version of either cluster tree estimate remains consistent under mild uniform continuity assumptions. The main finite-sample pruning result of Theorem VII.5 requires even milder conditions on the density than required for consistency.

More recent work [18] has extended the results here, showing that a variant of our first algorithm (Figure 3) is also a consistent estimator of the cluster tree for densities supported on a Riemannian submanifold of \(\mathbb{R}^d\), with rates of convergence depending only on the dimension of that manifold.

III. Algorithms and results

The first algorithm we consider in this paper is a generalization of Wishart’s scheme and of single linkage, shown in Figure 3. It has two free parameters: \(k\) and \(\alpha\). For practical reasons, it is of interest to keep these as small as possible. We provide finite-sample convergence rates for all \(1 \leq \alpha \leq 2\) and we can achieve \(k \sim d\log n\) if \(\alpha \geq \sqrt{2}\). Our rates for \(\alpha = 1\) force \(k\) to be much larger, exponential in \(d\). It is an open problem to determine whether the setting \((\alpha = 1, k \sim d\log n)\) yields consistency.

Conceptually, the algorithm creates a series of graphs \(G_r = (V_r, E_r)\) satisfying a nesting property: \(r \leq r' \Rightarrow V_r \subseteq V_r'\) and \(E_r \subseteq E_r'\). A point is admitted into \(G_r\) only if it has \(k\) neighbors within distance \(r\); when \(r\) is small, this picks out the regions of highest density, roughly. The edges of \(G_r\) are between all pairs of points within distance \(\alpha r\) of each other.
Algorithm 1

1) For each $x_i$ set $r_k(x_i) = \min\{r : B(x_i, r) \text{ contains } k \text{ data points}\}$.
2) As $r$ grows from 0 to $\infty$:
   a) Construct a graph $G_r$ with nodes $\{x_i : r_k(x_i) \leq r\}$. Include edge $(x_i, x_j)$ if $\|x_i - x_j\| \leq \alpha r$.
   b) Let $C_n(r)$ be the connected components of $G_r$.

Fig. 3. An algorithm for hierarchical clustering. The input is a sample $X_n = \{x_1, \ldots, x_n\}$ from density $f$ on $\mathcal{X}$. Parameters $k$ and $\alpha$ need to be set. Single linkage is ($\alpha = 1, k = 2$). Wishart suggested $\alpha = 1$ and larger $k$.

Algorithm 2

1) For each $x_i$ set $r_k(x_i) = \min\{r : B(x_i, r) \text{ contains } k \text{ data points}\}$.
2) As $r$ grows from 0 to $\infty$:
   a) Construct a graph $G_r^{\text{NN}}$ with nodes $\{x_i : r_k(x_i) \leq r\}$. Include edge $(x_i, x_j)$ if:
      \[
      \|x_i - x_j\| \leq \alpha \max(r_k(x_i), r_k(x_j)) \quad \text{k-NN graph}
      \|x_i - x_j\| \leq \alpha \min(r_k(x_i), r_k(x_j)) \quad \text{mutual k-NN graph}
      \]
   b) Let $C_n(r)$ be the connected components of $G_r^{\text{NN}}$.

Fig. 4. A cluster tree estimator based on the $k$-nearest neighbor graph.

In practice, the only values of $r$ that matter are those corresponding to interpoint distances within the sample, and thus the algorithm is efficient. A further simplification is that the graphs $G_r$ don’t need to be explicitly created. Instead, the clusters can be generated directly using Kruskal’s algorithm, as is done for single linkage.

The second algorithm we study (Figure 4) is based on the $k$-nearest neighbor graph of the samples. There are two natural ways to define this graph, and we will analyze the sparser of the two, the mutual $k$-NN graph, which we shall denote $G_r^{\text{NN}}$. Our results hold equally for the other variant.

One way to think about the second hierarchical clustering algorithm is that it creates the $k$-nearest neighbor graph on all the data samples, and then generates a hierarchy by removing points from the graph in decreasing order of their $k$-NN radius $r_k(x_i)$. The resulting graphs $G_r^{\text{NN}}$ have the same nodes as the corresponding $G_r$, but have potentially fewer edges: $E_r^{\text{NN}} \subseteq E_r$. This makes them more challenging to analyze.

Much of the literature on density-based clustering refers to clusters not by the radius $r$ at which they appear, but by the “corresponding empirical density”, which in our case would be $\lambda = k/(nv_d r^d)$, where $v_d$ is the volume of the unit ball in $\mathbb{R}^d$. The reader who is more comfortable with the latter notation should mentally substitute $G[\lambda]$ whenever we refer to $G_r$. We like using $r$ because it is directly observed rather than inferred. Consider, for instance, a situation in which the underlying density $f$ is supported on a low-dimensional submanifold of $\mathbb{R}^d$. The two cluster tree algorithms continue to be perfectly sensible, as does $r$; but the inferred $\lambda$ is misleading.

A. A notion of cluster salience

Suppose density $f$ is supported on some subset $\mathcal{X}$ of $\mathbb{R}^d$. We will find that when Algorithms 1 and 2 are run on data drawn from $f$, their estimates are consistent in the sense of Definition II.3. But an even more interesting question is, what clusters will be identified from a finite sample? To answer this, we need a notion of salience.

The first consideration is that a cluster is hard to identify if it contains a thin “bridge” that would make it look disconnected in a small sample. To control this, we consider a “buffer zone” of width $\sigma$ around the clusters.

Definition III.1. For $Z \subset \mathbb{R}^d$ and $\sigma > 0$, write $Z_\sigma = Z + B(0, \sigma) = \{y \in \mathbb{R}^d : \inf_{z \in Z} \|y - z\| \leq \sigma\}$.

$Z_\sigma$ is a full-dimensional set, even if $Z$ itself is not.

Second, the ease of distinguishing two clusters $A$ and $A'$ depends inevitably upon the separation between them. To keep things simple, we’ll use the same $\sigma$ as a separation parameter.

Definition III.2. Let $f$ be a density supported on $\mathcal{X} \subset \mathbb{R}^d$. We say that $A, A' \subset \mathcal{X}$ are $(\sigma, \epsilon)$-separated if there exists $S \subset \mathcal{X}$ (separator set) such that (i) any path in $\mathcal{X}$ from $A$ to $A'$ intersects $S$, and (ii) $\sup_{x \in S} f(x) < (1 - \epsilon) \inf_{x \in A \cup A'} f(x)$.

Under this definition, $A_\sigma$ and $A'_\sigma$ must lie within $\mathcal{X}$, otherwise the right-hand side of the inequality is zero. $S_\sigma$ need not be contained in $\mathcal{X}$.

B. Consistency and rate of convergence

We start with a result for Algorithm 1, under the settings $\alpha \geq \sqrt{2}$ and $k \sim d \log n$. The analysis section also has results for $1 \leq \alpha \leq 2$ and $k \sim (2/\alpha)^d d \log n$. The result states general saliency conditions under which a given level $\lambda$ of the cluster tree is recovered at level $r(\lambda)$ of the estimator. The mapping $r$ is of the form $\left(\frac{k}{nv_d r^d}\right)^{1/d}$ (see Definition IV.4), where $v_d$ is the volume of the unit ball in $\mathbb{R}^d$.

Theorem III.3. There is an absolute constant $C$ such that the following holds. Pick any $0 < \delta, \epsilon < 1$, and run Algorithm 1 on a sample $X_n$ of size $n$ drawn from $f$, with settings $\sqrt{2} \leq \alpha \leq 2$ and $k \geq C \cdot \frac{d \log n}{\epsilon^2 \log^2 \frac{1}{\delta}}$. 
Then there is a mapping \( r : [0, \infty) \to [0, \infty) \) such that the following holds with probability at least \( 1 - \delta \). Consider any pair of connected subsets \( A, A' \subset X \) such that \( A, A' \) are \((\sigma, \epsilon)\)-separated for \( \epsilon \) and some \( \sigma > 0 \). Let \( \lambda = \inf_{x \in A \cup A'} f(x) \). If \( n \geq \frac{4}{\epsilon^2} \left( 1 + \frac{3}{\delta} \right) \), then:

1) Separation. \( A \cap X_n \) is disconnected from \( A' \cap X_n \) in \( G_r(x) \).
2) Connectedness. \( A \cap X_n \) and \( A' \cap X_n \) are each connected in \( G_r(x) \).

The two parts of this theorem — separation and connectedness — are proved in Sections IV-A and IV-B, respectively.

A similar result holds for Algorithm 2 under stronger requirements on \( k \).

**Theorem III.4.** Theorem III.3 applies also to Algorithm 2, provided the following additional condition on \( k \) is met: \( k \geq n \log n \cdot \log 1/\epsilon \), where \( \lambda = \sup_{x \in A} f(x) \).

In the analysis section, we give a lower bound (Lemma V.3) that shows why this dependence on \( \lambda/\epsilon \) is needed.

Finally, we point out that these finite-sample results imply consistency (Definition II.3): as \( n \to \infty \), take \( k_n = (d \log n)/\epsilon^2 \) with any schedule of \( \{e_n\} \) such that \( e_n \to 0 \) and \( k_n/n \to 0 \). Under mild uniform continuity conditions, any two connected components \( A, A' \) of \( \{f \geq \lambda\} \) are \((\sigma, \epsilon)\)-separated for some \( \sigma, \epsilon > 0 \) (see Appendix B); thus they are identified given large enough \( n \).

**IV. Analysis of Algorithm 1**

A. Separation

Both cluster tree algorithms depend heavily on the radii \( r_k(x) \): the distance within which \( x \)'s nearest \( k \) neighbors lie (including \( x \) itself). The empirical probability mass of \( B(x, r_k(x)) \) is \( k/n \). To show that \( r_k(x) \) is meaningful, we need to establish that the mass of this ball under density \( f \) is also roughly \( k/n \). The uniform convergence of these empirical counts follows from the fact that balls in \( \mathbb{R}^d \) have finite VC dimension, \( d + 1 \).

We also invoke uniform convergence over half-balls: each of these is the intersection of a ball with a halfspace through its center. Using uniform Bernstein-type bounds, we derive basic inequalities which we use repeatedly.

**Lemma IV.1.** Assume \( k \geq d \log n \), and fix some \( \delta > 0 \). Then there exists a constant \( C_\delta \) such that with probability \( > 1 - \delta \), we have that, first, every ball \( B \subset \mathbb{R}^d \) satisfies the following conditions:

\[
\begin{align*}
  f(B) & \geq \frac{C_\delta d \log n}{n} \quad \implies \quad f_n(B) > 0 \\
  f(B) & \geq \frac{k}{n} + \frac{C_\delta}{n} \sqrt{kd \log n} \quad \implies \quad f_n(B) \geq \frac{k}{n} \\
  f(B) & \leq \frac{k}{n} - \frac{C_\delta}{n} \sqrt{kd \log n} \quad \implies \quad f_n(B) < \frac{k}{n}
\end{align*}
\]

Here \( f_n(B) = |X_n \cap B|/n \) is the empirical mass of \( B \), while \( f(B) \) is its probability under \( f \). Second, for every half-ball \( H \subset \mathbb{R}^d \):

\[
f(H) \geq \frac{C_\delta d \log n}{n} \quad \implies \quad f_n(H) > 0.
\]

We denote this uniform convergence over balls and half-balls as event \( E_\delta \).

**Proof.** See Appendix C. Here \( C_\delta = 2C_\alpha \log(2/\delta) \), where \( C_\alpha \) is the absolute constant from Lemma A.4.

We will typically preface other results by a statement like “Assume \( E_\delta \).” It is to be understood that \( E_\delta \) occurs with probability at least \( 1 - \delta \) over the random sample \( X_n \), where \( \delta \) is henceforth fixed. The constant \( C_\delta \) will keep appearing throughout the paper.

For any cluster \( A \subset X \), there is a certain scale \( r \) at which every data point in \( A \cap X_n \) appears in \( G_r \). What is this \( r \)?

**Lemma IV.2.** Assume \( E_\delta \). Pick any set \( A \subset X \), and let \( \lambda = \inf_{x \in A} f(x) \). If \( r < \sigma \) and \( v_d d \lambda \geq \frac{k}{n} + \frac{C_\delta}{n} \sqrt{kd \log n} \), then \( G_r \) contains every point in \( A \cap X_n \).

**Proof.** Any point \( x \in A \cap X_n \) has \( f(B(x, r)) \geq v_d d \lambda \); and thus, by Lemma IV.1, has at least \( k \) neighbors within radius \( r \).

In order to show that two separate clusters \( A \) and \( A' \) get distinguished in the cluster tree, we need to exhibit a scale \( r \) at which every point in \( A \) and \( A' \) is active, but there is no path from \( A \) to \( A' \).

**Lemma IV.3.** Assume \( E_\delta \). Suppose sets \( A, A' \subset X \) are \((\sigma, \epsilon)\)-separated by set \( S \), and let \( \lambda = \inf_{x \in A \cup A'} f(x) \). Pick \( 0 < r < \sigma \) such that

\[
\frac{k}{n} + \frac{C_\delta}{n} \sqrt{kd \log n} \leq v_d d \lambda < \left( \frac{k}{n} - \frac{C_\delta}{n} \sqrt{kd \log n} \right) \cdot \frac{1}{1 - \epsilon}.
\]

Then:

(a) \( G_r \) contains all points in \( (A \cap X_n) \cap X_n \).
(b) \( G_r \) contains no points in \( S \cap X_n \).
(c) If \( r < 2\sigma/(\alpha + 2) \), then \( A \cap X_n \) is disconnected from \( A' \cap X_n \) in \( G_r \).

**Proof.** Part (a) is directly from Lemma IV.2. For (b), any point \( x \in S \) has \( f(B(x, r)) < v_d d \lambda \cdot (1 - \epsilon) \); and thus, by Lemma IV.1, has strictly fewer than \( k \) neighbors within distance \( r \).

For (c), since points in \( S \) are absent from \( G_r \), any path from \( A \) to \( A' \) in that graph must have an edge across \( S \). But any such edge has length at least \( 2(\sigma - r) > \alpha r \) and is thus not in \( G_r \).

**Definition IV.4.** Define \( r(\lambda) \) to be the value of \( r \) for which \( v_d d \lambda = \frac{k}{n} + \frac{C_\delta}{n} \sqrt{kd \log n} \).

**Corollary IV.5.** The conditions of Lemma IV.3 are satisfied by \( r = r(\lambda) \) if \( r < 2\sigma/(\alpha + 2) \) and \( k \geq 4C_\delta^2 (d/\epsilon^2) \log n \).

B. Connectedness

We need to show that points in \( A \) (and similarly \( A' \)) are connected in \( G_r(\lambda) \). First we state a simple bound (proved in Appendix D) that works if \( \alpha = 2 \) and \( k \sim d \log n \); later we consider smaller \( \alpha \).
Lemma IV.6. Assume $E_0$. Let $A$ be a connected set in $\mathcal{X}$ with $\lambda = \inf_{x \in A_\mu} f(x)$. Suppose $1 \leq \alpha < 2$. Then $A \cap X_n$ is connected in $G_r$ whenever $r \leq 2\sigma/(2 + \alpha)$ and

$$v_{d^r} d \lambda \geq \max \left\{ \left( \frac{2}{\alpha} \right)^d C_\delta d \log n + \frac{k}{n} + C_\delta n \sqrt{kd \log n} \right\}.$$

Comparing this to the definition of $r(\lambda)$, we see that choosing $\alpha = 1$ would entail $k \geq 2^d$, which is undesirable. We can get a more reasonable setting of $k \sim d \log n$ by choosing $\alpha = 2$, but we’d like $\alpha$ to be as small as possible. A more refined argument shows that $\alpha \approx \sqrt{2}$ is enough.

Theorem IV.7. Assume $E_0$. Let $A$ be a connected set in $\mathcal{X}$ with $\lambda = \inf_{x \in A_\mu} f(x)$. Suppose $\alpha > \sqrt{2}$. Then $A \cap X_n$ is connected in $G_r$ whenever $r \leq \max\{2, \sigma/\alpha\}$ and

$$v_{d^r} d \lambda \geq \max \left\{ \frac{4C_\delta d \log n}{n} + \frac{k}{n} + C_\delta n \sqrt{kd \log n} \right\}.$$

Proof. Recall that a half-ball is the intersection of an open ball and a half-space through the center of the ball. Formally, it is defined by a center $\mu$, a radius $r$, and a unit direction $u$:

$$\{z \in \mathbb{R}^d : \|z - \mu\| < r, (z - \mu) \cdot u > 0\}.$$

We will describe any such set as “the half-ball of $B(\mu, r)$ in direction $u$”. If the half-ball lies entirely in $A_\mu$, its probability mass is at least $(1/2)v_{d^r} d \lambda$. By uniform convergence bounds (Lemma IV.1), if $v_{d^r} d \lambda \geq (4C_\delta d \log n)/n$, then every such half-ball within $A_\mu$ contains at least one data point.

Pick any $x, x' \in A \cap X_n$; there is a path $P$ in $A$ with $x \sim r x'$. We’ll identify a sequence of data points $x_0 = x, x_1, x_2, \ldots$, ending in $x'$, such that for every $i$, point $x_i$ is active in $G_r$ and $\|x_i - x_i+1\| \leq \alpha r$. This will confirm that $x$ is connected to $x'$ in $G_r$.

To begin with, recall that $P$ is a continuous function from $[0, 1]$ into $A$. For any point $y \in \mathcal{Y}$, define $N(y)$ to be the portion of $[0, 1]$ whose image under $P$ lies in $B(y, r)$; that is, $N(y) = \{0 \leq z \leq 1 : P(z) \in B(y, r)\}$. If $y$ is within distance $r$ of $P$, then $N(y)$ is nonempty. Define $\pi(y) = P(\sup N(y))$, the furthest point along the path within distance $r$ of $y$ (Figure 5, left).

The sequence $\{x_i\}$ is defined iteratively; $x_0 = x$, and for $i = 0, 1, 2, \ldots$:

- If $\|x_i - x\| \leq \alpha r$, set $x_{i+1} = x'$ and stop.
- By construction, $x_i$ is within distance $r$ of path $P$ and hence $N(x_i) \neq \emptyset$.

- Let $B$ be the open ball of radius $r$ around $\pi(x_i)$. The half of $B$ in direction $x_i - \pi(x_i)$ contains a data point; this is $x_{i+1}$ (Figure 5, right).

The process eventually stops since each $\pi(x_{i+1})$ is further along path $P$ than $\pi(x_i)$; formally, $\sup N(x_{i+1}) > \sup N(x_i)$. This is because $\|x_{i+1} - \pi(x_i)\| < r$, so by continuity of the function $P$, there are points further along $P$ (beyond $\pi(x_i)$) whose distance to $x_{i+1}$ is still less than $r$. Thus $x_{i+1}$ is distinct from $x_0, x_1, \ldots, x_i$. Since there are finitely many data points, the process must terminate, so the sequence $\{x_i\}$ constitutes a path from $x$ to $x'$.

Each $x_i$ lies in $A_r \subseteq A_{r-r}$ and is thus active in $G_r$ under event $E_0$ (Lemma IV.2). Finally, the distance between successive points is

$$\begin{align*}
\|x_i - x_{i+1}\|^2 &= \|x_i - \pi(x_i) + \pi(x_i) - x_{i+1}\|^2 \\
&= \|x_i - \pi(x_i)\|^2 + \|\pi(x_i) - x_{i+1}\|^2 - 2(x_i - \pi(x_i)) \cdot (x_{i+1} - \pi(x_i)) \\
&\leq 2r^2 \leq \alpha^2 r^2,
\end{align*}$$

where the second-last inequality is from the definition of half-ball.

To complete the proof of Theorem III.3, take $k \geq 4C_\delta^2 (d/\epsilon^2) \log n$. The relationship that defines $r = r(\lambda)$ (Definition IV.4) then implies

$$\frac{k}{n} \leq v_{d^r} d \lambda \leq \frac{k}{n} \left( 1 + \frac{\epsilon}{2} \right).$$

This shows that clusters at density level $\lambda$ emerge when the growing radius $r$ of the cluster tree algorithm reaches roughly $(k/(\lambda v_{d^r} n))^{1/d}$. In order for $(\sigma, \epsilon)$-separated clusters to be distinguished, the one additional requirement of Lemma IV.3 and Theorem IV.7 is that $r = r(\lambda)$ be at most $\sigma/2$; this is what yields the final lower bound on $n$.

**V. ANALYSIS OF ALGORITHM 2**

The second cluster tree estimator (Figure 4), based on the $k$-nearest neighbor graph of the data points, satisfies the same guarantees as the first, under a more generous setting of $k$.

Let $G_r^{NN}$ be the $k$-NN graph at radius $r$. We have already observed that $G_r^{NN}$ has the same vertices as $G_r$, and a subset of its edges. Therefore, if clusters are separated in $G_r$, they are certainly separated in $G_r^{NN}$: the separation properties of Lemma IV.3 carry over immediately to the new estimator. What remains is to establish a connectedness property, an
analogue of Theorem IV.7, for these potentially much sparser graphs.

A. Connectivity properties

As before, let \( f \) be a density on \( \mathcal{X} \subset \mathbb{R}^d \). Let \( \Lambda = \sup_{x \in \mathcal{X}} f(x) \); then the smallest radius we expect to be dealing with is roughly \( (k/(nv_d \Lambda))^{1/d} \). To be safe, let’s pick a value slightly smaller than this, and define \( r_o = (k/(2nv_d \Lambda))^{1/d} \).

We’ll first confirm that \( r_o \) is, indeed, a lower bound on the radii \( r_k(\cdot) \).

**Lemma V.1.** Assume \( E_o \). If \( k \geq 4C_3^2 d \log n \), then \( r_k(x) > r_o \) for all \( x \).

**Proof.** Pick any \( x \) and consider the ball \( B(x, r_o) \). By definition of \( r_o \),

\[
f(B(x, r_o)) \leq v_d r_o^d \Lambda = \frac{k}{2n} \leq \frac{k}{n} - \frac{C_3}{n} \sqrt{kd \log n}
\]

where the last inequality is from the condition on \( k \). Under \( E_o \) (Lemma IV.1), we then get \( f_n(B(x, r_o)) < k/n \); therefore \( r_k(x) > r_o \).

Now we present an analogue of Theorem IV.7.

**Theorem V.2.** Assume \( E_o \). Let \( A \) be a connected set in \( \mathcal{X} \), with \( \lambda = \inf_{x \in A} f(x) \). Suppose \( \alpha \geq \sqrt{2} \). Then \( A \cap X_n \) is connected in \( G_r^{\text{NN}} \) whenever \( r + r_o \leq \sigma \) and

\[
v_d r_o^d \Lambda \geq \frac{k}{n} - \frac{C_3}{n} \sqrt{kd \log n}
\]

and

\[
k \geq \max \left\{ \frac{\lambda}{\Lambda} \cdot 8C_3 d \log n, 4C_3^2 d \log n \right\}.
\]

**Proof.** We’ll consider events at two different scales: a small radius \( r_o \), and the potentially larger radius \( r \) from the theorem statement.

Let’s start with the small scale. The lower bound on \( k \) yields

\[
v_d r_o^d \Lambda = \frac{k}{2n} \Lambda \geq \frac{4C_3 d \log n}{n}.
\]

As in the proof of Theorem IV.7, this implies that every half-ball of radius \( r_o \) within \( A_o \) contains at least one data point.

Let \( x \) and \( x' \) be any two points in \( A \cap X_n \). As in Theorem IV.7, we can find a finite sequence of data points \( x = x_0, x_1, \ldots, x_p = x' \) such that for each \( i \), two key conditions hold: (i) \( ||x_i - x_{i+1}|| \leq \alpha r_o \) and (ii) \( x_i \) lies within distance \( r_o \) of \( A \).

Now let’s move to a different scale \( r \leq \sigma - r_o \). Since each \( x_i \) lies in \( A_{r_o} \subseteq A_{\sigma - r} \), we know from Lemma IV.2 that all \( x_i \) are active in \( G_r^{\text{NN}} \). The edges \( (x_i, x_{i+1}) \) are also present, because

\[
||x_i - x_{i+1}|| \leq \alpha r_o \leq \min\{r_k(x_i), r_k(x_{i+1})\}
\]

using Lemma V.1 and the bound on \( k \). Hence \( x \) is connected to \( x' \) in \( G_r^{\text{NN}} \).

It is straightforward to check that \( r(\lambda) \) is always \( \geq r_o \), and Theorem III.4 follows immediately.

**B. A lower bound on neighborhood cardinality**

The result for \( k \)-nearest neighbor graphs requires a larger setting of \( k \) than our earlier result; in particular, \( k \) needs to exceed the ratio \( \Lambda/\lambda \). We now show that this isn’t just a looseness in our bound, but in fact a necessary condition for these types of graphs.

Recall that the mutual \( k \)-NN graph contains all the data points, and puts an edge between points \( x \) and \( x' \) if \( ||x - x'|| \leq \alpha \min\{r_{k}(x), r_{k}(x')\} \) (the \( \alpha \) is our adaptation). We will assume \( 1 \leq \alpha \leq 2 \), as is the case in all our upper bounds.

**Lemma V.3.** Pick any \( \lambda > 0 \), any \( \Lambda > 32\lambda \), and any \( k \leq \Lambda/(64\lambda) \). Then there is a density \( f \) on \( \mathcal{X} \subset \mathbb{R} \) such that \( f(x) \leq \alpha \) for all \( x \in \mathcal{X} \), and with the following property: for large enough \( n \), when \( n \) samples are drawn i.i.d. from \( f \), the resulting mutual \( k \)-NN graph (with \( 1 \leq \alpha \leq 2 \)) is disconnected with probability at least \( 1/2 \).

**Proof.** Consider the density shown in Figure 6, consisting of two dense regions, \( A \) and \( C \), bridged by a less dense region \( B \). Each region is of width \( L = 1/(\lambda + 2\Delta) \). We’ll show that the mutual \( k \)-NN graph of a sample from this distribution is likely to be disconnected. Specifically, with probability at least \( 1/2 \), there will be no edges between \( A \) and \( B \cup C \).

To this end, pick any \( n \geq \Lambda/\lambda \), and define \( \Delta = 1/(4n) \). Consider the leftmost portion of \( B \) of length \( \Delta \). The probability that a random draw from \( f \) falls in this region is \( \Delta = 1/\Lambda(4n) \). Therefore, the probability that no point falls in this region is \( (1 - 1/(4n))^n \geq 3/4 \). Call this event \( E_1 \).

Next, divide \( A \) into intervals of length \( 2\Delta, 2\Delta, 4\Delta \), and so on, starting from the right. We’ll show that with probability at least \( 3/4 \), the right half of each such interval contains at least \( k + 1 \) points; call this event \( E_2 \). To see why, let’s focus on one particular interval, say that of length \( 2\Delta \). The probability that a random point falls in the right half of this interval is \( 2^{-\Delta} \Delta \geq 2^{4+3}k/n \). Therefore, the number of points in this region is \( 2^{4+3}k \) in expectation, and by a Chernoff bound, is \( \geq k + 1 \) except with probability \( < \exp(-2^{4+3}k) \). Taking a union bound over all the intervals yields an overall failure probability of at most \( 1/4 \).
With probability at least 1/2, events $E_1$ and $E_2$ both occur. Whereupon, for any point in $A$, its nearest neighbor in $B \cup C$ is at least twice as far as its $k$ nearest neighbors in $A$. Thus the mutual $k$-NN graph has no edges between $A$ and $B \cup C$. □

This constraint on $k$ is unpleasant, and it would be interesting to either find mild smoothness assumptions on $f$, or better, modified notions of $k$-NN graph, that render it unnecessary.

VI. LOWER BOUND

We have shown that the two cluster tree algorithms distinguish pairs of clusters that are $(\sigma, \epsilon)$-separated. The number of samples required to capture clusters at density $\geq \lambda$ is, by Theorem III.3,

$$O\left(\frac{d}{v_d(\sigma/2)^2\lambda \epsilon^2} \log \frac{d}{v_d(\sigma/2)^2\lambda \epsilon^2}\right).$$

We’ll now show that this dependence on $\sigma, \lambda$, and $\epsilon$ is optimal. The only room for improvement, therefore, is in constants involving $d$.

**Theorem VI.1.** Pick any $0 < \epsilon < 1/2$, any $d > 1$, and any $\sigma, \lambda > 0$ such that $\lambda v_d(1-\sigma)^d < 1/120$. Then there exist: an input space $X \subset \mathbb{R}^d$; a finite family of densities $F = \{f_i\}$ on $X$; subsets $A_i, A'_i, S_i \subset X$ such that $A_i$ and $A'_i$ are $(\sigma, \epsilon)$-separated by $S_i$ for density $f_i$ and $\inf_{x \in A_i \cap A'_i} f_i(x) \geq \lambda$, with the following additional property.

Consider any algorithm that is given $n \geq 100$ i.i.d. samples $X_n$ from some $f_i \in F$ and, with probability at least 3/4, outputs a tree in which the smallest cluster containing $A_i \cap X_n$ is disjoint from the smallest cluster containing $A'_i \cap X_n$. Then

$$n \geq \frac{C_2}{v_d\sigma^d\lambda \epsilon^2 d^{1/2}} \log \frac{1}{v_d\sigma^d\lambda \epsilon^2 d^{1/2}}$$

for some absolute constant $C_2$.

**Proof.** Given the parameters $d, \sigma, \epsilon, \lambda$, we will construct a space $X$ and a finite family of densities $F = \{f_i\}$ on $X$. We will then argue that any cluster tree algorithm that is able to distinguish $(\sigma, \epsilon)$-clusters must be able, when given samples from some $f_i$, to determine the identity of $i$. The sample complexity of this latter task can be lower-bounded using Fano’s inequality (Appendix E): it is $\Omega((\log |F|)/\theta)$, for

$$\theta = \max_{i \neq j} K(f_i, f_j),$$

where $K(\cdot, \cdot)$ is Kullback-Leibler divergence.

The support $X$. The support $X$ is made up of two disjoint regions: a cylinder $X_0$, and an additional region $X_1$ which serves as a repository for excess probability mass. $X_1$ can be chosen as any Borel set disjoint from $X_0$. The main region of interest is $X_0$ and is described as follows in terms of a constant $c > 1$ to be specified. Pick $\tau > 1$ so that $\tau^{d-1} \leq 2$ and $(\tau - 1)\sigma \leq (2v_d\lambda)^{-1/d}$. Let $B_{d-1}$ be the unit ball in $\mathbb{R}^{d-1}$, and let $\tau \sigma B_{d-1}$ be this same ball scaled to have radius $\tau \sigma$. The cylinder $X_0$ stretches along the $x_1$-axis; its cross-section is $\tau \sigma B_{d-1}$ and its length is $4(c+1)\sigma$ for some $c > 1$ to be specified: $X_0 = [0, 4(c+1)\sigma] \times \tau \sigma B_{d-1}$. Here is a picture of it:

A family of densities on $X$. The family $F$ contains $c-1$ densities $f_1, \ldots, f_{c-1}$ which coincide on most of the support $X$ and differ on parts of $X_0$. Each density $f_i$ is piecewise constant as described below. Items (ii) and (iv) describe the pieces that are common to all densities in $F$.

(i) Density $\lambda(1-\epsilon)$ on $(4\sigma i + \sigma, 4\sigma i + 3\sigma) \times \tau \sigma B_{d-1}$.

(ii) Balls of mass $1/(2\epsilon c)$ centered at locations $4\sigma, 8\sigma, \ldots, 4\sigma c$ along the $x_1$-axis: each such ball is of radius $(\tau - 1)\sigma$, and the density on these balls is $1/(2v_d(\tau - 1)\sigma^d) \geq \lambda$. We refer to these as mass balls.

(iii) Density $\lambda$ on the remainder of $X_0$: this is the union of the cylinder segments $[0, 4\sigma i + \sigma] \times \tau \sigma B_{d-1}$ and $[4\sigma i + \sigma, 4\sigma i + 3\sigma, 4(c+1)\sigma] \times \tau \sigma B_{d-1}$ minus the mass balls. Since the cross-sectional area of the cylinder is $v_d(\tau - 1)\sigma^{d-1}$, the total mass here is at most $\lambda\tau^{d-1}v_d(\tau - 1)(4(c+1) - 2) \leq 8\lambda v_d - \sigma^{d}(c+1)$.

(iv) The remaining mass is at least $1/2 - 8\lambda v_d - \sigma^{d}(c+1)$; we will be careful to choose $c$ so that this is nonnegative. This mass is placed on $X_1$ in some fixed manner that does not vary between densities in $F$.

Here is a sketch of $f_i$. The low-density region of width $2\sigma$ is centered at $4\sigma i + 2\sigma$ on the $x_1$-axis, and contains no mass balls.

For any $i \neq j$, the densities $f_i$ and $f_j$ differ only on the cylindrical sections $(4\sigma i + \sigma, 4\sigma i + 3\sigma) \times \tau \sigma B_{d-1}$ and $(4\sigma j + \sigma, 4\sigma j + 3\sigma) \times \tau \sigma B_{d-1}$, which are disjoint, contain no mass ball, and each have volume $2\tau^{d-1}v_d - \sigma^d$. Thus

$$K(f_i, f_j) = 2\tau^{d-1}v_d - \sigma^d \left(\log \frac{\lambda}{\lambda(1-\epsilon)} + \lambda(1-\epsilon) \log \frac{\lambda(1-\epsilon)}{\lambda}\right)$$

$$= 2\tau^{d-1}v_d - \sigma^d \lambda(-\log(1-\epsilon)) \leq 8\ln 2 v_d - \sigma^d \lambda \epsilon^2$$

(using $\ln(1-\epsilon) \geq -2\epsilon$ for $0 < \epsilon \leq 1/2$). This is an upper bound on the $\theta$ in the Fano bound.

**Clusters and separators.** Now define the clusters and separators as follows: for each $1 \leq i \leq c-1$,

- $A_i$ is the tubular segment $[\sigma, 4\sigma i] \times (\tau - 1)\sigma B_{d-1}$.
- $A'_i$ is the tubular segment $[4\sigma(i+1), 4(e+1)\sigma - \sigma] \times (\tau - 1)\sigma B_{d-1}$, and
- $S_i = \{4\sigma i + 2\sigma\} \times \tau \sigma B_{d-1}$ is the cross-section of the cylinder at location $4\sigma i + 2\sigma$. 
Thus $A_i$ and $A_i'$ are $d$-dimensional sets while $S_i$ is a $(d-1)$-dimensional set. It can be seen that, for density $f_i$, $A_i$ and $A_i'$ are $(\sigma, \epsilon)$-separated, and $\inf_{x \in A_i \cup A_i'} f_i(x) \geq \lambda$.

Now that the various structures are defined, we still need to argue that if an algorithm is given a sample $X_n$ from some $f_i$ (where $i$ is unknown), and is able to separate $A_i \cap X_n$ from $A_i' \cap X_n$, then it can effectively infer the identity of $i$. This has sample complexity $\Omega((\log c)/\theta)$.

Let's set $c$ to be a small constant, say $c = 6$. Then, even a small sample $X_n$ of $n \geq 100$ points is likely (with probability at least $3/4$, say), to contain points from all of the $c$ mass balls, each of which has mass $1/(2c)$. Suppose the algorithm even knows in advance that the underlying density is one of the $c-1$ choices in $F$, and is subsequently able (with probability at least $3/4$) to separate $A_i$ from $A_i'$. To do this, it must connect all the points from mass balls within $A_i$, and all the points from mass balls within $A_i'$, and yet keep these two groups apart. In short, this algorithm must be able to determine (with overall probability at least $1/2$) the segment $(4\sigma + 4\sigma i + 3\sigma)$ of lower density, and hence the identity of $i$.

We can thus apply Fano’s inequality to conclude that we need

$$n > \frac{1}{\theta} \log(c-1) - 1 \geq \frac{1}{8\epsilon^3 \lambda^2} \ln 2 \geq \frac{C_2}{\min_{x \in f_i} f_i(x)}$$

for some absolute constant $C_2$. The last equality comes from the formula $v_d = \pi^{d/2}/\Gamma((d/2) + 1)$, whereupon $v_d = O(v_d^{1/2})$.

This is almost the bound in the theorem statement, short a logarithmic term. To finish up, we now switch to a larger value of $c$:

$$c = \left\lceil \frac{1}{16v_{d-1} \sigma^d \lambda} - 1 \right\rceil,$$

and apply the same construction. We have already established that we need $n = \Omega(c/\epsilon^2)$ samples, so assume $n$ is at least this large. Then, for small enough $\epsilon$, it is very likely that when the underlying density is $f_i$, the sample $X_n$ will contain the four point masses at $4\sigma$, $4\sigma i$, $4\sigma(i+1)$, and $4(c+1)\sigma$. Therefore, the clustering algorithm must connect the point at $4\sigma$ to that at $4\sigma i$ and the point at $4\sigma(i+1)$ to that at $4(c+1)\sigma$, while keeping the two groups apart. Therefore, this algorithm can determine $i$. Applying Fano’s inequality gives $n = \Omega((\log c)/\theta)$, which is the bound in the theorem statement.

\[\Box\]

VII. PRUNING

Hartigan’s notion of consistency (Definition II.3) requires distinct clusters to be distinguished, but does not guard against fragmentation within a cluster. Consider, for instance, the density shown in Figure 6. Under Hartigan-consistency, in the limit, the cluster tree must include a cluster that contains all of $A$ and a separate, disjoint cluster that contains all of $C$. But the tree is allowed to break $A$ into further subregions. To be concrete, suppose we draw a sample from that density and receive four points from each of $A$ and $C$. Figure 7, left, shows a possible cluster tree on these samples that meets the consistency requirement. However, we’d prefer the one on the right. Formally we want to avoid or remove false clusters as defined below.

**Definition VII.1.** Let $A_n$ and $A_n'$ be the vertices of two separate connected components (potentially at different levels) in the cluster tree returned by an algorithm. We call $A_n$ and $A_n'$ false clusters if they are part of the same connected component of the level set $\{x : f(x) \geq \min_{x' \in A_n \cup A_n'} f(x')\}$.

This problem is generally addressed in the literature by making assumptions about the size of true clusters. Real clusters are assumed to be large in some sense, for instance in terms of their mass [4], or excess mass [19]. However, relying on size can be misleading in practice, as is illustrated in Figure 8. It turns out that, building on the results of the previous sections, there is a simple way to treat spurious clusters independent of their size.

\[1\]The excess mass of a component $A$ at level $\lambda$ is generally defined as $\int_A (f(x) - \lambda) \, dx$. 
greater. Looking at a nearby level will require a bit more separation between two sets \(A, B\). Separation procedures other than the ones discussed here. The main connected and thus detect the situation.

The above intuition is likely to extend to cluster tree procedures other than the ones discussed above. We now need to show that \(A \cap X_n\) connected at some nearby level in the tree.

A. Intuition

The pruning procedure of Figure 9 consists of a simple lookup: it reconnects components at level \(r\) if they are part of the same connected component at some level \(r' > r\), where \(r'\) is a function of a tuning parameter \(\tilde{c} \geq 0\). The larger \(\tilde{c}\) is, the more aggressive the pruning.

The pruning procedure builds upon the same intuition as for the procedure of [17]. However, it differs in its ability to handle either of the two cluster tree algorithms, and moreover works under significantly milder conditions than those of [17]. The intuition is the following. Suppose \(A_n, A_n' \subset \mathbb{X}_n\) are not connected at some level \(r\) in the empirical tree (before pruning), but ought to be: they belong to the same connected component \(A \in \mathcal{C}(\lambda)\), where \(\lambda = \min \{f(x) : x \in A_n \cup A_n'\}\). Then, key points from \(A\) that would have connected them are missing at level \(r\) in the empirical tree (Figure 8). These points have \(r_k(x)\) greater than \(r\), but probably not much greater. Looking at a nearby level \(r' > r\), we will find \(A_n, A_n'\) connected and thus detect the situation.

The above intuition is likely to extend to cluster tree procedures other than the ones discussed here. The main requirement on the cluster tree estimate is that points in \(A\) (as discussed above) be connected at some nearby level in the tree.

B. Separation

The pruning procedure increases connectivity, but we must make sure that it isn’t too zealous in doing so: clusters that are sufficiently separated should not be merged. We now will require a bit more separation between two sets \(A\) and \(A'\) in order to keep them apart in the empirical tree. As might be expected, how much more separation depends on the pruning parameter \(\tilde{c}\). The higher \(\tilde{c}\), the more aggressive the pruning, and the greater the separation requirement for detecting distinct clusters. The following lemma builds on Corollary IV.5.

Lemma VII.2. Assume \(E_o\). Consider two sets \(A, A' \subset \mathbb{X}\), and let \(\lambda = \inf_{x \in A_n \cup A_n'} f(x)\). Suppose there exists a separator set \(S\) such that

- Any path in \(\mathbb{X}\) from \(A\) to \(A'\) intersects \(S\).
- \(\inf_{x \in S_p} f(x) < (1 - 2\tilde{c})\lambda - \tilde{c}\).

Then \(A \cap X_n\) and \(A' \cap X_n\) are in separate connected components of \(\mathbb{C}_n(r(\lambda))\) after pruning, provided

\[
v_d(2\sigma/(\alpha + 2))^d((1 - \epsilon)\lambda - \tilde{c}) > \frac{k}{n} + \frac{C_\delta}{n} \sqrt{kd \log n}.
\]

Proof. Let \(r\) denote \(r(\lambda)\) and recall from the definitions of \(r(\lambda)\) and \(\lambda_r\) (Figure 9) that

\[
\lambda_r = \lambda \left(\frac{k}{n} + \frac{C_\delta}{n} \sqrt{kd \log n}\right)^{-1} - \tilde{c} \\
\geq \left(1 - 2\frac{C_\delta}{\sqrt{\lambda_r} \sqrt{k}} \sqrt{kd \log n}\right) \lambda - \tilde{c} \geq (1 - \epsilon)\lambda - \tilde{c}.
\]

The final term, call it \(\lambda'\), is \(\geq 0\) by the hypotheses of the lemma. Since \(\lambda_r \geq \lambda'\) we have \(r(\lambda') \geq r(\lambda_r)\). Thus we just have to show that \(A \cap X_n\) and \(A' \cap X_n\) are in separate connected components of \(\mathbb{C}_n(r(\lambda'))\). To this end, notice that, under our assumptions on \(A\) and \(A'\), these two sets belong to separate connected components of \(\{x \in \mathbb{X} : f(x) \geq \lambda'\}\); in fact

\[
\inf_{x \in \mathbb{R}_+} f(x) \leq (1 - 2\epsilon)\lambda - \tilde{c} \leq (1 - \epsilon)\lambda'.
\]

Moreover, the final requirement of the lemma statement can be rewritten as \(r(\lambda') < 2\sigma/(\alpha + 2)\). The argument of Lemma IV.8(c) then implies that \(A \cap X_n\) is disconnected from \(A' \cap X_n\) in \(\mathbb{C}_n(r(\lambda'))\) and thus in \(\mathbb{C}_n(r(\lambda_r))\), and hence also at level \(r\) after pruning.

C. Connectedness

We now turn to the main result of this section, namely that the pruning procedure reconnects incorrectly fragmented clusters. Recall the intuition detailed above. We first have to argue that points with similar density make their first appearance at nearby levels \(r\) of the empirical tree. From the analysis of the previous sections, we know that a point \(x\) is present at level \(r(f(x))\), roughly speaking. We now need to show that it cannot appear at a level too much smaller than this.

These assertions about single points are true only if the density doesn’t vary too dramatically in their vicinity. In what follows, we will quantify the smoothness at scale \(\sigma\) by the constant

\[
L_\sigma = \sup_{\|x - x'\| \leq \sigma} |f(x) - f(x')|.
\]

Lemma VII.3. Assume \(E_o\). Pick any \(x\) and let \(f_\sigma(x) = \inf_{x' \in B(x, \sigma)} f(x')\). Suppose

\[
v_d(\sigma/2)^d(f_\sigma(x) + L_\sigma) \geq \frac{k}{n} - \frac{C_\delta}{n} \sqrt{kd \log n},
\]

we then have

\[
v_d(\sigma/2)^d(f_\sigma(x) + L_\sigma) \geq \frac{k}{n} - \frac{C_\delta}{n} \sqrt{kd \log n}.
\]

Proof. Consider any \(r\) such that

\[
v_d(\sigma/2)^d(f_\sigma(x) + L_\sigma) \leq \frac{k}{n} - \frac{C_\delta}{n} \sqrt{kd \log n}.
\]

Thus we just

\[
\inf_{x \in \mathbb{R}_+} f(x) \leq (1 - 2\epsilon)\lambda - \tilde{c} \leq (1 - \epsilon)\lambda'.
\]

Moreover, the final requirement of the lemma statement can be rewritten as \(r(\lambda') < 2\sigma/(\alpha + 2)\). The argument of Lemma IV.8(c) then implies that \(A \cap X_n\) is disconnected from \(A' \cap X_n\) in \(\mathbb{C}_n(r(\lambda'))\) and thus in \(\mathbb{C}_n(r(\lambda_r))\), and hence also at level \(r\) after pruning.

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Proof. Consider any \(r\) such that

\[
v_d(\sigma/2)^d(f_\sigma(x) + L_\sigma) \leq \frac{k}{n} - \frac{C_\delta}{n} \sqrt{kd \log n}.
\]
Pruning of level $r$.

- Set
  \[ \lambda_r = \frac{1}{v_d r^d} \left( \frac{k}{n} - \frac{C_{\delta}}{n} \sqrt{kd \log n} \right) - \hat{c}. \]
- Connect any two components of $C_n(r)$ that belong to the same connected component in $C_n(r(\max(\lambda_r, 0)))$.

Algorithm 1 or 2 is run on a sample $X_n$ of size $n$ drawn from $f$, with settings

\[ \sqrt{\frac{2}{\alpha}} = 2 \quad \text{and} \quad k \geq C \cdot \frac{d \log n}{\epsilon^2} \cdot \log^2 \frac{1}{\delta}, \]

followed by the pruning procedure with parameter $\hat{c}$.

Then the following holds with probability at least $1 - \delta$.

Define

\[ \lambda_0 = \frac{k}{v_d (\sigma/2)^d} \cdot \frac{1 + \epsilon}{1 - \epsilon} + \frac{\hat{c}}{1 - \epsilon}. \]

or in the case of Algorithm 2, the maximum of this quantity and $(A/k)(\log n \cdot \log(1/\delta))$, where $A = \sup_{x \in \mathcal{X}} f(x)$.

Then any $\lambda \leq \hat{c}$.

The first part of the above theorem (recovery of true clusters) implies that the pruned tree remains a consistent estimator of the cluster tree, under the same asymptotic conditions as those for Theorem III.3 and Theorem III.4, and the additional condition that $\hat{c} \to 0$.

The second part of the theorem states some general conditions on $\lambda$ and $\lambda_0$ under which false clusters are removed. To better understand these conditions, let’s consider the simple case when $f$ is Hölder-smooth: that is, there exist constants $L, \beta > 0$ such that for all $x, x' \in \mathcal{X}$,

\[ |f(x) - f(x')| \leq L \|x - x'\|^\beta. \]

Consider

\[ \sigma = \frac{\hat{c}}{L^{1/\beta}} \quad \text{so that we have} \quad \sup_{\|x - x'\| \leq \hat{c}} |f(x) - f(x')| \leq \hat{c}. \]

Then any $\lambda > 4\hat{c}$.

The separation and connectedness results of this section can now be combined into the following theorem.

Theorem VII.5. There is an absolute constant $C$ such that the following holds. Pick any $0 < \delta, \epsilon < 1$ and $\hat{c} > 0$. Assume

Then $r \leq \sigma/2$, implying $f(B(x, r)) \leq v_d r^d (f(x) + L_\sigma)$. Using the first inequality and Lemma IV.1, we have $f_n(B(x, r)) < k/n$, that is $r < r_\lambda(x)$.

Next, by combining the above lower-bound on $r_k(x)$ with our previous results on connectedness for both types of algorithms, we obtain the following pruning guarantees.

Lemma VII.4. Assume that event $E_\lambda$ holds, and that $\hat{c} \geq L_\sigma$. Let $A_n$ and $A_n'$ denote two disconnected sets of vertices of $G_r$ or $G_r^{NN}$ after pruning, for some $r > 0$. Define $\lambda = \inf_{x \in A_n \cup A_n'} f(x)$. Then $A_n$ and $A_n'$ are disconnected in the level set $\{x \in \mathcal{X} : f(x) \geq \lambda\}$ if the following two conditions hold: first,

\[ v_d (\sigma/2)^d \lambda - L_\sigma \geq \frac{k}{n} + \frac{C_{\delta}}{n} \sqrt{kd \log n}, \]

and second,

\[ k \geq \left\{ \begin{array}{ll}
4C_{\delta}d \log n & \text{for } G_r \\
\max(4C_{\delta}^2d \log n, (A/\lambda)8C_{\delta}d \log n) & \text{for } G_r^{NN}
\end{array} \right. \]

Proof. Let $A$ be any connected component of $\{x \in \mathcal{X} : f(x) \geq \lambda\}$. We’ll show that $A \cap X_n$ is connected in $G_r$ (or $G_r^{NN}$) after pruning, from which the lemma follows immediately.

Define $\lambda_\sigma = \inf_{x \in A_n} f(x) \geq \inf_{x \in A} f(x) - L_\sigma \geq \lambda - L_\sigma$. Recall from Definition IV.4 that $r(\lambda_\sigma)$ is the value of $r$ for which $v_d r^d \lambda_\sigma = \frac{k}{n} + \frac{C_{\delta}}{n} \sqrt{kd \log n}$. The first condition in the lemma statement thus implies that $r(\lambda_\sigma) \leq \sigma/2$. The second condition, together with Theorem IV.7 or Theorem V.2, implies that $A \cap X_n$ is connected at level $r(\lambda_\sigma)$ of $G_r$ or $G_r^{NN}$.

Next we show that $r(\lambda_\sigma) \leq r(\hat{c})$, by showing that $\lambda_\sigma \geq \hat{c}$. Again by the first condition on $k$, Lemma VII.3 holds for every $x \in A$, implying with little effort that

\[ \lambda_\sigma \geq \frac{1}{v_d r^d} \left( \frac{k}{n} - \frac{C_{\delta}}{n} \sqrt{kd \log n} \right) - L_\sigma \geq \frac{1}{v_d r^d} \left( \frac{k}{n} - \frac{C_{\delta}}{n} \sqrt{kd \log n} \right) - \hat{c} = \hat{\lambda}_r. \]

Thus $A \cap X_n$ is connected at level $r(\hat{\lambda}_r) \geq r(\lambda_\sigma)$ of $G_r$ (or $G_r^{NN}$), and thus is reconnected when pruning at level $r$. □
by choosing $\bar{\epsilon}$ as a function of $k$ (e.g. $k = \Theta(\log^3 n)$ and $\bar{\epsilon} = \Theta(1/\sqrt{n})$).

Finally, remark that under the above smoothness assumption and choice of $\sigma, \epsilon$, we can further guarantee that all false clusters are removed. We only need to reconnect all components at levels where the minimum $f$ value is at most $4\bar{\epsilon}$. By Lemma VII.3, for $k$ in the above range, we have $r_k(x) \geq \left(k/10nv_0\bar{\epsilon}\right)^{1/d}$ when $f(x) \leq 4\bar{\epsilon}$. Thus, we just need to reconnect all components at levels $r > \left(k/10nv_0\bar{\epsilon}\right)^{1/d}$, and prune all other levels as discussed above. This then guarantees that all false clusters are removed with high probability, while also ensuring that the estimator remains consistent.

VIII. Final remarks

Both cluster tree algorithms are variations on standard estimators, but carefully control the neighborhood size $k$ and make use of a novel parameter $\alpha$ to allow more edges at every scale $r$. The analysis relies on $\alpha$ being at least $\sqrt{2}$, and on $k$ being at least $d\log n$. Is it possible to dispense with $\alpha$ (that is, to use $\alpha = 1$) while maintaining this setting of $k$?

There remains a discrepancy of $2^d$ between the upper and lower bounds on the sample complexity of building a hierarchical clustering that distinguishes all $(\sigma, \epsilon)$-separated clusters. Can this gap be closed, and if so, what is needed, a better analysis or a better algorithm?

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REFERENCES


APPENDIX

A. Plug-in estimation of the cluster tree

One way to build a cluster tree is to return $C_{f_n}$, where $f_n$ is a uniformly consistent density estimate.

Lemma A.1. Suppose estimator $f_n$ of density $f$ (on space $\mathcal{X}$) satisfies $\sup_{x \in \mathcal{X}} |f_n(x) - f(x)| \leq \epsilon_n$. Pick any two disjoint sets $A, A' \subset \mathcal{X}$ and define $\Xi = \inf_{x \in A, y \in A'} f(x)$ and $\xi = \sup_{x \in P, \epsilon \leq \Xi} \inf_{x \in P} f(x)$. If $\Xi - \xi > 2\epsilon_n$ then $A, A'$ lie entirely in disjoint connected components of $C_{f_n}(\Xi - \epsilon_n)$.

Proof. $A$ and $A'$ are each connected in $C_{f_n}(\Xi - \epsilon_n)$. But there is no path from $A$ to $A'$ in $C_{f_n}(\lambda)$ for $\lambda > \Xi + \epsilon_n$.

The problem, however, is that computing the level sets of $f_n$ is usually not an easy task. Hence we adopt a different approach in this paper.

B. Consistency

The following is a straightforward exercise in analysis.

Lemma A.2. Suppose density $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and is zero outside a compact subset $\mathcal{X} \subset \mathbb{R}^d$. Suppose further that for some $\lambda$, $\{x \in \mathcal{X} : f(x) \geq \lambda\}$ has finitely many connected components, among them $A \neq A'$. Then there exist $\sigma, \epsilon > 0$ such that $A, A'$ are $(\sigma, \epsilon)$-separated.

Proof. Let $A_1, A_2, \ldots, A_k$ be the connected components of $\{f \geq \lambda\}$, with $A_1 = A$ and $A_k = A_2$.

First, each $A_i$ is closed and thus compact. To see this, pick any $x \in \mathcal{X} \setminus A_i$. There must be some $x'$ on the shortest path from $x$ to $A_i$ with $f(x') < \lambda$ (otherwise $x \in A_i$). By continuity of $f$, there is some ball $B(x', r)$ on which $f < \lambda$; thus this ball doesn’t touch $A_i$. Then $B(x, r)$ doesn’t touch $A_i$.

Next, for any $i \neq j$, define $\Delta_{ij} = \inf_{x \in A_i, y \in A_j} \|x - y\|$ to be the distance between $A_i$ and $A_j$. We’ll see that $\Delta_{ij} > 0$. Specifically, define $g : A_i \times A_j \rightarrow \mathbb{R}$ by $g(a, a') = \|a - a'\|$. 

Since $g$ has compact domain, it attains its infimum for some $a \in A_i, a' \in A_j$. Thus $\Delta_{ij} = \|a - a'\| > 0$.

Let $\Delta = \min_{i\neq j} \Delta_{ij} > 0$, and define $S$ to be the set of points at distance exactly $\Delta/2$ from $A$: $S = \{x \in X : \inf_{y \in A} \|x - y\| = \Delta/2\}$. $S$ separates $A$ from $A'$. Moreover, it is closed by continuity of $\| \cdot \|$, and hence is compact. Define $\lambda_0 = \sup_{x \in S} f(x)$. Since $S$ is compact, $f$ (restricted to $S$) is maximized at some $x_0 \in S$. Then $\lambda_0 = f(x_0) < \lambda$.

To finish up, set $\delta = (\lambda - \lambda_0)/3 > 0$. By uniform continuity of $f$, there is some $\sigma > 0$ such that $f$ doesn’t change by more than $\delta$ on balls of radius $\sigma$. Then $f(x) \leq \lambda_0 + \delta = \lambda - 2\delta$ for $x \in S_\sigma$ and $f(x) \geq \lambda - \delta$ for $x \in A_\sigma \cup A'_\sigma$.

Thus $S$ is a $(\sigma, \delta/(\lambda - \delta))$-separator for $A, A'$.

C. Proof of Lemma IV.1

We start with a standard generalization result due to Vapnik and Chervonenkis; the following version is a paraphrase of Theorem 5.1 of [20].

**Theorem A.3.** Let $G$ be a class of functions from $X$ to $\{0, 1\}$ with VC dimension $d < \infty$, and $\mathbb{P}$ a probability distribution on $X$. Let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Suppose $n$ points are drawn independently at random from $\mathbb{P}$; let $\mathbb{E}_n$ denote expectation with respect to this sample. Then for any $\delta > 0$, with probability at least $1 - \delta$, the following holds for all $g \in G$:

\[
-\min(\beta_n \sqrt{\mathbb{E}_n g}, \beta_n^2 + \beta_n \sqrt{\mathbb{E}_n g}) \\
\leq \mathbb{E}_n g - \mathbb{E} g \\
\leq \min(\beta_n^2, \beta_n^2 \sqrt{\mathbb{E}_n g}, \beta_n \sqrt{\mathbb{E}_n g}),
\]

where $\beta_n = \sqrt{(4/n)(d \log n + \log(8/\delta))}$.

By applying this bound to the class $G$ of indicator functions over balls (or half-balls), we get the following:

**Lemma A.4.** Suppose $X_n$ is a sample of $n$ points drawn independently at random from a distribution $f$ over $X$. For any set $Y \subset X$, define $f_n(Y) = |X_n \cap Y|/n$. There is a universal constant $C_\sigma > 0$ such that for any $\delta > 0$, with probability at least $1 - \delta$, for any ball (or half-ball) $B \subset \mathbb{R}^d$:

- $f(B) \geq \frac{C_\sigma}{n} (d \log n + \log \frac{1}{\delta})$ implies $f_n(B) > 0$.
- $f(B) \geq \frac{C_\sigma}{n} (d \log n + \log \frac{1}{\delta} + \sqrt{k(d \log n + \log \frac{1}{\delta})})$ implies $f_n(B) \geq k/n$.
- $f(B) < \frac{C_\sigma}{n} (d \log n + \log \frac{1}{\delta} + \sqrt{k(d \log n + \log \frac{1}{\delta})})$ implies $f_n(B) < k/n$.

**Proof.** The VC dimension of balls in $\mathbb{R}^d$ is $d + 1$, while that of half-balls (each the intersection of a ball and a halfspace) is $O(d)$. The following statements apply to either class.

The bound $f(B) - f_n(B) \leq \beta_n \sqrt{f(B)}$ from Theorem A.3 yields $f(B) > \beta_n^2 \implies f_n(B) > 0$. For the second bound, we use $f(B) - f_n(B) \leq \beta_n^2 + \beta_n \sqrt{f_n(B)}$. It follows that

\[
f(B) \geq \frac{k}{n} + \beta_n^2 + \beta_n \sqrt{\frac{k}{n}} \implies f_n(B) \geq \frac{k}{n}.
\]

For the last bound, we rearrange $f(B) - f_n(B) \geq -(\beta_n^2 + \beta_n \sqrt{f(B)})$ to get

\[
f(B) < \frac{k}{n} - \beta_n^2 - \beta_n \sqrt{\frac{k}{n}} \implies f_n(B) < \frac{k}{n}.
\]

Lemma IV.1 now follows immediately, by taking $k \geq d \log n$. Since the uniform convergence bounds have error bars of magnitude $(d \log n)/n$, it doesn’t make sense, when using them, to take $k$ any smaller than this.

D. Proof of Lemma IV.6

Consider any $x, x' \in A \cap X_n$. Since $A$ is connected, there is a path $P$ in $A$ with $x P_x' x'$. Fix any $0 < \gamma < 1$. Because the density of $A_\sigma$ is lower bounded away from zero, it follows by a volume and packing-covering argument that $A$, and thus $P$, can be covered by a finite number of balls of diameter $\gamma r$. Thus we can choose finitely many points $z_1, z_2, \ldots, z_k \in P$ such that $x = z_0, x' = z_k$ and $\|z_i - z_j\| \leq \gamma r$.

Under $E_n$ (Lemma IV.1), any ball centered in $A$ with radius $(\alpha - \gamma)r/2$ contains at least one data point if

\[
v_d \left((\alpha - \gamma)r \right)^d \lambda \geq C_d d \log n \left(\frac{2}{n}\right).
\]

(A.1)

Assume for the moment that this holds. Then, every ball $B(z_i, (\alpha - \gamma)r/2)$ contains at least one point; call it $x_i$.

By the upper bound on $r$, each such $x_i$ lies in $A_\tau$; therefore, by Lemma IV.1, the $x_i$ are all active in $G_r$. Moreover, consecutive points $x_i$ are close together:

\[
\|x_{i+1} - x_i\| \leq \|x_{i+1} - z_{i+1}\| + \|z_{i+1} - z_i\| + \|z_i - x_i\| \leq \alpha r.
\]

Thus all edges $(x_i, x_{i+1})$ exist in $G_r$, whereby $x$ is connected to $x'$ in $G_r$.

All this assumes that equation (A.1) holds for some $\gamma > 0$. Taking $\gamma \to 0$ gives the lemma.

E. Fano’s inequality

Consider the following game played with a predefined, finite class of distributions $F = \{f_1, \ldots, f_\ell\}$, defined on a common space $A$:

- Nature picks $I \in \{1, 2, \ldots, \ell\}$.
- Player is given $n$ i.i.d. samples $X_1, \ldots, X_n$ from $f_i$.
- Player then guesses the identity of $I$.

Fano’s inequality [21], [22] gives a lower bound on the number of samples $n$ needed to achieve a certain success probability. It depends on how similar the distributions $f_i$ are: the more similar, the more samples are needed. Define $\theta = \frac{1}{\ell} \sum_{i,j=1}^\ell K(f_i, f_j)$ where $K(\cdot)$ is KL divergence. Then $n$ needs to be $\Omega((\log \ell)/\theta)$. Here’s the formal statement.

**Theorem A.5 (Fano).** Let $g : A^n \to \{1, 2, \ldots, \ell\}$ denote Player’s computation. If Nature chooses $I$ uniformly at random from $\{1, 2, \ldots, \ell\}$, then for any $0 < \delta < 1$,

\[
n \leq \frac{(1 - \delta)(\log_2 \ell) - 1}{\theta} \implies \Pr(g(X_1, \ldots, X_n) \neq I) \geq \delta.
\]