Contracts: The Theory of Dynamic Principal-Agent Relationships and the Continuous-Time Approach

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1. Introduction

Agency conflicts arise in many situations: between shareholders and firm management, between customers and service providers, between the electorate and politicians, etc. Early groundbreaking work on agency problems, such as Mirrlees (1976), Holmstrom (1979) has focused on static models. The basic conclusions of static models are about designing the best measures of performance, and the optimal amount of “skin in the game” that the agent should have to resolve agency conflicts optimally. For example, Jensen and Meckling (1976) consider a setting where a manager can take a project with negative NPV = -A, but which generates private benefits, “perks,” of B. The manager will not take the project if the fraction \( \alpha \) of the firm he owns satisfies

\[
B - \alpha A \leq 0 \iff \alpha \geq B/A.
\]

Stylized static models can be, nevertheless, limiting, in that they miss the effects that arise only in dynamic models, and also they cannot be calibrated to our dynamic world.

Jensen and Murphy (1990) document empirically that dynamic effects are important. They measure the sensitivities of both CEO’s flow compensation and CEO’s total wealth to firm value. Flow compensation, which includes salary, bonuses, and newly granted stock and options, measures static effects. Existing stock ownership and previously granted options measure dynamic effects. They find that on average, the CEO’s wealth goes up by $3.25 for each $1000 added to shareholder value, and of this amount $2.65 is due to existing stock and options.

There are also many theoretical reasons why dynamic effects are important. Lazear (1979) argues that commitment to increasing wages can help organizations retain workers and maintain long-term relationships between firms and workers. Radner (1985) shows that as the agent becomes more patient, it is possible to attain

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efficiency in long-term relationships by aggregating outcomes across many periods and forming more precise statistics about the agent’s effort.² Holmstrom and Milgrom (1987) develop a model, in which the optimal contract linearly aggregates output across periods to determine the agent’s pay. More generally, Fudenberg, Holmstrom and Milgrom (1990) show that in many settings, it is possible to allow the agent’s wealth to summarize information about the agent’s past performance in such a way that the optimal contract is implemented. This insight suggests a link between financing constraints and optimal contracts. In this case, the amount of financial slack that a firm has, cash plus its additional borrowing capacity, summarizes the past performance of the firm’s management team. These applications typically model the firm’s management team as one agent.

The characterization of optimal contracts in dynamic single-agent settings is a challenging problem. The space of contracts is complex - the agent’s compensation can be a function of the entire history of performance. Moreover, the problem consists of one dynamic optimization problem embedded in another. The principal is maximizing his objective while recognizing that the agent is also searching for the best dynamic effort strategy within the contract. However, the complexity of this problem is reduced significantly using recursive methods that treat the agent’s continuation payoff as a state variable, see Spear and Srivastava (1987). Effectively, the problem of finding an optimal contract is reduced to dynamic programming.

Solving this dynamic programming problem is difficult in discrete time, although Phelan and Townsend (1991) propose an iterative method that relies on solving a large number of linear programming problems. In contrast, in continuous-time the solution can be characterized by ordinary differential equations, using optimal stochastic control. This yields the problem of finding the optimal contract significantly more tractable, and solutions, commensurately clearer. Sections 2, 3 and 4 of this article summarize the recursive simplification of the optimal dynamic contracting problem and its solution based on the continuous-time model of Sannikov (2008).³ The continuous-time approach has the advantages of tractability (which stems from the differential equation that characterizes the optimal contract), clarity (discrete-time models get messy very quickly) and computing power.

One has to realize that, from both descriptive and prescriptive points of view, solving for the optimal contract is just one method of analyzing agency problems. In practice, one has to take into account many considerations outside theoretical models. Because of that, it is even better to understand not just the optimal contract, but the whole range of contracts that are close to optimal. That is, it is

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² See Fudenberg, Levine and Maskin (1994) for an analogous result for repeated games with public monitoring.

³ There are many other versions of continuous-time agency models, including those of Sung (2005), Cvitanic, Wan and Zhang (2006 and 2009), Westerfield (2006), Williams (2007) and Adrian and Westerfield (2009). See Cvitanic and Zhang (2011) for a fantastic overview of many of these models.
important to understand contract features that are especially important in order for contracts to be close to optimal.

By understanding the range of contracts that come close to attaining maximal profit, one has the flexibility to incorporate considerations of practical importance without significant sacrifices in theoretical efficiency of contracts. This broader question is addressed in Fong and Sannikov (2010). In Section 5 we review several results that allow us to compare how far away a given contract is from the optimal one, and what contract features are particularly important for approximate optimality.4

One field where the tractability of continuous-time methods has proved particularly fruitful is corporate finance. Several papers explore how financing frictions can arise from an agency problem, and derive implications on capital structure, default, dividends and investment. Examples of these papers are DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin and Rochet (2007), Piskorski and Westerfield (2008), Biais, Mariotti, Rochet and Villeneuve (2011), DeMarzo, Fishman, He and Wang (2009), He (2009), Hoffmann and Pfeil (2009), Piskorski and Tchystyi (2010), Zhu (2011) and many others.5 These papers treat the firm’s management as one agent, and link the agent’s continuation payoff to the firm’s financing constraint.6 Section 6 reviews several corporate finance applications of dynamic agency models.

Section 7 allows for the noise in the agent’s performance measure to be partially observable, and asks when it is optimal to filter out the observable component completely when rewarding the agent. Section 8 concludes.

2. The model

This section presents a classic version of the repeated principal-agent problem. The timeline is $t \in \{1, 2, \ldots \}$ in discrete time, and $t \in [0, \infty)$ in continuous time. Before employment starts, the principal offers a state-contingent contract to the agent, which specifies payments $c_t \in [0, \infty)$ to the agent at each moment of time, as a function of the past output history. Then the agent chooses effort $a_t \in [0, \bar{a}]$. We will normalize effort, so that $a_t$ is the expected output given effort $a_t$. The agent’s utility at time $t$ is given by

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4 The question of how close to optimality one can get using simple contracts and mechanisms has been recognized widely in literature. Most relevant here is the literature on optimal dynamic taxation, which compares the efficiency of fully optimal tax policies and certain simple tax policies, such as linear. For example, see Golosov, Troshkin and Tsyvinski (2010) and Farhi and Werning (2010).


6 There is also extensive discrete-time literature that links financing constraints to agency problems, including papers such as Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007a) and DeMarzo and Fishman (2007b).
\[ u(c_t) - h(a_t), \]

where \( u(c) \) is an increasing concave utility function and \( h(a) \) is an increasing convex function describing the cost of effort. Assume that \( u(0) = 0, u'(c) \to 0 \) as \( c \to \infty, h(0) = 0 \) and \( h'(0) > 0 \). The agent discounts future utility at rate \( r \).

In discrete-time models, the agent’s average expected utility is defined as

\[
E \left[ r \sum_{i=1}^{\infty} \frac{u(c_i) - h(a_i)}{(1 + r)^i} \right].
\]

In continuous time, \( t \in [0, \infty) \) and the agent’s average expected utility is

\[
E \left[ r \int_0^{\infty} e^{-r}(u(c_t) - h(a_t))dt \right].
\]

The principal does not observe the agent’s effort \( a_t \) directly, but only observes output that depends on effort and noise.

In discrete time, denote by \( q_t \) the realization of output at time \( t \), and by \( P(q | a) \) the probability distribution of output given effort (then \( a = E[q | a] \)). Let

\[
X_t = \sum_{i=1}^{t} q_i
\]

denote the total output produced up to time \( t \). Then the path \( \{X_s, s = 1, 2, \ldots t\} \) summarizes the history of output. In discrete time it is useful to include a public randomization device, i.e., consider public histories \( h^t = \{x_0, x_1, x_1, \ldots x_{t-1}, x_{t-1}\} \) that include an i.i.d. sequence of continuous random variables \( x_0, x_1 \ldots \). In terms of timing, we shall assume that in each period \( t \) the agent is paid \textit{before} the output \( q_t \) is realized, so that both the agent’s effort and his compensation at time \( t \) are determined by the same information.\(^7\)

In continuous time, the dependence of output on the agent’s effort is modeled as

\[
dX_t = a_t \, dt + \sigma \, dZ_t,
\]

where \( Z_t \) is a Brownian motion. That is, the total incremental output depends on the agent’s current effort and noise. A public history \( h^t \) in continuous time is the path \( \{X_s, s \leq t\} \).

The principal can commit to any contract that specifies how the agent’s pay \( c_t \) depends on the entire history \( h^t \) (subject to technical measurability conditions in

\[\text{\footnotesize \ref{footnote}}\]

\[\text{\footnotesize \footnote{This assumption is a bit cleaner than the more standard assumption that the agent is paid after output in period \( t \) is realized.}}\]
continuous time). If the agent can get at least his reservation utility $u$ from the contract offered by the principal, he accepts it and chooses an optimal effort strategy, given the compensation policy $\{c_t \}$. The most basic version of the principal-agent problem assumes that the agent is tied to the principal forever once he accepts the contract, but the methods easily extend to settings where the agent may quit or be fired.

Formally, the problem of finding the optimal contract is a constrained maximization problem. We will consider a problem in which the objective function is the principal’s expected profit

$$
\max_{\{c_t, a_t\}} E^a \left[ r \int_0^\infty e^{-rt} (a_t - c_t) dt \right],
$$

We consider a constraint that the agent has to get a required level of utility $W_0$\(^8\)

$$
E^a \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(a_t)) dt \right] = W_0
$$

and a set of incentive constraints for the agent’s strategy

$$
E^{\tilde{a}} \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(\tilde{a}_t)) dt \right] \leq E^a \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(a_t)) dt \right]
$$

for any alternative strategy $\{\tilde{a}_t\}$. (IC)

The superscripts over expectations $E^a$ and $E^{\tilde{a}}$ highlight that the agent’s strategy affects the probability distribution over the paths of output, and thus over compensation realizations. Thus, each expectation in the incentive constraint depends on the agent’s effort directly, as it enters the cost of effort $h(a_t)$, and indirectly through its effect on the probability distribution over the paths of $X_t$.

Also, note that even though the contract specifies just the agent’s pay, the principal optimizes over both the compensation rule $\{c_t\}$ and his anticipation of the agent’s response $\{a_t\}$, to make sure that the agent’s strategy used to compute the principal’s profit is an optimal response to the contract.

The principals’ problem is a difficult one, because of a large space of possible contracts and the complexity of the agent’s incentive constraints. The agent himself is a dynamic optimizer. Thus, the principal’s problem consists of two dynamic optimization problems embedded in one another.

\(^8\) As it will become clear later, it is convenient to work with a constraint that the agent’s utility is exactly $W_0$, as opposed to greater than or equal to $u$. Once we solve the problem for all $W_0 \geq u$, we can then always choose $W_0$ optimally.
What simplifies the principal’s problem significantly is the one-shot deviation principle - the agent’s incentive constraints hold for all alternative strategies \( \{ \tilde{a}_i \} \) if they hold just for strategies that differ from \( \{ a_i \} \) for an instant. Thus, the one-shot deviation principle reduces the set of incentive constraints.

Furthermore, gains and losses from one-shot deviations are easy to evaluate by seeing how sensitive the agent’s continuation payoff

\[
W_t = E_t^a \left[ r \int_t^\infty e^{-r(s-t)} (u(c_s) - h(a_s)) \, ds \right]
\]

is to output in a given period.

Equipped with this insight, we prove that there is a one-to-one correspondence between (1) contracts \( \{ c_i \} \) together with anticipated incentive-compatible effort responses \( \{ a_i \} \) and (2) controlled processes \( \{ W_t \} \) that satisfy appropriate one-shot deviation conditions. We will see how this works exactly in the next section. The implication of this reasoning is that we can now solve the principal’s problem using standard methods from optimal stochastic control.

### 3. Reducing the principal’s problem to an optimal stochastic control problem.

In this section we reduce the principal’s problem in the \textit{continuous-time} model to optimal stochastic control. With this goal in mind, we use the one-shot deviation principle to focus on instantaneous incentive constraints, express these constraints in terms of the agent’s continuation payoff \( W_t \), and then finally prove that there is a correspondence between incentive-compatible contracts and appropriate controlled processes \( W_t \). The analysis of the discrete-time model is completely analogous, except that it requires randomization at the beginning of each period.

The agent’s continuation payoff \( W_t \) in a given contract after a given history of output \( \{ X_s, s \leq t \} \) depends on the agent’s future compensation \( \{ c_s, s \geq t \mid X_s, s \leq t \} \) and on his future effort strategy \( \{ a_s, s \geq t \} \). Recall that \( W_t \) is defined as

\[
W_t = E_t^a \left[ r \int_t^\infty e^{-r(s-t)} (u(c_s) - h(a_s)) \, ds \right].
\]

For any fixed contract \( \{ c_s, t \geq 0 \} \) and strategy \( \{ a_t, t \geq 0 \} \), \( W_t \) can be viewed as a stochastic process that unfolds as the history of output \( \{ X_s, s \leq t \} \) is realized. The following proposition provides a representation of \( W_t \) as a diffusion process. Diffusions have continuous sample paths (i.e. no jumps) characterized by their drifts and volatilities. The representation (1) is proved with the Martingale Representation Theorem, which applies because information arrives via a Brownian filtration in our model.
**Proposition 1.** (Sannikov, 2008). For any pair \(\{c_t, a_t\}\) with finite utility for the agent, (a) the evolution of the agent’s continuation payoff \(W_t\) through time can be written as

\[
dW_t = r (W_t - u(c_t) + h(a_t)) \, dt + r Y_t (dX_t - a_t \, dt)
\]

for some process \(Y_t\) in \(L^*\), and (b) \(W_t\) satisfies the transversality condition.

\[
\lim_{y \to 
- \infty} E_t^y [e^{-rT} W_{T+y}] = 0 \text{ almost everywhere}.^9
\]

Conversely, a process \(W_t\) that follows (1) and satisfies the transversality condition is the agent’s continuation value.

Two important points about equation (1) are worth noting. First, when the agent takes effort \(a_t\),

\[
dX_t - a_t \, dt = \sigma \, dZ_t,
\]

so that \(dX_t - a_t \, dt\) is zero in expectation. It follows that \(r (W_t - u(c_t) + h(a_t))\) is the drift of \(W_t\).

The form of the drift follows from the definition of \(W_t\) as the agent’s future expected payoff. If we think of \(W_t\) as what the principal owes to the agent, then

\[
r W_t - r(u(c_t) - h(a_t))
\]

accrues the interest on debt net of the flow of repayments. The agent receives payoff flow of \(u(c_t) - h(a_t)\) through consumption utility net of the cost of effort. The transversality condition has to hold if the debt is eventually repaid.

Second, since we are looking at contracts and strategies where the agent’s compensation and effort are determined by output \(X_t\), the agent’s continuation payoff \(W_t\) is also determined by output. The term \(r Y_t\) expresses the sensitivity of the agent’s continuation payoff to output.

Proposition 1 also provides a converse statement. Among all processes \(W_t\) computed according to (1), given any starting value and given any volatility process \(Y_t\), there is only one that satisfies the transversality condition: the agent’s continuation payoff. That is, any process computed according to (1) other than the agent’s continuation value would violate the transversality condition, for any choice of \(Y_t\). The reason is that whenever the value \(\hat{W}_t\) owed to the agent does not correspond to the actual value delivered, then no matter how the principal gambles with his debt (through \(Y_t\)), he cannot avoid violating the transversality condition.

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^9 A process \(\{Y\}\) is in \(L^*\) if \(E \left[ \int_0^t Y_s^2 \, ds \right] < \infty \text{ for all } t\).
To sum up, Proposition 1 says that for any contract \(\{c_t\}\) and a response strategy \(\{a_t\}\), with finite payoff to the agent, exactly one process \(W_t\) satisfies (1) and the transversality condition, and that process \(W_t\) is the agent’s continuation value.

**Proof of Proposition 1.** Fix a contract \(\{c_t\}\) and a strategy \(\{a_t\}\), and define

\[
V_t = E_t^r \left[ r \int_0^\infty e^{-rt}(u(c_s) - h(a_s))ds \right] = r \int_0^t e^{-rt}(u(c_s) - h(a_s))ds + e^{-rt}W_t
\]  

(2)

Since \(V_t\) is a martingale under the strategy \(\{a_t\}\), by the Martingale Representation Theorem there exists a process \(Y_t\) in \(L^r\) such that

\[
dV_t = rY_t e^{-rt} (dX_t - a_t) dt.
\]

Differentiating (2) with respect to \(t\), we find that

\[
dV_t = r e^{-rt} (u(c_t) - h(a_t)) dt - r e^{-rt} W_t + e^{-rt} dW_t = r Y_t e^{-rt} (dX_t - a_t) dt \Rightarrow
\]

\[
dW_t = r (W_t - u(c_t) + h(a_t)) dt + r Y_t (dX_t - a_t) dt.
\]

The transversality condition holds by the Dominated Convergence Theorem.\(^\text{10}\)

To prove the converse, suppose that the process \(W_t\) follows (1) and satisfies the transversality condition. Then the process \(V_t\) defined above is a martingale when the agent is following the strategy \(\{a_t\}\), since \(dV_t = r Y_t e^{-rt} (dX_t - a_t) dt\). Therefore,

\[
W_0 = V_0 = E[V_0] = E \left[ r \int_0^t e^{-rt}(u(c_s) - h(a_s))ds \right] + E[e^{-rt}W_t].
\]

Taking the limit \(t \to \infty\), we find that \(W_0\) is the agent’s continuation payoff, since the transversality condition holds. A similar argument shows that \(W_t\) is the agent’s continuation payoff at any time \(t > 0\). QED

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\(^\text{10}\) Note that \(r \int_0^\infty e^{-rs}(u(c_s) - h(a_s))ds \to r \int_0^\infty e^{-rs}(u(c_s) - h(a_s))ds\), and the sequence is dominated by

\[
-\infty < -r \int_0^\infty e^{-rs} h(a)ds \leq r \int_0^\infty e^{-rs}(u(c_s) - h(a_s))ds \leq r \int_0^\infty e^{-rs} u(c_s)ds < \infty,
\]

so \(E[e^{-rt}W_t] = E \left[ r \int_0^\infty e^{-rs}(u(c_s) - h(a_s))ds \right] - E \left[ r \int_0^t e^{-rs}(u(c_s) - h(a_s))ds \right] \to 0\) as \(t \to \infty\). A similar argument shows that \(\lim_{t \to \infty} E_t[ e^{-rt}W_{t+1} ] = 0\) for any \(t > 0\).
The agent’s incentives are related to the coefficient $Y_t$ in (1), the sensitivity of the agent’s continuation payoff to output. A continuous-time version of the one-shot deviation principle in discrete time states that the agent’s strategy $\{a_t\}$ is optimal if and only if at each moment of time, the agent maximizes the expected impact of effort on his continuation value net of the cost of effort, i.e.

$$- r h(a_t) + r Y_t a_t$$

Proposition 2 shows that if this one-shot condition holds at all times, then any dynamic deviation strategy $\{\tilde{a}_t\}$ is suboptimal. The main idea of the proof is to look at deviations that follow $\{\tilde{a}_t\}$ until time $t$ and $\{a_t\}$ thereafter. Then $W_t$ measures the agent’s continuation value after time $t$, and $r Y_t (\tilde{a}_t - a_t)$ measures the incremental expected change of the agent’s continuation value as a result of an additional deviation at time $t$. When any incremental deviation hurts the agent, inductively it follows that the whole deviation strategy $\{\tilde{a}_t\}$ is worse than $\{a_t\}$. Conversely, if the one-shot condition fails, then Proposition 2 identifies a strategy that is superior to $\{a_t\}$.

**Proposition 2.** (Sannikov, 2008) The agent’s strategy $\{a_t\}$ is optimal (i.e. satisfies (IC)) if and only if

$$\forall \tilde{a}, \ t \in [0, \infty), \ Y_t a_t - h(a_t) \geq Y_t \tilde{a} - h(\tilde{a}) \quad (3)$$

for the process $Y_t$ that represents the agent’s continuation value under strategy $\{a_t\}$ in contract $\{c_t\}$ in (1).

**Proof.** Consider

$$\tilde{V}_t = r \int_0^t e^{-r(s)} (u(c_s) - h(\tilde{a}_s)) ds + e^{-r(t)} W_t,$$

the agent’s expected payoff from following strategy $\{\tilde{a}_t\}$ until time $t$, and strategy $\{a_t\}$ thereafter. Differentiating with respect to $t$, we find that

$$d\tilde{V}_t = re^{-r(t)} (u(c_t) - h(\tilde{a}_t)) dt - re^{-r(t)} W_t + e^{-r(t)} (r(W_t - u(c_t) + h(\tilde{a}_t)) dt + r Y_t (dX_t - a_t dt))$$

$$\begin{align*}
&= re^{-r(t)} (h(a_t) - h(\tilde{a}_t)) dt + e^{-r(t)} Y_t (dX_t - a_t dt)
\end{align*}$$

When the agent is deviating to $\tilde{a}_t$ for an additional moment, then $dX_t = \tilde{a}_t dt + \sigma dZ_t$, and

$$d\tilde{V}_t = \underbrace{re^{-r(t)} (h(a_t) - h(\tilde{a}_t)) dt + e^{-r(t)} Y_t (\tilde{a}_t - a_t) dt}_{\leq 0 \text{ if (3) holds}} + re^{-r(t)} Y_t \sigma dZ_t.$$

Thus, when condition (3) holds then
\[ E^a \left[ \int_0^t e^{-r(s)} (u(c_s) - h(\tilde{a}_s)) ds \right] + \frac{E^a \left[ e^{-rT} W_T \right]}{e^{-rT} h(\tilde{a})} = E^a \left[ \tilde{V}_T \right] \leq \tilde{V}_0 = W_0 = E^a \left[ \int_0^\infty e^{-r(s)} (u(c_s) - h(a_s)) ds \right]. \]

Taking the limit \( t \to \infty \), we find that

\[ E^a \left[ \int_0^\infty e^{-r(s)} (u(c_s) - h(\tilde{a}_s)) ds \right] \leq E^a \left[ \int_0^\infty e^{-r(s)} (u(c_s) - h(a_s)) ds \right], \]

since

\[ E^a \left[ \int_0^t e^{-r(s)} (u(c_s) - h(\tilde{a}_s)) ds \right] \to E^a \left[ \int_0^\infty e^{-r(s)} (u(c_s) - h(\tilde{a}_s)) ds \right]. \]

At the same time, if (3) fails on a set of positive measure, then let

\[ \tilde{a}_t = \argmax Y_t a' - h(a'). \]

Under strategy \( \{ \tilde{a}_t \} \), the drift of \( \tilde{V}_t \) is nonnegative and positive on a set of positive measure, so for large enough \( t \),

\[ E^a \left[ \int_0^t e^{-r(s)} (u(c_s) - h(\tilde{a}_s)) ds \right] + \frac{E^a \left[ e^{-rT} W_T \right]}{E^a \left[ \tilde{V}_T \right]} > E^a \left[ \int_0^\infty e^{-r(s)} (u(c_s) - h(a_s)) ds \right], \]

where the left hand side is the agent’s payoff from following \( \{ \tilde{a}_t \} \) until time \( t \), and switching to \( \{ a_t \} \) thereafter. Thus, if (3) fails, strategy \( \{ a_t \} \) is suboptimal.\(^{11}\) QED

Since \( h(a) \) is a strictly convex function, there is a unique effort level \( a(Y) \) that maximizes \( Y a - h(a) \) for any \( Y \). We then say that sensitivity \( Y \) enforces action \( a(Y) \). Together with Proposition 1, this insight allows us to reduce the problem of finding the optimal contract to an optimal stochastic control problem, with controls \( Y_t \) and \( c_t \). We summarize this key result in the following theorem.

**Theorem 1.** There is a one-to-one correspondence between contract - incentive-compatibility effort strategy pairs \( \{ c_0, a_t \ t \geq 0 \} \), which give the agent a finite utility level, and controlled processes

\[ dW_t = r (W_t - u(c_t) + h(a(Y_t))) dt + r Y_t \sigma dZ_t \]  \hspace{1cm} (4)

\(^{11}\) Note that the strategy \( \{ \tilde{a}_t \} \) we just constructed is generally not optimal, because the process \( \tilde{V}_t \) that represents the agent’s continuation value under \( \{ c_t, \tilde{a}_t \} \) differs from \( Y_t \). Thus, \( \tilde{a}_t \) does not need to maximize \( \tilde{V}_t a' - h(a') \).
with controls \( \{ c_t, Y_t \} \), that satisfy the transversality condition \( \lim_{t \to \infty} E_t^r \left[ e^{-r t} W_{t+1} \right] = 0 \). The objective function of the control problem, which corresponds to the principal’s profit under the corresponding contract, has the flow of profit of \( a(Y_t) - c_t \) and the discount rate \( r \).

Proof. For any incentive-compatible contract \( \{ c, a \} \), the representation (1) provides the values of the control \( Y_t \). Together \( \{ c_t, Y_t \} \) drive the controlled process \( W_t \), and \( a_t = a(Y_t) \) by (3) as required.

Conversely, if \( W_t \) is any controlled process given by (4), then by Proposition 1, \( W_t \) is the agent’s continuation value under the contract \( \{ c_t, a_t = a(Y_t) \} \), which is incentive-compatible by the condition (3) of Proposition 2. QED

It is important to note that in discrete time, a result analogous to Theorem 1 also holds, and it is the basis for the recursive formulation of the repeated principal-agent problem in Spear and Srivastava (1987). We state the following discrete-time result without proof.

**Theorem 1D.** There is a one-to-one correspondence between discrete-time contract and incentive-compatible strategy pairs \( \{ c_t(h^t), a_t(h^t) \} \) with finite payoff to the agent and maps

\[
c_t(h^t), a(h^t), W(h^t)
\]

such that

\[
E[ W(h^{t+1}) \mid h_t, a(h^t) ] - W(h^t) = r (W(h^t) - u(c(h^t)) + h(a(h^t))) \quad \text{(PK)}
\]

and

\[
E[ W(h^{t+1}) \mid h_t, a(h^t) ] - r h(a(h^t)) \approx E[ W(h^{t+1}) \mid h_t, \bar{a} ] - r h(\bar{a}) \quad \text{(IC)}
\]

for any \( \bar{a} \in [0, \bar{a}] \), and an appropriate transversality condition holds.

If there is public randomization, then \( W(h^t) \) in Theorem 1D denotes the agent’s continuation payoff after the randomizing device \( x_{t-1} \) at the beginning of period \( t \) is realized.

4. **Solving the Optimal Stochastic Control Problem**

In this section, we discuss the solution of the optimal stochastic control problem presented in Theorem 1. Both in discrete and in continuous time, the problem can be solved using dynamic programming and the Bellman equation. The principal’s value function, his expected future profit under the optimal policy, is a function of the current state:

\[
F(W_t) = \max_{\{ c, Y, x_{t+1} \} } E_t^{(a(Y_t))} \left[ \frac{\int_t^\infty e^{-r s}(a(Y_s) - c_s)ds}{W_t} \right],
\]

subject to the transversality conditions and the law of motion (1) of \( W_t \).
Function $F(W)$ and the optimal policy $\{c(W), Y(W)\}$ can be found using a Bellman equation. An important discrete-time method to solve the Bellman equation is due to Phelan and Townsend (1991), who propose an iterative algorithm that involves solving multiple linear programming problems. It is based on the following version of the Bellman equation:

$$
F_T(w) = \max_{\Pi(a,q,c,w') \in T} \sum_{A \times Q \times C \times W_{T-1}} \Pi(a,q,c,w')(r(q - c) + F_{T-1}(w')),
$$

where $\Pi(a,q,c,w')$ is the joint probability distribution over the agent’s effort, output and consumption in this period, and the agent’s continuation value $w'$ in the next period. Variables $w$ and $w'$ denote the agent’s continuation values before the randomizing device at the beginning of the corresponding period is realized. Variable $T$ indexes the iteration, and $A, Q, C$ and $W_{T-1}$ are discretized sets of effort, output and consumption levels and the agent’s continuation values.

The constraints for the Bellman equation are

$$
\sum_{A \times Q \times C \times W_{T-1}} \Pi(a,q,c,w') = 1 \quad \text{The probabilities add up to 1}
$$

$$
\forall a,q,c,w', \Pi(a,q,c,w') \geq 0 \quad \text{The probabilities are nonnegative}
$$

$$
\forall a, \hat{q}, \hat{c} \sum_{W_{T-1}} \Pi(\hat{a}, \hat{q}, \hat{c}, w') = P(\hat{q} \mid \hat{a}) \sum_{Q \times C \times W_{T-1}} \Pi(\hat{a},q,c,w')
$$

The joint probability distribution over $a, q$ and $c$ is consistent with the conditional probabilities $P(q \mid a)$ with which output $q$ is realized given effort $a$\(^{12}\)

$$
\sum_{A \times Q \times C \times W_{T-1}} w' \Pi(a,q,c,w') - w = r \sum_{A \times Q \times C \times W_{T-1}} (w - u(c) + h(a))\Pi(a,q,c,w') \quad \text{(PK)}
$$

$$
\forall a, \hat{a}, \sum_{Q \times C \times W_{T-1}} (w' - rh(a))\Pi(a,q,c,w') \geq \sum_{Q \times C \times W_{T-1}} (w' - rh(\hat{a})) \frac{P(q \mid \hat{a})}{P(q \mid a)} \Pi(a,q,c,w') \quad \text{(IC)}
$$

We have to comment about pluses and minuses of this computational approach. A big plus is that the objective function and all the constraints are linear in

\(^{12}\) If the payment to the agent in period $t$ can be made conditional on output in that period, as in Phelan and Townsend (1991), then this condition has to be replaced with a weaker condition

$$
\forall \hat{a}, \hat{q} \sum_{C \times W_{T-1}} \Pi(\hat{a}, \hat{q},c,w') = P(\hat{q} \mid \hat{a}) \sum_{Q \times C \times W_{T-1}} \Pi(\hat{a},q,c,w'),
$$

and condition (IC), with

$$
\forall a, \hat{a}, \sum_{Q \times C \times W_{T-1}} (w' + ru(c) - rh(a))\Pi(a,q,c,w') \geq \sum_{Q \times C \times W_{T-1}} (w' + ru(c) - rh(\hat{a})) \frac{P(q \mid \hat{a})}{P(q \mid a)} \Pi(a,q,c,w').
$$
probabilities \( \Pi(\cdot) \), which are choice variables in this optimization problem. Thus, the problem of solving the Bellman equation is reduced to a series of **linear programming** problems, which can be mechanically coded on a computer. The minuses are, first, that the computational burden is quite large. The linear programming problem has high dimensionality, which equals the product of grid sizes of \( A, Q, C \) and \( W_{T-1} \). Also, one has to solve this linear program multiple times - once for each value of \( w \in W_T \), for each iteration \( T \). The second disadvantage is that it is difficult to identify regularities in the principal’s policy, as it is presented as a collection of 4-dimensional arrays.

In continuous time, the Bellman equation is much more tractable, easier to solve numerically, and the solution is easier to interpret. In addition, it is amenable to analytical comparative statics.\(^\text{13}\)

In general, the **value function** and the **optimal policy** for the stochastic control problem (5) is characterized by the HJB equation

\[
rf(W) = \max_{c,y} r(a(y) - c) + r(W - u(c) + h(a(y)))F'(W) + \frac{1}{2}r^2 y^2 \sigma^2 F''(W).
\]

A solution to this equation defines functions \( c(W) \) and \( y(W) \) that solve the maximization problem in (6) and suggests that controls that solve problem (5) must be \( c_t = c(W_t) \) and \( Y_t = y(W_t) \). Under these controls, the process

\[
G_t = r \int_0^t e^{-r(t-s)}(a(Y_s) - c_s)ds + e^{-rt} F(W_t)
\]

is a martingale and, under any other controls, (6) implies that \( G_t \) is a supermartingale.\(^\text{14}\) Therefore, for an arbitrary control policy \( \{c_t, Y_t\} \),

\[
\lim_{t \to \infty} E\left[ r \int_0^t e^{-r(t-s)}(a(Y_s) - c_s)ds \right] + \lim_{t \to \infty} E\left[ e^{-rt} F(W_t) \right] = \lim_{t \to \infty} E[G_t] \leq G_0 = F(W_0)
\]

with equality for the controls \( c_t = c(W_t) \) and \( Y_t = y(W_t) \).\(^\text{15}\) Thus, (a) any policy such that \( \lim_{t \to \infty} E[e^{-rt} F(W_t)] = 0 \) attains profit weakly less than \( F(W_0) \) and (b) if \( \lim_{t \to \infty} E[e^{-rt} F(W_t)] = 0 \) for the policy \( c_t = c(W_t) \) and \( Y_t = y(W_t) \), then that policy attains

---

\(^\text{13}\) See DeMarzo and Sannikov (2006), who derive comparative statics analytically by directly differentiating the HJB equation.

\(^\text{14}\) By Ito’s lemma, the drift of \( G_t \) is

\[
e^{-rt}\left( r(a(Y_t) - c_t) - rF(W_t) + r(W_t - u(c_t) + h(a(Y_t)))F'(W_t) + \frac{1}{2}r^2 y^2 \sigma^2 F''(W_t) \right) \leq 0.
\]

with equality if and only if \( c_t = c(W_t) \) and \( Y_t = y(W_t) \).
profit $F(W_0)$. We see that (8) allows us to verify optimality, subject to technical conditions.

To solve (6) and understand the structure of the optimal policy, one has to separate the cases when $y = 0$ and when $y \neq 0$. We conjecture that whenever $y = 0$ is optimal, then $F(W)$ coincides with the retirement profit function, which is attainable by a policy that gives the agent a constant wage, i.e. $F_0(u(c)) = -c$.\textsuperscript{16} Otherwise $y > 0$, and equation (6) can be transformed to\textsuperscript{17}

$$
F''(W) = \min_{y>0,c} \frac{F(W) - a(y) + c - F'(W)(W - u(c) + h(a(y)))}{ry^2/2}.
$$

Define function $F$ as follows:

**Definition:** Let $F$ be the function that solves equation (9) from the boundary condition $F(0) = F_0(0) = 0$, with the largest slope $F'(0) > 0$ such that $F(W) = F_0(W)$ for some $W > 0$. Assume that such a slope exists.\textsuperscript{18}

The following theorem demonstrates that function $F$ solves the HJB equation on the feasible domain $[0, \infty)$, characterizes the principal’s value function on $[0, W]$, and serves as an upper bound for the principal’s value function for $W > W$.

**Theorem 2.** (Sannikov, 2008) Function $F$ defined above is concave and satisfies the HJB equation for all $W \geq 0$.

For any $W_0 \in [0, W]$, it is the principal’s value function under the optimal contract. The optimal contract is characterized by payments $c_t = c(W_t)$ and recommended effort $a_t = a(y(W_t))$, where the $W_t$ is the controlled process that starts at $W_0$ and follows

$$
dW_t = r(W_t - u(c_t) + h(a_t)) \, dt + r y(W_t) \, (dX_t - a_t \, dt),
$$

\textsuperscript{16} This conjecture no longer holds if the principal and agent have different discount rates.

\textsuperscript{17} Because maximization is performed over $y \in (0, \infty)$, which is not a compact set, the solution to (9) is defined only as long as $(W, F(W), F'(W))$ stay in the domain such that

$$
\min_{y,c} F(W) - a + c - F'(W)(W - u(c) + h(a)) \leq 0
$$

and 

$$
F(W) - 0 + c - F'(W)(W - u(c) + h(0)) > 0 \quad \text{for all } c \geq 0.
$$

It can be shown that any solution of (9) with $F(0) = 0$ that has a lower slope at 0 than the principal’s value function $F$ eventually stops existing when it reaches the latter boundary. As the proof of Theorem 2, no solution that satisfies $F(0) = 0$ reaches the former boundary.

\textsuperscript{18} If not, i.e. the solution with boundary conditions $F(0) = F'(0) = 0$ stays weakly above $F_0(W)$, then, because $F(W) < 0$ for all $W > 0$ and function $F(W)$ is an upper bound on the principal’s profit, there is no contract with positive profit for the principal. In this case, the principal prefers to not hire the agent.
until the stopping time $\tau$ when $W_t$ hits $0$ or $\bar{W}$. After time $\tau$, the agent is paid a fixed wage $c_\tau$ such that $u(c_\tau) = W_\tau$, and the agent puts effort $0$.

For any $W_0 > \bar{W}$, function $F(W)$ is negative and is an upper bound on the principal’s value function. Thus, there is no profitable contract with positive profit to the principal in that range.

We illustrate Theorem 2 with a computed example, where $u(c) = \sqrt{c}$, $h(a) = a^2$, and $r=5\%$. The left panel of the following figure presents the principal’s profit $F(W)$ computed for the volatilities of output $\sigma = 0.5$ (blue), $\sigma = 1$ (purple) and $\sigma = 2$ (red). The green curve presents the first-best profit, which arises in the limit as $\sigma \to 0$, and the black curve presents the retirement profit $F_0(W)$. The right panel presents the agent’s effort and consumption under the optimal contract.\(^{19}\)

---

\(^{19}\) In this example, because the cost of effort does not satisfy the technical condition $h' (0) > 0$, point $\bar{W}$, is no longer finite, but rather function $F(W)$ asymptotes to $F_0(W)$ as $W \to \infty$. 

---

\*Proof of Theorem 2.* To see that function $F$ is concave, first note that

$$F'''(0) < \frac{F'(0) - \bar{a} - F'(0)h(\bar{a})}{ry^2\sigma^2/2} < 0,$$

---
where \( y = h'(\Omega) \) is the sensitivity that enforces action \( \Omega \). Second, once concave near \( W = 0 \), the solution can never become convex. Otherwise, \( F''(W) \) would reach 0 at some point \( W > 0 \). However, it is easy to check that any solution with \( F''(W) = 0 \) at some point \( W \) is a straight line.

Second, certainly function \( F \) satisfies (6) if we restrict maximization to choices \( y > 0 \). We need to check that \( F \) satisfies (6) if we also allow \( y \leq 0 \), which enforce \( a = 0 \), i.e. that

\[
rf(W) \geq -rc + r(W - u(c))F'(W) + \frac{1}{2} r^2 y^2 F''(W). \tag{10}
\]

Since \( F''(W) < 0 \), the right hand side is maximized when \( y = 0 \), and condition (10) is equivalent to the statement: For any point \( W \), the tangent to \( F(W) \) at \( W \) passes weakly above \( F_0(W) \). That is certainly true because \( F \) is a concave function that is always greater than \( F_0 \) by construction. Note that the tangents to \( F \) at \( W = 0 \) and \( \overline{W} \) touch \( F_0 \), so that \( y = 0 \) also maximizes (6) at those two points.

Because the contract defined in Theorem 2 chooses controls that are maximizers in (6), and because \( W_t \) satisfies the transversality condition (indeed, stays bounded), the controls correspond to a contract that attains profit \( F(W_0) \) by (8). Moreover, (8) also implies that this contract is better than any other contract such that

\[
\lim_{t \to \infty} E[e^{-rt} F(W_t)] = 0.
\]

More generally, if \( h'(a) > 0 \), then it can be easily shown that there is a point \( W^* > \overline{W} \) at which \( F_0(W^*) \) is first-best profit. Then, for an arbitrary contract, if \( \tau^* \) is the first time when \( W_t \) hits \( \tau \) then

\[
E \left[ r \int_0^\infty e^{-rt} (a(Y_t) - c_s) ds \right] \leq E \left[ r \int_0^\tau e^{-rt} (a(Y_t) - c_s) ds \right] + E[e^{-r\tau} F(W^*)] = E[G_*] \leq G_0 = F(W_0),
\]

because \( E \left[ r \int_0^\tau e^{-rt} (a(Y_t) - c_s) ds \right] \leq E[e^{-r\tau} F_0(W^*)] \leq E[e^{-r\tau} F(W^*)] \). Note also that

\[
\lim_{t \to \infty} E[1_{\tau^* > t} e^{-rt} F(W_t)] = 0.
\]

This argument shows that \( F(W_0) \) is an upper bound on the principal’s profit also if \( W_0 > \overline{W} \). However, due to a possible violation of the transversality condition, profit \( F(W_0) \) may not be attainable when \( W_0 > \overline{W} \). QED

5. Approximately Optimal Contracts

In this section, we discuss methods to (1) evaluate how far a given contract is from the optimum and (2) design approximately optimal contracts, which are simple, or which satisfy certain contracting rigidities. Fong and Sannikov (2010) argue that because many considerations of practical importance may be outside a theoretical model, it is important to understand not only the optimal contract, but also the whole range of contracts that are close to optimal. For example, in some
environments, *downward wage rigidity* is an important consideration. Employers are reluctant to cut wages, as it is argued that decreasing wages hurts employee morale (see Bewley (1998)). In other environments, contract design can be limited in other ways, and industry standards may require the use of specific contractual forms. For example, options, stock grants and bonuses are used for CEO compensation. Likewise, incentives for portfolio managers are often created by performance fees, which give them a share of return above a certain benchmark. While common contractual forms offer significant flexibility, they may not accommodate all theoretically possible fully contingent contracts.

Fong and Sannikov (2010) develop a methodology to characterize approximately optimal contracts and identify contract features that are particularly important for optimality. They analyze separately cases when the informational problem is small, such as in jobs where workers perform routine tasks that are easy to monitor, and when it is large, such as in the cases of CEOs and portfolio managers.

**Small \( \sigma \).** When the noise parameter \( \sigma \) is small, there is a huge multiplicity of contractual forms that attain efficiency. For example, it is possible to attain efficiency with contracts that exhibit downward wage rigidity, and even with contracts that pay the agent a *constant wage* until a stopping time when the agent is fired.

Many asymptotically efficient contracts (including, surprisingly, contracts that pay the agent a constant wage until termination) can be constructed using the idea of *selling* the firm to the agent. That is, the agent ends up absorbing all of the risk of realized output, possibly net of some hedgeable risk. By the law of large numbers it is possible to compensate the agent with very smooth consumption, and to keep the agent employed as long as it is efficient. That is, when \( \sigma \) is small, the probability of inefficient termination is minimal.

**Large \( \sigma \).** When measures of the agent’s performance are sufficiently noisy, then inefficiency is significant and the trade-off between incentives and insurance is important. While contracts with nondecreasing wages become highly inefficient, many simple forms of performance-sensitive arrangements (such as options, stock and bonuses) can do quite well. The reason is that small deviations from the optimal contract with respect to the rate of compensation or pay-performance sensitivity have only second-order effects on optimality. Thus, a simple candidate contract can be a crude approximation of the optimal contract, yet be close to the optimal contract in terms of efficiency.

Below we present two results from Fong and Sannikov (2010), which compare profit under an arbitrary contract to that from the optimal contract. Theorem 3, which requires knowledge of the optimal contract, effectively characterizes the class of approximately optimal contracts. In contrast, Theorem 4 allows us to evaluate a given contract even if we do not know the optimal contract. The idea of Theorem 4 applies beyond the model of Section 2, to settings where we can characterize the
agent’s incentives (as we did in Theorem 1) but cannot compute the optimal contract.

**Theorem 3.** (Fong-Sannikov, 2010) Let $F(W)$ be the principal’s value function under the optimal contract. Define

$$H(c, Y, W) = a(Y) - c - F(W) + F'(W) \left(W - u(c) + h(a(Y))\right) + \frac{1}{2} F''(W) r Y^2 \alpha^2,$$

so that by the HJB equation (6), $\max_{c,Y} H(c, Y, W) = 0$. Then the principal’s profit from an arbitrary contract $C'$, in which the agent’s continuation value follows (4), is

$$F(W_0) + E^C \left[\int_0^\infty e^{-r s} H(c_s, Y_s, W_s) ds\right]. \quad (11)$$

Equation (6) implies that under the optimal contract, $H(c_0, Y_0, W_t) = 0$ after all output histories, so that $F(W_0)$ is the principal’s profit. Under an arbitrary contract $C'$, $H(c_0, Y_0, W_t) \leq 0$ and the principal’s profit is not more than $F(W_0)$. A contract is almost as efficient as the optimal contract if $H(c_0, Y_t, W_t)$ is close to zero almost all the time.

**Corollary.** If contract $C'$ satisfies $H(c_0, Y_0, W_t) > -\epsilon$ after all output histories, it attains the principal’s profit that is within $\epsilon$ of the optimal contract profit.

**Proof of Theorem 3.** For an arbitrary contract $C'$, consider the process

$$G_t = r \int_0^t e^{-r s} (a(Y_s) - c_s) ds + e^{-r t} F(W_t) - r \int_0^t e^{-r s} H(c_s, Y_s, W_s) ds.$$

Differentiating with respect to $t$, it is easy to see that $G_t$ is a martingale, so

$$F(W_0) = G_0 = \lim_{t \to \infty} E[G_t] = E^C \left[\int_0^\infty e^{-r s} (a(Y_s) - c_s) ds\right] - E^C \left[\int_0^\infty e^{-r s} H(c_s, Y_s, W_s) ds\right].$$

Therefore, the profit under contract $C'$ is given by (11). QED

Theorem 3 can be easily adapted to many other settings that boil down to a stochastic control problem. However, an essential requirement behind the result is that the principal’s value function $F(W_t)$ under the optimal policy has to be known. There are many settings, where the optimal policy is difficult to find, one may conjecture a natural simple policy that may be close to optimal. In this case, is it possible to prove formally that the candidate policy is close to optimal, without the knowledge of the optimal policy itself?

It turns out that it is possible, and Theorem 4 lays out a general method to do that. It assumes that the recursive structure of the problem is given by Theorem 1, considers a candidate contract $C'$ that attains the value function $\tilde{F}(W_t)$, and delivers an upper bound on the profit from the optimal contract relative to $C'$. The bound
easily extends to problems with other recursive structures, including problems in which the recursive structure is based on a set of necessary incentive constraints (e.g. first-order incentive constraints). Of course, in that case, one has to verify that the contract \( C' \) is fully incentive compatible.

**Theorem 4.** (Fong-Sannikov, 2010) Consider a candidate contract \( C' \), in which \( W_t \) follows (4) and the principal’s value function is given by \( \tilde{F}(W_t) \). Let

\[
K(W) = \max_{c,y} a(y) - c - \tilde{F}(W) + (W - u(c) + h(a(y)))\tilde{F}'(W) + \frac{1}{2} \sigma^2 \tilde{F}''(W),
\]

and let \( \bar{K} = \max_W K(W) \). Then the principal’s profit under the optimal contract \( C \) is bounded from above by

\[
\tilde{F}(W_0) + E^C \left[ r \int_0^\infty e^{-rs} K(W_s) ds \right] \leq \tilde{F}(W_0) + \bar{K}.
\]

The bound \( \tilde{F}(W_0) + \bar{K} \) is useful when we do not know the optimal contract, since in those situations we cannot evaluate the expectation \( E^C \) under the optimal contract \( C \). Also, note that \( K(W) \geq 0 \), since the choices from contract \( C' \) on the right hand side of (12) yield 0.

**Proof.** Consider the process

\[
G_t = r \int_0^t e^{-rs} (a(Y_s) - c_s) ds + e^{-rs} \tilde{F}(W_t) - r \int_0^t e^{-rs} K(W_s) ds.
\]

Then \( G_t \) is a supermartingale under any contract, from the definition of \( K(W) \). In particular, under the optimal contract \( C \),

\[
\tilde{F}(W_0) = G_0 \geq \lim_{t \to \infty} E[G_t] = E^C \left[ \int_0^\infty e^{-rs} (a(Y_s) - c_s) ds \right] - E^C \left[ \int_0^\infty e^{-rs} K(W_s) ds \right].
\]

This proves the bound (13). QED

In the basic setting of Section 2, because we know the optimal contract, we can use Theorem 3 to investigate the sensitivity of expected profit to contract features. Specifically, we ask two questions:

1. For any value of \( W \), what rates of compensation \( c \) and target effort levels \( a \) are approximately optimal, in the sense that \( H(c, Y, W) \geq -\epsilon \)?

2. What density over the state space of \( W \) does the expectation \( E^C \) in Theorem 3 imply?

In the following figure, we illustrate the range within which \( a \) can vary, while the choice of \( c \) matches that in the optimal contract, such that \( H(c, Y, W) \geq -0.001 \), for our example above with \( \sigma = 2 \). We also do the same for \( c \).
We see that the acceptable error bounds are generally quite wide, except near termination. In particular, the bounds suggest that when the agent’s continuation value is low, it is important to restrict the agent’s compensation.

Furthermore, the values of $H(c, Y, W)$ are most important in parts of the state space where the implied density related to the expectation $E^C$ is large. This density is different for different contracts $C'$, and can be computed using the Kolmogorov Forward Equation

$$\frac{\partial}{\partial t} f(t, W) = -\frac{\partial}{\partial W} [\mu(W) f(t, W)] + \frac{\partial^2}{\partial W^2} [\sigma^2(W) f(t, W)],$$

with boundary conditions $f(0, W) = \delta_{W_0} (W)$ (the Dirac delta function) and $f(t, 0) = f(t, \overline{W}) = 0$ for the absorbing boundaries at 0 and $\overline{W}$. In this equation, $\mu(W)$ is the drift of $W_t$ in contract $C'$, and $\sigma(W)$ is the volatility. Since we are interested in the expectation (11) over the state space, it is useful to calculate the function

$$\tilde{f}(W) = \int_0^\infty e^{-rt} f(t, W) dt.$$

Then

$$E^C \left[ r \int_0^\infty e^{-rs} H(c(W_s), Y(W_s), W_s) ds \right] = \int_0^{\overline{W}} H(c(W), Y(W), W) \tilde{f}(W) dW$$

for any contract $C'$ that is recursive in $W$. Thus, the density $\tilde{f}(W)$ indicates which parts of the state space are particularly important for how well the contract $C'$ does relative to the optimal contract. If $H(c(W), Y(W), W)$ happens to be significantly below 0 in a region where the density $\tilde{f}(W)$ is thin, the negative impact on the principal’s profit will be minimal.
We conclude that there is a large class of approximately optimal contracts, because small deviations away from the optimum have second-order effect on profitability (except at corners). Theorem 3 allows us to perform sensitivity analyses, to identify the characteristics that are particularly important for contract optimality. These concern the agent’s compensation and pay-performance sensitivity, as well as the probability distribution over the state space that indicates the portions of the state space that carry the greatest weight.

6. Applications in Corporate Finance

The continuous-time approach to dynamic agency problem has proved particularly fruitful in corporate finance applications. The companion paper of Biais, Mariotti and Rochet (2012) provides an excellent overview of the connections between discrete and continuous-time models in corporate finance. Among other things, that paper explains the sequences of discrete-time models that lead in the limit to Brownian or Poisson models in continuous time.

Here we take advantage of the methodology developed in Sections 2, 3 and 4 to overview two models directly set in continuous time: those of DeMarzo and Sannikov (2006) and DeMarzo, Fishman, He and Wang (2011) (hereafter DS and DFHW).20 DFHW build upon the DS model. For pedagogical purposes and to emphasize methodology, we take a version of the DS model (under special assumptions) that is a special case of the DFHW model, and present these two models side-by-side instead of in sequence.

In both models, the agent can lower the firm’s cash flows through private actions that generate private benefits to the agent. Specifically, the agent’s action at time $t$, $a_t \in [0, \infty)$, affects the cash flows according to

$$d\hat{X}_t = K_t((\mu - i_t)dt + \sigma dZ_t) - a_t dt,$$

where $K_t$ the firm’s capital and $i_t$ is the investment rate per unit of capital. Action $a_t$, which could be interpreted as cash flow diversion, generates private benefit to the agent at rate $\lambda a_t$, where $\lambda \in (0, 1]$. In this case, the cash flow observed by the principal, $\hat{X}_t$, is interpreted as the reported cash flow.

In DFHW capital evolves according to the standard equation

$$dK_t = (\Phi(i_t) - \delta)K_t dt,$$

where $\delta$ is the depreciation rate and $\Phi$ is an investment function with adjustment costs such that $\Phi(0) = 0$, $\Phi' > 0$ and $\Phi'' < 0$.

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20 Biais, Mariotti, Plantin and Rochet (2007) prove the convergence of discrete-time optimal contracts to the continuous-time ones, in a model that is similar to that of DS.
DS assume that the scale of the firm is fixed, so that $i_t = 0$, $\delta = 0$ and $K_t = 1$ at all times.

In both models, the principal and agent are risk-neutral and have discount rates $r$ and $\gamma > r$ respectively. The agent has limited liability, so payments to the agent cannot be negative.

A contract specifies, as functions of histories $\{\hat{x}_{s,t}, s \leq t\}$, the amount of cash that the agent is allowed to keep $\{dC_t\}$ as well as the termination time $\tau$. In the event of termination, the agent gets his outside option of $R = 0$, and the principal receives value $qK$ from the firm’s assets. If there is investment, as in DFHW, we have to assume that $q \geq 1/\Phi'(-\infty)$, since quick disinvestment yields $1/\Phi'(-\infty)$ per unit of capital. We also assume that the first-best solution involves running the project forever, and never terminating, i.e.

$$q < \sup_i \frac{\mu - i}{r - \Phi(i) + \delta} < \infty.$$  

The optimal contract maximizes the principal’s profit, subject to a constraint that the agent’s expected payoff is $W_0$ at time 0, and a set of incentive-compatibility constraints.

$$\max_{C, \tilde{\gamma}} rE^{a=0} \left[ \int_0^\tau e^{-rt} (d\hat{X}_t, -dC_t) dt + e^{-rt} qK \right] \text{s.t.}$$

$$E^{a=0} \left[ \gamma \int_0^\tau e^{-rt} dC_t \right] = W_0 \quad \text{and} \quad E^{\tilde{\gamma}} \left[ \gamma \int_0^\tau e^{-rt} (dC_t, + \lambda \tilde{a}_t, dt) \right] \leq W_0$$

for any deviation strategy $\{\tilde{a}_t\}$. Note that, unlike in the problem posed in Section 2, we right away took $\{a_t = 0\}$ as a part of the solution. A contract that enforces an action strategy $\{a_t\}$ of the agent, such that $a_t > 0$ on a positive-measure set, cannot be optimal. Indeed, it can be shown that given any contract in which the agent diverts cash flows for private benefits, there exists a more profitable incentive-compatible contract in which the principal pays the agent directly instead.

By the logic of Section 3, there is a one-to-one correspondence between contracts that satisfy the constraints, and controlled processes

$$dW_t = \gamma(W_t dt - dC_t) + \gamma Y_t (d\hat{X}_t - K_t (\mu - i_t) dt) \quad \text{and} \quad dK_t = (\Phi(i_t) - \delta) K_t dt,$$  

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21 The setting of DS more generally allows for an arbitrary outside option $R$.

22 Here we deviate slightly from the notation of DS and DFHW and follow that of Section 2. Specifically, we normalize the agent’s payoff by $\gamma$.  

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such that \( Y_t \geq \lambda \) enforces actions \( a_t = 0 \). Here the controls are \( \{C_t, \ i_t, \ Y_t\} \), and the termination time \( \tau \) is the earliest time when \( W_t \) reaches 0.

This problem has convenient scale-invariance properties because \( K_t \) scales everything, including the outside options at time \( \tau \) (the agent receives 0). Therefore, it is natural to conjecture the principal’s value function of the form \( F(W,K) = f(w) K \), where \( w = W/K \). Then by Ito’s lemma \( w_t = W_t/K_t \) follows

\[
dw_t = (\gamma - \Phi(i_t) + \delta)\, w_t\, dt - \gamma\, dC_t/K_t + \gamma\, Y_t\, \sigma\, dZ_t.
\]

The HJB equation is

\[
rf(W,K) = \max_{\delta C/dt} \left[ r(\mu - i)K - rdC/dt + E[dF(W,K)]/dt \right] \Rightarrow
\]

\[
rf(w) = \max_{\delta C/dt} \left[ r(\mu - i) + (\Phi(i) - \delta)f(w) + (\gamma - \Phi(i) + \delta)wf'(w) + \frac{1}{2}\gamma^2\lambda^2\sigma^2 f''(w) - (r + \gamma f'(w))\frac{dC}{K,dt} \right].
\]

It can be shown that the solution to this equation is a concave function, so that it is optimal to set \( Y = \lambda \). Moreover, letting \( \overline{w} \) be the lowest value such that \( f'(\overline{w}) = -r/\gamma \), it follows that on the interval \([0, \overline{w}]\), \( r + \gamma f'(w) > 0 \). Thus, it is optimal to set \( dC = 0 \) whenever \( w \in [0, \overline{w}] \). It follows, the function \( f(w) \) can be determined from the ordinary differential equation

\[
rf(w) = \max_i \left[ r(\mu - i) + (\Phi(i) - \delta)f(w) + (\gamma - \Phi(i) + \delta)wf'(w) + \frac{1}{2}\gamma^2\lambda^2\sigma^2 f''(w) \right] \quad (14)
\]

with boundary conditions \( f(0) = r\overline{g} \) and \( f'(\overline{w}) = -r/\gamma \). To solve for \( f(w) \), we also impose the super-contact condition \( f''(\overline{w}) = 0 \), which we explain below.

The optimal investment rate in (14) is determined by the first-order condition

\[
\Phi(i) (f(w) - wf'(w))/r = 1,
\]

where \( (f(w) - wf'(w))/r \) can be interpreted as the marginal Tobin’s Q of the firm’s capital. Since \( f \) is a concave function, the marginal Tobin’s Q is increasing in \( w \).

**Theorem 5.** (DFHW) Equation (14) has a unique solution that satisfies boundary conditions \( f(0) = r\overline{g}, \ f'(\overline{w}) = -r/\gamma \) and \( f''(\overline{w}) = 0 \). The solution is concave. Denote by \( \iota : [0, \overline{w}] \to R \) the investment function that solves the maximization problem in (14).

Then the optimal contract can be written in terms of state variables

\[
dW_t = \gamma(W_t dt - dC_t) + \gamma\lambda(d\hat{X}_t - K_t(\mu - i_t)dt) \quad \text{and} \quad K_t
\]

where the rate of investment is \( i_t = \iota(W_t/K_t) \) and compensation is paid only when \( W_t = \overline{w}K_t \) in such a way that \( W_t \) never becomes greater than \( \overline{w}K_t \), after time 0. Termination occurs at time \( \tau \) when \( W_t \) reaches 0 for the first time.
The super-contact condition can be explained as follows. It turns out that equation (14) can be used to compute the principal’s profit for any contract of the form of Theorem 5, with an arbitrary (and not just optimal) choice of the payout point $\bar{w}$. Thus, the optimal choice of $\bar{w}$ boils down to finding the maximal solution of (14) that satisfies $f(0) = rg$ and $f'(\bar{w}) = -r/\gamma$.

If $f''(\bar{w}) \neq 0$, e.g. $f''(\bar{w}) < 0$, then $f'(\bar{w} + \epsilon) < -r/\gamma$ and another solution of (14) with a slightly higher slope $f'(0)$ also reaches slope $-r/\gamma$ at some point $\bar{w} + \epsilon'$. This is true because solutions to (14) are continuous in the initial condition $f'(0)$. Thus, using the point $\bar{w} + \epsilon'$ instead of $\bar{w}$ to compensate the agent results in higher profit to the principal. We can interpret $f''(\bar{w}) = 0$ as the first-order condition for the optimal choice of $\bar{w}$.

For the setting of DS, where the scale of the firm is fixed, equation (14) reduces to

$$rf(w) = r\mu + \gamma \frac{w}{\mu} f'(w) + \frac{1}{2} \gamma^2 \lambda^2 \sigma^2 f''(w).$$

(15)

Using (15), the super-contact condition is equivalent to $f(w) = \mu - w$. The following figure illustrates function $f(w)$ for the DS model.

Agency models like those of Biais, Mariotti, Plantin and Rochet (2007) (hereafter BMPR), DS and DFHW provide one explanation to financing frictions that exist in practice. The variable $w_t$ in the optimal contract can be interpreted as the amount of financial slack of the firm. In practice that could correspond to the firm’s cash reserves, its access to credit, and marketable securities on the balance sheet that can be sold to generate extra cash.

The optimal contract of Theorem 5 implies that (1) inefficient liquidation is more likely after the firm suffers losses, even when these losses are uncorrelated with future profitability and (2) investment is positively related to past performance. Also, (3) the optimal contract is consistent with the evidence of DeAngelo, DeAngelo
and Stulz (2006), who document a strong positive relationship between the firms’ retained earnings and dividend payouts. Finally, agency models predict that managers should be compensated with restricted stock, so that they have exposure to the firm’s cash flow risk.

To illustrate the practical implications of the optimal contract, papers such as BMPR and DS propose an implementation of the optimal contract using standard securities. The implementation is not unique, e.g. BMPR create a map between \( w_t \) and the firm’s cash reserves, while DS map \( w_t \) into a credit line balance. Below we describe the implementation of DS.

DS map the interval of continuation values \([0, \bar{w}]\) into a credit line, with point \( \bar{w} \) corresponding to balance zero, and \( 0 \) corresponding to balance \( \bar{w} / (\gamma \lambda) \). The scaling coefficient \( \gamma \lambda \) is chosen so that each cash flow subtracts exactly one dollar from the credit line balance, i.e.

\[
d\left(\frac{\bar{w} - w_t}{\gamma \lambda}\right) = \gamma \lambda \left(\frac{\bar{w} - w_t}{\gamma \lambda}\right) + \left(\mu - \frac{\bar{w}}{\lambda}\right) dt - \frac{d\hat{X}_t}{\text{cash flow}} + \frac{dC_t}{\lambda}.
\]

In this expression, interest \( \gamma \) is charged on the credit line balance, positive cash flows reduce the balance, and dividends, paid when the credit line balance is zero, add to the balance. The term \( \mu - \bar{w} / \lambda \) can be interpreted as coupons on long-term debt, paid to investors who help to start the project.

In this implementation, the credit line balance changes with the cash flows. Excess cash flows are paid out as dividends only when the credit line is fully paid off. The agent gets a fraction \( \lambda \) of the dividends, \( dC_t \), while outside investors get the rest. Thus, the agent is compensated with a fraction \( \lambda \) of the firm’s equity. Termination is triggered when the firm runs out of credit, i.e. the credit line balance reaches the credit limit of \( \bar{w} / (\gamma \lambda) \).

7. **Aggregate vs. Idiosyncratic Risk**

One essential assumption behind the benchmark agency model of Section 2, and the applications of Section 6, is that the noise \( dZ_t \) that obscures output is idiosyncratic and unobservable. If the noise is partially observable, then the logic of Holmstrom (1979) suggests that it is optimal to filter out the aggregate observable component of noise entirely, e.g. by evaluating the agent relative to a benchmark. This logic is only partially true. It turns out that it is optimal to expose the agent to some aggregate risk if this risk affects the agency problem, e.g. if it is correlated with the project’s future profitability or the liquidation value of the project. DFHW, Piskorksi and Tchistyi (2010) and Hoffmann and Pfeil (2009) demonstrate this point using models, in which Poisson shocks can change the value of the project. Here, we will illustrate the same point by adding observable Brownian shocks to the model of DFHW.
Consider a variation of the DFHW setting in which the productivity of the firm’s capital is subject to observable aggregate shocks, i.e.

\[ dK_t = (\Phi(i_t) - \delta)K_t dt + \sigma^K dZ^K_t. \]

The Brownian motion \( dZ^K_t \) also affects the firm’s cash flows, which follow

\[ d\hat{X}_t = K_t((\mu - i_t)dt + \sigma dZ_t + \sigma' dZ^K_t) - a_t dt. \]

As before, it is optimal to enforce action \( a_t = 0 \). However, now the law of motion of the agent’s continuation value can depend on both the cash flows and aggregate shocks,

\[
\frac{dW_t}{E[dF(W,K)]/dt} = \gamma(W_t dt - dC_t) + \gamma Y_t (d\hat{X}_t - K_t((\mu - i_t)dt + \sigma dZ_t + \sigma' dZ^K_t)) - \frac{\gamma Y_t K_t dZ^K_t}{K_t((\sigma dZ + \sigma' dZ^K_t)}
\]

Incentive compatibility requires that \( Y_t = \lambda_t \), and allows for an arbitrary value of \( Y^K_t \). The agent is fully hedged against aggregate shocks if \( Y^K_t = -Y_t \sigma' \).

We guess that the principal’s value function has the form \( F(W,K) = f(w) K \), where \( w = W/K \). Then by Ito’s lemma \( w_t = W_t/K_t \) follows

\[
dw_t = (\gamma - \Phi(i_t) + \delta + (\sigma^K)^2) w_t dt - \gamma dC_t/K_t + \gamma Y_t \sigma dZ_t + (\gamma Y_t \sigma' + Y^K_t) - \sigma^K w_t \) dZ^K_t
\]

- \( \gamma (Y_t \sigma' + Y^K_t) \sigma^K dt. \)

The HJB equation is

\[
rF(W,K) = \max_{dC,i} r(\mu - i)K - r dC / dt + E[dF(W,K)] / dt \Rightarrow
\]

\[
r_f(w) = \max_{dC,i} r(\mu - i) + (\Phi(i) - \delta) f(w) + (\gamma - \Phi(i) + \delta)wf'(w) - (r + \gamma f'(w)) \frac{dC}{K_t dt}
\]

\[
+ \frac{1}{2} \left( \gamma^2 Y^2 \sigma^2 + (\gamma(Y \sigma' + Y^K) - \sigma^K w)^2 \right) f''(w).
\]

Now, consider the function \( f(w) \) that solves the HJB equation from DFHW,

\[
r_f(w) = \max_i r(\mu - i) + (\Phi(i) - \delta) f(w) + (\gamma - \Phi(i) + \delta)wf'(w) + \frac{1}{2} \gamma^2 \lambda^2 \sigma^2 f''(w).
\]

with boundary conditions \( f(\theta) = r_0 \), \( f'(\infty) = -r \gamma \) and \( f''(\infty) = 0 \).

Since \( f(w) \) is concave, and since

\[
\min_{Y \geq \lambda, Y^K} \gamma^2 Y^2 \sigma^2 + (\gamma(Y \sigma' + Y^K) - \sigma^K w)^2 = \gamma^2 \lambda^2 \sigma^2,
\]
it follows that function $f$ also solves (16), and that $Y = \lambda$ and $Y^K = \sigma^K w / \gamma - \lambda \sigma^t$ solve the maximization problem there.

Thus, under the optimal contract $w_t$, follows

$$dw_t = (\gamma - \Phi(i_t) + \delta) w_t \, dt - \gamma \, dC_t / K_t + \gamma \lambda \sigma \, dZ_t,$$

i.e. it does not depend on observable shocks $Z_t^K$. However, the agent's continuation payoff $W_t$ does depend on observable shocks and follows

$$dW_t = \gamma(W_t \, dt - dC_t) + \gamma \lambda K_t \sigma \, dZ_t + \sigma^K W_t \, dZ_t^K.$$

We conclude that it is optimal to expose the agent to an appropriate amount of observable aggregate risk when that risk is correlated with the project's value. It is optimal to filter out the observable shocks completely from the agent's continuation value only if the observable risk does not affect the project, e.g. if $\sigma^K = 0$.

8. Conclusions

To conclude, we summarize briefly the discrete and continuous-time approaches to optimal contracts, and then outline a few areas open for research.

Probably two most prominent results from discrete-time analysis of principal-agent models are

1. The optimal fully history-dependent contract can be found within a much smaller class of recursive contracts, using dynamic programming

2. It is possible to achieve first-best outcomes in the limit as the agent becomes more patient, or as the agent's effort can be observed more and more precisely

However, the exact characterization of the optimal contract in discrete time is difficult. In contrast, by modeling the principal-agent interaction in continuous time, it is possible to derive tractable characterizations of optimal contracts using ordinary differential equations. These characterizations can be used to do comparative statics analytically and to compute contracts quite easily. Moreover, continuous-time solutions make the properties of the optimal contract much more transparent.

These advantages of the continuous-time approach to dynamic contracting have made it an attractive tool to use in applications. Continuous-time has been particularly fruitful in corporate finance, although there are also important applications in public finance (Farhi and Werning (2010)), macroeconomics (Williams (2010)) and health economics (Fong (2009)). In corporate finance, the incentive variable, the agent's continuation payoff, has been linked to the amount of financial slack of the firm. This link highlights the relationship between financing frictions and agency problems.
Existing work motivates several directions to future research. First, it is interesting to investigate how incentives applied externally to an organization affect organizational culture and cooperation among organization members. This question is important, because corporate finance applications use agency theory as if one agent managed the firm. Sometimes the agent is interpreted as the CEO, but often it is the entire management team. If the firm might go bankrupt due to underperformance, then the management has incentives to work together to avoid that. How do these incentives work? The seminal contribution of Holmstrom (1982) suggests that when team members are rewarded based on a performance measure that depends on the sum of team-members efforts, and when it is not possible to monitor individual contributions to collective output, then team members have incentives to free ride on their peers’ efforts. However, even in the absence of an external monitoring technology, the ability of team members to monitor each other can restore equilibrium with optimal effort if teams interact over multiple periods. Moreover, if the whole team has enough “skin in the game,” and if team members observe each others’ efforts well enough and have means to reward and punish each other, then the best equilibrium in the team results in efficient effort. Many puzzles still remain. For example, empirical literature has documented evidence that there can be significant differences in productivity among similar organizations. Chew, Bresnahan and Clark (1990) document persistent performance differences amongst airline meal providers, with best and worst differing by a factor of 3.

Second, there is a puzzling scarcity of “indexation” in contracts observed in practice. For example, most CEOs are compensated based on their firm’s absolute performance in absolute terms, not relative to firms in the same industry. The use of benchmarks creates less noisy measures of performance. For example, in the static model of Holmstrom (1979), it is optimal to filter out as much noise as possible from the performance measure that reflects the agent’s effort. In dynamic models, it may be optimal to expose the agent to some observable risk if it interacts with the agency problem, as we discussed in Section 7. In either case, optimal contracting theory suggests that benchmarking should be used at least to some extent.

There is also prevalence of contracting on prices in practice, which leads to additional interesting effects in a general equilibrium setting. Many papers that study macroeconomic impacts of financial frictions, such as Kiyotaki and Moore (1997), He and Krishnamurthy (2011) and Brunnermeier and Sannikov (2011), document the emergence of endogenous risk. This happens when the agents’

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24 For example, Murphy (1999) documents that only a small percentage of companies (perhaps 15%) set CEO bonuses based on comparisons to a peer group, and the use of indexed options in CEO compensation is even more rare. However, there is also some evidence that over time, changes in individual CEO pay are positively correlated to company performance relative to the overall market.
actions affect prices, creating aggregate risk that affects other agents through contracts. The economy may enter a crisis regime when the constrained agents are simultaneously hit by an aggregate shock that cannot be hedged. This absence of hedging is somewhat at odds with optimal contracting theory.

Third, more research is needed to understand how (approximately) optimal contracts can be implemented with standard securities. While BMPR and DS provide two different implementations, the incentive properties of many securities observed in practice have not been explored in enough depth. For example, the maturity structure of debt has important implications on incentives. Diamond and Rajan (2001) demonstrate that short-term financing provides strong incentives that discipline bank managers. In contrast, long-term debt creates the debt overhang problem when there is a positive probability of default, thus weakening the management’s incentives. The dynamic incentive properties of many securities, and their relationship to optimal contracts, are not sufficiently understood. In particular, it is interesting to study how approximately optimal contracts can be implemented using simple standard securities - and the results of Section 5 can help us in answering this question.

Finally, there many variations of the classic dynamic agency problem that do not have clear solutions. For example, the agent’s effort could have a persistent effect on future profit, as in Hopenhayn and Jarque (2007), and noise can be autocorrelated as in DeMarzo and Sannikov (2011). Likewise, the agent’s private information, another source of moral hazard, could be correlated over time as in Farhi and Werning (2010). Also, there may be an adverse selection problem in addition to moral hazard, as in Sannikov (2007) or Cvitanic, Wan and Yang (2011). This is an exciting area of research, which is full of complexities. While fully optimal contracts may be difficult to find, it may be that the study of approximately optimal contracts in these settings, using techniques similar to those we discussed in Section 5, would produce clear qualitative predictions about effective ways to deal with dynamic private information.

Bibliography


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