Learning, Termination, and Payout Policy in Dynamic Incentive Contracts

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Abstract

We study a principal-agent setting in which both the principal and the agent learn about future profitability from output, and the firm has the option to abandon the project if profitability is too low. Because output also depends on the agent’s effort, deviations by the agent interfere with learning and mislead the principal. As a result, the agent’s incentives to exert effort are decreasing in his information rent, the extra benefit the agent derives from knowing profitability exceeds the principal’s beliefs. The optimal contract captures the transition from a young, financially constrained firm to a mature firm that pays dividends. In the young-firm phase, poor performance permanently raises the firm’s threshold for termination, as doing so lowers the agent’s information rents. In the mature-firm phase, dividends are smooth relative to earnings and the termination threshold is fixed. Dividend smoothing results because current cash flow surprises are optimally used to adjust financial slack in line with the perceived change in profitability. In the special case in which profitability reflects the agent’s pure private ability, we demonstrate a simple implementation in which the agent is compensated via a fixed fraction of the firm’s equity.

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In many types of agency relationships output is informative about both the agent’s current actions and the future quality of the project itself. Company earnings depend on both current managerial effort and exogenous factors that affect the firm’s fundamentals. Mortgage payments reflect the borrower’s effort to save and exogenous shocks to income as well as nondiscretionary expenses.\(^1\) The performance of a worker may depend on both the worker’s effort and skill. While such situations are common in practice, most agency models do not allow current outcomes to carry information about future profitability. The standard literature on dynamic moral hazard assumes that the principal knows the correct distribution of future output, whereas in many settings in practice the agent may mislead the principal about the future by choosing an off-equilibrium effort level. When the agent may mislead the principal, his incentives to deviate depend not only on his immediate private benefits, but also on his *information rents*, the incremental payoff that the agent can earn from his private information.

We consider a setting in which a risk neutral agent manages a firm or project on behalf of a risk neutral principal or outside investors. The firm has the option to abandon the project if profitability, which is unknown and stochastic, is too low. Both the agent and principal update their beliefs about profitability based on the firm’s current cash flows. The agent may shirk or divert these cash flows for private benefit, potentially distorting the principal’s perception of profitability and creating additional information rents. These rents arise because by reducing reported cash flows today, the agent also lowers the principal’s expectations regarding future profitability.

Our paper attempts to understand how the principal can optimally design and a long-term contract which both provides incentives and controls the agent’s information rents. Methodologically, the paper contributes along three dimensions

1. recognizing the link between information rents and incentives in an environment with learning about future profitability, i.e. formulating the necessary first-order

\(^1\) See Piskorski and Tchistyi (2010).
incentive constraints, and identifying a recursive representation of information rents;\(^2\)

(2) characterizing the properties of optimal control of information rents in a full-commitment contract using the recursive structure, i.e. identifying the distortions that arise from the desire to relax the agent’s incentive constraint; and,

(3) proposing a verification method for the validity of the first-order approach.

We focus on a specific application which is important in corporate finance, embedding the agency problem into the classic real option environment (see Dixit and Pindyck (1994)). In our model, the uncertain profitability of the firm provides the principal with an abandonment option – if profitability is too low, it is optimal to liquidate the firm. There are well-understood parameters that determine the first best abandonment threshold – profitability, its volatility, and the liquidation value. To that we add an agency problem, with main parameter \(\lambda\) which determines the private benefit the agent can extract from reducing the project’s cash flows. The magnitude of the agency problem also depends on the volatility of the firm’s cash flows, which determines the amount of noise hiding the agent’s actions. The two problems become linked because profitability cannot be directly observed, but must be inferred from the firm’s reported cash flows.

We show that the optimal contract has two phases: transitory, during which the agent receives no payments and efficiency of the firm’s liquidation decision gradually improves, and stationary, in which payouts are made and the firm’s liquidation threshold is fixed. The dynamics in the stationary payout phase is characterized in closed form, even though the termination threshold is generally not first best. Optimal contract dynamics mimic the empirical transition from a young to a mature firm. A key prediction of our model is that payouts, initiated in the mature phase, are smooth relative to the firm’s earnings, with current payouts depending only on the firm’s expected future profitability and not the

\(^2\) Recursive methods to solve for dynamic contracts were pioneered by Rogerson (1985), Spear and Srivastava (1987), Phelan and Townsend (1991) and have been used more recently in Albuquerque and Hopenhayn (2001), DeMarzo and Sannikov (2006), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007a, 2007b), Biais, Mariotti, Plantin and Rochet (2007) and Biais, Mariotti, Rochet and Villeneuve (2010). See also Stokey and Lucas (1989) and Ljungqvist and Sargent (2004).
current earnings “surprise.” Intuitively, a positive earnings surprise increases the firm’s perceived profitability, so it is optimal to use the extra cash to increase financial slack (rather than payouts) in order to postpone liquidation. Because dividend changes reflect permanent changes to profitability, they are persistent and have substantial implications for firm value. Cash balances and the level of financial slack track the expectation of future profitability.

We consider two important settings which differ based on the information rents inherent in the agent’s outside option. In the case in which the firm’s profitability derives purely from the agent’s ability, it is natural to presume that the agent can continue to capture the same information rents even after leaving the firm – i.e. his expected payoff increases with his true ability similarly both inside and outside the firm. Because termination cannot be used to control the agent’s information rents, the principal does not benefit from distorting this decision, and the optimal contract has a simple form we call a first-best targeting contract: payments to the agent are deferred until the first best can be obtained. We show also that the contract can be implemented in a natural way – the agent is compensated with equity, the firm makes payouts only after cash reserves (or financial slack) exceeds a target based on profitability, and the project is liquidated only if reserves are exhausted.

When a component of profitability is firm-specific, however, the agent’s information rents will diminish after leaving the firm. In that case, termination can be used to control the agent’s information rents, and we show that the optimal contract will involve permanent distortions. Again, payments are deferred until the firm matures, but the mature firm will adopt a liquidation threshold that is above the first best. This ultimate liquidation threshold

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3For empirical support for such smoothing, seeLintner (1956), Fama and Babiak (1968), as well as more recent studies by Allen and Michaely (2003) and Brav et al. (2005). Lambrecht and Myers (2012) consider an alternative agency model to derive a smooth payout policy. In their model, long-term incentive contracts are not possible, but managers are disciplined by the threat that shareholders can replace them if they are willing to pay a cost proportional to the value of the firm's equity. To avoid this threat, managers must pay dividends that are proportional to any rents they consume. Dividend smoothing then follows from managers' desire to smooth their own consumption stream. While our long-run dividend process is similar to theirs, their model does not capture the endogenous transition from a non-dividend paying young firm to a mature dividend paying firm, which is a key feature of our analysis.

4This payout policy is therefore also consistent with the evidence that firms’ cash and leverage positions are strongly influenced by past profitability, even when firms are financially unconstrained (see, e.g., Fama and French (2002)).
increases monotonically during the “young” phase, and the increase is more rapid when performance is poor. In effect, the agent is punished for early poor performance with a permanently higher standard for liquidation. Distorting future termination in this way is optimal because it reduces the agent’s information rents early on.

*Relationship to the Literature*

Our application of dynamic agency with learning in a real option environment is both natural and important, and the simple nature of the optimal contract, where we have a closed form solution for valuation and payouts in the mature phase, should prove useful both empirically and as a concrete illustration of the underlying theory. More broadly, the core of our analysis applies to a wide class of environments with persistent asymmetric information. There is a huge interest in recent literature to such problems. Examples include dynamic price discrimination of Battaglini (2005), dynamic optimal taxation of Golosov, Troshkin and Tsyvinski (2013), managerial compensation of Sannikov (2010) and Garrett and Pavan (2012), and a general mechanism design perspective of Athey and Segal (2013) and Pavan, Segal and Toikka (2014). The interest is not surprising – static models of private information have had a huge impact on our thinking (e.g. Akerlof (1970), Spence (1973), etc.) – but ultimately we live in a dynamic world and the question of how private information comes about is of prime economic importance. Unfortunately, with dynamic private information, complexity builds – and tractability falls – quickly in the analysis. As a general, tractable methodology has been slow to emerge, to make progress, researchers have built special environments that yield solvable models.

For example, Bergemann and Hege (1998, 2005) design an attractive environment in which the principal and agent wait for the arrival of a single success. There is learning, as each failure raises the probability that the project is bad, and the agent’s deviations generate private information as in our setting. They focus on short-term contracts, and Horner and Samuelson (2013) take the analysis further by studying equilibria when bargaining power belongs to the principal. Halac, Kartik and Liu (2015) analyze optimal dynamic contracts and combine moral hazard, learning and adverse selection in a related model. In their setting with commitment, they find that only the interaction of all three frictions precludes efficiency. A related family of models considers Markov environments with two states,


Multiple environments, often carefully constructed with the goal of obtaining an analytical solution, make it difficult to identify common themes – e.g. it is difficult to bridge the methods and conclusions from two-state Markov models and with those based on exponential utility. The accessibility of the field is thus limited, since the simplifications employed are particular to a given environment, and do not easily extend broadly.

In this paper we elaborate a framework that we hope will have broader applicability, thereby enhancing our common understanding of these environments, and bringing it closer to the richness required for empirical work. In particular, our analysis develops according to the following structure: (1) a definition of information rent as the local sensitivity of payoffs to private information (see Myerson (1981)), and an expression of local incentive constraints in terms of information rents, (2) a recursive representation of information rents as a stochastic expectation, (3) a characterization of the distortions that arise in the optimal control of information rents, and (4) a verification method for global incentive compatibility in the conjectured optimal contract or equilibrium. Of these milestones, our theoretical innovations concern particularly the last two, e.g. we develop methods to prove analytically the properties of the principal’s value function that allow for a sharp characterization of distortion accumulations via a multiplier on information rents. In our setting, this multiplier has a natural representation as a long-run termination threshold.

We view these four points as unifying threads in this literature. While the precise form of the incentive constraint will differ depending on whether the agent reports a hidden endowment, distorts the principal’s beliefs via cash diversion, or invests in unobservable human capital, information rent invariably enters the constraint, and conceptually can be defined in any environment with private information. We can make use of a recursive
representation for information rents regardless of whether we design contracts or analyze Markov equilibria (which is why we combined both strands of literature in the discussion above). Likewise, verification of global incentive compatibility can involve conditions on how information rents evolve – see Williams (2011), Sannikov (2014) or Cisternas (2014) – regardless of whether we design long-term contracts with commitment or analyze short-term contracts. Only point (3), the optimal control of information rents, is specific to a setting in which some commitment is available (e.g. in dynamic contracts, but not Markov equilibria). The direction of the resulting distortions depends on how information rents enter the incentive constraint – e.g. whether they weaken incentives as in our setting, or strengthen them as in Sannikov (2014) – but nonetheless this step shares common features since solutions to control problems have common properties regardless of what kind of a control problem it is.

We have taken care to elaborate these steps not because they are new, but because they are often obscured. It is possible to write the incentive constraint without defining information rents, and all too tempting to plug the equilibrium value of information rents into the incentive constraint up front. Doing so, however, obscures the relationship between models with different types of information rents. Likewise, many papers in literature do not draw a conceptual distinction between the agent’s and principal’s optimization; while completely correct and efficient in a given environment, this lack of common structure makes it difficult for a consistent framework to emerge.

Indeed, there appears to be a “tradeoff” in the literature between highly structured models with sharp characterizations of the optimal contract but little general structure, and general papers which develop a recursive structure but do not always produce clean characterizations. Papers such as Kapicka (2013) and Farhi and Werning (2013) are fairly clear about the recursive structure, and they are also quite general. Likewise, Fernandes and Phelan (2000) is a classic paper that identifies recursive structure in general settings

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5 Equilibrium models of information rents generally appear separate, as they do not anticipate the control of information rents through contract design. E.g. see career concerns of Holmstrom (1999) or the “ratchet effect” of Laffont and Tirole (1988). See also Baron and Besanko (1987), and Freixas, Guesnerie, and Tirole (1985), for related results.
where the first-order approach may fail, but one which is difficult to use to derive analytical properties of the optimal contract.

We follow this approach in structuring our analysis. We characterize the solution of the agent’s problem using methods also employed by Williams (2008), but our biggest innovations are in characterizing the properties of distortions in the optimal contract. Generally, “distortions” is an abstract term, but in our application it acquires a very concrete meaning in terms of an adjustment of the termination threshold away from first best. The contract given a fixed level of distortions is characterized in closed form. Thus, we identify a simple representation of complex state variable dynamics in the optimal contract. We are able to prove analytically key properties of the optimal contract – e.g. monotonicity of the accumulation of distortions – by exploring properties of the principal’s value function. One implication is that the optimal contract exhibits *irreversibility*, as poor short-term performance has effects that cannot be reversed by good performance later on. This result distinguishes our model from standard dynamic agency models in which sufficiently high current performance can “make up” for early shortfalls.6

Finally, we provide convenient *on equilibrium path* conditions to verify global incentive compatibility. In this way, we improve upon existing literature, which suggests verification procedures based on off-equilibrium paths. For example, Pavan, Segal, Toikka (2014) develop an integral monotonicity condition, and Farhi and Werning (2013) verify their contract by computing the agent’s off-equilibrium path value function explicitly. We can formulate our on-path conditions because we were able to explicitly characterize an upper bound on the agent’s off-path payoff (i.e. after deviations). Moreover, our sufficiency condition is broad enough to allow us to verify not just the optimal contract, but also other simpler contracts.

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6 In standard agency models without learning, a single variable – the agent’s continuation value – fully summarizes his performance. Good performance resets the agent’s continuation value, independently of early outcomes (see, e.g. DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007a, 2007b) and Sannikov (2008)). Even in models with learning, it is common that sufficiently high current outcomes resolve any belief discrepancy and thus “erase” any long-run consequences of early outcomes (see, e.g., Tchistyi (2013), Kwon (2013), and Faingold and Vasama (2014)).
The closest papers to ours are Prat and Jovanic (2014), He et al. (2014), and Tchistyi (2014). The first two papers are in an exponential utility framework and there is no endogenous termination of the contract. In Prat and Jovanic, effort costs are linear and the agent’s ability is constant and so learning converges over time; their focus is on how learning makes commitment less valuable – the volatility of the agent’s wage is higher as a result of information rents – and that surplus improves with tenure as uncertainty about ability declines. He et al. consider (as we do) a stationary learning environment, but with convex effort costs. They show that an optimal contract controls information rents by distorting future effort downward. Tchistyi (2014) shares our risk-neutral setting in which termination of the agent is a vital aspect of the contract. In his model profitability follows a two-state Markov process; he focuses on an implementation with “performance pricing,” in which termination is accelerated by charging an increasing interest rate on profit shortfalls.

The paper is organized as follows. In the next section, we present the model of an agency problem in a standard real option environment, and in Section 2 we present the first-best solution of the real option problem alone. Section 3 defines information rents and expresses a local incentive compatibility constraint in terms of information rents. There is value to reducing the agent’s information rents – this helps incentive compatibility – and so Section 4 characterizes optimal contracts that constrain the agent’s information rents. We also show that with a sufficiently high payoff to the agent, which depends on information rents, we get the first-best contract. Section 5 analyzes the particularly tractable “pure private ability” environment, in which the principal cannot control the agent’s information rents – in this case distortions are temporary and eventually first-best is attained. In Section 6, we allow for firm-specific profitability and analyze the irreversible, dynamic distortions the principal will use to reduce the agent’s information rents. We discuss comparative statics and present examples in Section 7. Section 8 concludes, and all proofs are in the appendix.
1. The Model

We consider a setting in which the firm is managed by a single agent, governed by a set of outside investors which we may think of collectively as “the principal.” The firm’s true profitability is changing and unknown, and current output provides information regarding future profits. In addition, there is a standard moral hazard problem within the firm in which the agent/manager chooses hidden effort at each moment, and may reduce effort and sacrifice expected output in order to enjoy private benefits. In this section we build a continuous time model that captures these effects, and formulate the optimal contracting problem.

1.1. Learning

Firm fundamentals at date $t$ are determined by the parameter $\delta_t^*$, which is related to the agent’s managerial skill and the firm’s productivity. The true value of $\delta_t^*$ is never known with certainty, but the principal and the agent both believe at time 0 that $\delta_0^* \sim N(\delta_0, \gamma_0)$. As a result of changing market conditions, the firm’s fundamentals evolve over time according to

$$d\delta_t^* = \sigma' dZ_t' \quad \text{with} \quad \sigma' > 0.$$

The parameter $\delta_t^*$ affects the firm’s cash flows according to

$$dX_t = (\delta_t^* - a_t)dt + \sigma dZ_t^*$$

where $a_t \geq 0$ is the agent’s unobservable action and $Z_t^*$ and $Z_t'$ are independent Brownian motions. We interpret $\delta_t^*$ as the efficient profit rate of the firm, and $a_t$ as the extent to

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7 Throughout the paper, we will adopt both “principal-agent” and “investor-manager” terminology, depending on context.

8 For a consideration of the role of learning about profitability and its importance for firm valuation, see Pastor and Veronesi (2003).

9 We can, for example, interpret $\delta_t^*$ as capturing the suitability of the agent’s managerial style and skills to current market conditions.
which the agent pursues private benefits (for example, by shirking or cash flow diversion). Volatility $\sigma > 0$ captures transient shocks to the firm’s cash flows.

Given actions $a_t$ by the agent, cash flows $X_t$ can be used as a signal to form beliefs about $\delta_t^*$. By the linear Kalman-Bucy filter, the belief about firm fundamentals based on information up to time $t$ is $\delta_t^* \sim N(\delta_t, \gamma_t)$, where

$$d\delta_t = \left(\gamma_t / \sigma^2\right)(dX_t - (\delta_t - a_t)dt) \quad \text{and} \quad d\gamma_t = \left(\sigma^2 - \gamma^2_t / \sigma^2\right)dt \quad (1)$$

For simplicity we focus the steady state of this problem, in which $\gamma_0 = \gamma_t = \sigma\sigma'$. In that case, (1) reduces to

$$d\delta_t = \nu(dX_t - (\delta_t - a_t)dt), \quad (2)$$

with $\nu = \sigma' / \sigma$. For the remainder of our analysis, the key parameters are the cash flow volatility $\sigma$ and the sensitivity $\nu$ of the profitability estimate (i.e. the optimal Kalman gain). The volatility of profitability is then given by $\sigma' = \nu\sigma$.

### 1.2. Moral Hazard and Off-Equilibrium Path Beliefs

We assume that both the principal and the agent are risk-neutral. We view the principal as representing outside investors with unlimited wealth, whereas the agent has no initial wealth.¹⁰ Both the agent and investors discount the future at rate $r > 0$, and the agent’s consumption cannot be negative. The agent’s limited liability prevents a general solution to the moral hazard problem in which the firm is simply sold to the agent.

When the agent works for the firm, there is moral hazard. We interpret the agent’s unobservable action $a_t \geq 0$ as the extent to which the agent diverts his own effort and resources from the firm to a preferred outside activity for private benefit. If $a_t > 0$, the agent gets a private benefit at rate of $\lambda a_t$. We assume it is more efficient to use the agent’s and

¹⁰ The assumption that the agent has no initial wealth is without loss of generality; equivalently, we can assume the agent has already invested any initial wealth in the firm.
firm’s resources inside the firm to generate profit rather than for the agent’s private benefit and thus $\lambda \in (0,1)$.

On the equilibrium path, the principal infers the firm’s fundamentals $\delta_t$ from her anticipation of the agent’s equilibrium incentives and actions. On path, the principal has the same beliefs about fundamentals as the agent, and from the perspective of both

$$dZ_t = \left( dX_t - (\delta_t - a_t) dt \right) / \sigma$$

is a Brownian motion, and beliefs

$$d\delta_t = v \sigma dZ_t$$

follow a martingale. However, the agent could deviate from the strategy that the principal expects him to follow. If so, then the principal incorrectly interprets the signals, leading to belief asymmetry off the equilibrium path. This hypothetical scenario matters for the agent’s incentives to deviate.

To understand the evolution of beliefs off the equilibrium path, consider the agent’s deviation from an equilibrium effort strategy $\{a_t\}$ to a different effort strategy $\{\hat{a}_t\}$. Then the principal will update her belief about firm fundamentals incorrectly according to (2), while the agent, who knows his past deviations, forms beliefs according to

$$d\hat{\delta}_t = v \left( dX_t - (\hat{\delta}_t - \hat{a}_t) dt \right).$$

(4)

Given the agent’s deviation of $\{\hat{a}_t\}$, the belief divergence evolves according to

$$d(\hat{\delta}_t - \delta_t) = v \left( (\hat{a}_t - a_t) - (\hat{\delta}_t - \delta_t) \right) dt.$$ (5)

In the absence of further deviations, the principal gradually learns about true fundamentals, i.e. the belief divergence decays exponentially at rate $v$. Alternatively, the agent can maintain the current belief divergence indefinitely by continuing to deviate by the amount $(\hat{\delta}_t - \delta_t)$. 


1.3. Outside Options

Investors contribute any required initial capital and in exchange receive the cash flows generated by the firm less any compensation paid to the agent. When the agent leaves the firm, the investors receive an expected final payoff of $\hat{L}(\delta, \hat{\delta})$, which we can interpret as a liquidation value, or alternatively, the payoff from replacing the agent. Note that while in principle the liquidation value can depend on the investors’ and agent’s beliefs, in equilibrium $\delta = \hat{\delta}$ and so it is sufficient to specify $L(\delta) \equiv \hat{L}(\delta, \hat{\delta})$.

When the agent departs, his outside option may also depend on both market beliefs about the firm’s fundamentals $\delta$ as well as his own estimate of profitability $\hat{\delta}$. Again, these estimates will be equal in equilibrium, but in this case, because we need to determine the agent’s incentive to deviate, it is important to specify also how the agent’s outside option will change if $\hat{\delta} \neq \delta$.

In general, if some component of the firm’s profitability is agent-specific and transferable, then the agent’s outside option can depend both on the public perception $\delta$ – which might, for example, determine his initial outside wage – as well as the agent’s potentially superior information $\hat{\delta}$ – which might impact his future performance and opportunity for advancement. We denote the agent’s outside option by $\hat{R}(\delta, \hat{\delta})$, which we assume to be linear.

In our analysis, we distinguish two important cases. First, with pure private ability, we assume $\delta$ represents the agent’s skill, and once fired, the agent can continue to earn the same benefit from his private information, so that $\partial \hat{R} / \partial \hat{\delta} = \lambda / r$. 11 Alternatively, when profitability is at least somewhat firm-specific, we assume the value of the agent’s private information is diminished once he leaves the firm, so that $\partial \hat{R} / \partial \hat{\delta} = (\lambda - \psi) / r$, where $\psi \in (0, \lambda]$ captures the degree of firm-specificity of $\hat{\delta}$.

11 E.g. the agent can continue to engage in his preferred outside activity post-termination, enjoying similar private benefits as when he was shirking within the firm.
Combining these, we have the following general representation for the agent’s outside option:

\[ \hat{R}(\delta, \hat{\delta}) = R(\delta) + \frac{\lambda - \psi}{r}(\hat{\delta} - \delta), \quad \psi \in [0, \lambda]. \]  \hspace{1cm} (6)

In this representation, \( R(\delta) \) represents the equilibrium outside option for the agent, when \( \delta = \hat{\delta} \). The agent may enjoy an additional gain from having private information, but only to the extent that it is not firm-specific.\(^{12}\)

Finally we make the following technical assumptions regarding the equilibrium outside options:

i) \( R(\delta) \) is linear, and \( R'(\delta) \in [0, \lambda / \nu + (\lambda - \psi) / r] \),

ii) \( R(\delta) + L(\delta) \) is linear, and \( R'(\delta) + L'(\delta) \in [0, 1 / r] \). \hspace{1cm} (7)

The assumption of linear outside options greatly simplifies our analysis without changing any qualitative results. Moreover, because the optimal contract will in general depend only on the outside option in a very narrow range, a linear approximation is likely to be reasonably accurate. The upper bound on the slope of the total liquidation value assures that liquidation is inefficient when profitability is high. Finally, we introduce an upper bound on the slope of the agent’s outside option; otherwise, when the agent’s outside option is too steep, the principal’s problem is no longer incentive provision but rather insuring the agent against market forces (see e.g. Grochulski and Zhang (2014)).

1.4. The Contracting Problem

The agent’s compensation is determined by a long-term contract. This contract specifies, based on the history of the firm’s cash flows, non-negative compensation for the agent while he manages the firm, as well as a rule determining if and when the agent is “fired.”

\(^{12}\) We will show that the only relevant deviations will involve \( \delta > \hat{\delta} \), so that private information will always be positive (it is therefore enough to assume linearity only in this range).
Formally, a contract is a pair \((c, \tau)\), where \(c\) is a non-negative \(X\)-measurable process that represents the agent’s compensation and \(\tau\) is an \(X\)-measurable stopping time.\(^{13}\)

A contract \((c, \tau)\) together with an \(X\)-measurable effort recommendation \(a\) optimally gives the agent an expected payoff of \(W_0\) if it maximizes the principal’s profit

\[
E \left[ \int_0^\tau e^{-rt} \left( \delta_t - a_t - c_t \right) dt + e^{-r\tau} L(\delta_\tau) \right] \tag{8}
\]

subject to

\[
W_0 = E \left[ \int_0^\tau e^{-rt} (\lambda a_t + c_t) dt + e^{-r\tau} R(\delta_\tau) \right], \tag{9}
\]

\[
W_t \equiv E \left[ \int_t^\tau e^{-r(s-t)} (\lambda a_s + c_s) ds + e^{-r(\tau-t)} R(\delta_\tau) \big| \delta_t \right] \geq R(\delta_t), \tag{10}
\]

and

\[
W_0 \geq E \left[ \int_0^\tau e^{-rt} (\lambda \hat{a}_t + c_t) dt + e^{-r\tau} \hat{R}(\delta_\tau, \hat{\delta}_\tau) \right] \text{ for any other strategy } \hat{a} \tag{11}
\]

In constraint (10), the variable \(W_t\) is called the agent’s continuation payoff. Contracts \((c, \tau)\) together with effort recommendations \(a\) that satisfy (10) are called individually rational. Contracts \((c, \tau)\) with effort recommendations \(a\) are called incentive-compatible if they satisfy (10) and (11).\(^{14}\) Note that because the agent’s problem is time-consistent, a

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\(^{13}\) We allow the principal to make lump-sum payments to the agent, but since those need not be used except at time 0, we write payments as flows to simplify notation. More generally, we could write compensation through a cumulative process \(\{C_t\}\), where \(dC_t = c_t dt\) when payments are continuous. Also for simplicity, we assume that compensation and termination are determined by the cash flow process, ruling out public randomization. We will later verify that public randomization does not improve the contract.

\(^{14}\) In the incentive compatibility constraint (11) we do not include the agent’s option to quit before time \(\tau\) because in the optimal contract the agent will always prefer to wait to be fired instead of quitting voluntarily. In particular, Lemma A in the Appendix shows that \(a_t = 0\) in the optimal contract. Given that, any belief divergence \(\hat{\delta}_t - \delta_t\) cannot be negative. Then the continuation strategy \(\hat{a}_t = a_t + \hat{\delta}_t - \delta_t\) maintains the divergence at its current level and gives the agent a payoff that exceeds the agent’s outside option:
profitable deviation after any history could be incorporated into \( \hat{a} \), and hence it is sufficient to write the incentive compatibility constraint (11) as of date 0 only.

Note that by varying \( W_0 > R(\delta_0) \), we can consider different divisions of bargaining power between the agent and the investors. For example, if the agent enjoys all the bargaining power due to competition among investors, then the agent will receive the maximal value of \( W_0 \) subject to the constraint that the investors’ payoff be at least equal to their initial investment. Alternatively, if investors have all of the bargaining power, then \( W_0 \) will be chosen to maximize (8).

2. The First-Best Solution

Before beginning our analysis of the contracting problem, it is useful to consider the first-best solution in this environment. Suppose effort is observable so that there is no agency problem. Then, because \( \lambda < 1 \), it is optimal to set \( a_t = 0 \) until liquidation. It remains to determine the optimal threshold \( \delta_L \) at which to liquidate. Because liquidation is irreversible, it is optimal to trigger liquidation when the expected profitability \( \delta_r \) reaches a critical level \( \delta_L \) that is below the “myopic” level \( \delta_m \), at which cash flows equal the interest on the total liquidation value (see (7) and Figure 1).

\[
W_r + E \left[ \int e^{-ct} \lambda (\hat{\delta}_r - \delta_r) dt + e^{-ct} \frac{\lambda - \psi}{r} (\hat{\delta}_r - \delta_r) \right] \geq R(\delta_r) + \frac{\lambda - \psi}{r} (\hat{\delta}_r - \delta_r) = \hat{R}(\delta_r, \hat{\delta}_r).
\]
Figure 1: First-best Liquidation Threshold and Value Function

\( r = 4.5\% , \ \nu \sigma = 0.27 , \ R + L = 37.5 + 5.6 \delta , \ \delta_w = 2.3 \)

This problem is a standard “abandonment option” problem whose solution is well known (see e.g. Dixit and Pindyck (1994)). To solve it, let \( \tau_L \) be the first time \( \delta_i = \delta_L \). Given stopping time \( \tau_L \), the total surplus generated by the project is

\[
V(\delta_0) \equiv E \left[ \int_0^{\tau_L} e^{-r \tau_L} d\tau_L + e^{-r \tau_L} \left( L(\delta_{\tau_L}) + R(\delta_{\tau_L}) \right) \right]
\]

\[
= \frac{\delta_0}{r} + E \left[ e^{-r \tau_L} \right] \left( L(\delta_L) + R(\delta_L) - \frac{\delta_L}{r} \right)
\]

We can calculate the terminal discount factor \( E[e^{-r \tau_L}] \) in closed form as:\(^{15}\)

\[
E \left[ e^{-r \tau_L} \right] = \exp \left( -\sqrt{2r} \left( \frac{\delta_0 - \delta_L}{\nu \sigma} \right) \right).
\]  \( (12) \)

Thus, the first-best liquidation threshold, which maximizes the total surplus \( V \), is given by

---

\(^{15}\) Given the law of motion (3), the terminal discount factor is the unique solution to the HJB equation \( rf(\delta) = \frac{1}{2} \nu^2 \sigma^2 f''(\delta) \) with boundary conditions \( f(\delta_L) = 0 \) and \( f(\infty) = 0 \).
\[ \delta_L = \arg \max_{\delta} \exp \left( \frac{\sqrt{2r}}{\nu \sigma} \delta \right) \left( L(\delta) + R(\delta) - \frac{\delta}{r} \right). \]  

(13)

It is straightforward to show that \( \delta_L < \delta_m \) and that \( \delta_L \) increases with \( r \) and decreases with volatility \( \nu \sigma \). With \( L + R \) linear, we have the explicit solution

\[ \delta_L = \delta_m - \frac{\nu \sigma}{\sqrt{2r}}. \]

As Figure 1 makes clear, the option to abandon the project increases its first-best value relative to the standard setting in which profitability is fixed and there is no learning.

For the remainder of our analysis we make the natural assumption that

\[ \delta_0 > \delta_L > 0. \]  

(14)

That is, initial productivity is sufficiently high that it is worthwhile to operate the firm, but if cash flows are disappointing it will be optimal to liquidate the firm before it becomes unprofitable (though this second assumption is mainly for convenience and is not crucial to the analysis).

### 3. Incentive Compatibility

Before we attempt to characterize the optimal contract, in this section we develop further the requirements of incentive compatibility. To provide incentives in our setting, we not only need to offset the direct private benefits the agent will obtain from deviating, but because the agent will become privately informed in the event he deviates, we must also offset his potential information rents. In this section we evaluate the magnitude of these information rents and develop a necessary first-order condition for incentive compatibility.

Recall that the agent receives private benefit \( \lambda \in (0,1] \) per dollar of lost output when he shirks or diverts cash flow. Because this activity is not efficient, it is natural to restrict attention to contracts in which effort is always optimal; that is, \( \{a_t = 0\} \). As we show formally in the appendix (see Lemma A), this restriction is without loss of generality: Given any contract with \( a_t > 0 \), there is an alternative contract in which \( a_t = 0 \) and the
private benefits $\lambda a$, the agent would have received are replaced with direct payments from the principal. In this way the agent’s payoff and incentives are unchanged, and the principal’s payoff (weakly) improves. Technically, this result stems from the linearity of the agent’s private benefits and the unbounded action space $a \in [0, \infty)$, which together guarantee that the agent’s incentives are unaltered by this transformation.

Given a contract, the agent’s continuation payoff $W_t$ represents the expected utility the agent will receive on the equilibrium path conditional on the history of realized and reported cash flows up to date $t$. In order to provide incentives to the agent, this continuation payoff must be sensitive to the firm’s reported cash flows. We denote by $\beta_t$ the strength of these incentives:

$$\beta_t \equiv \frac{dW_t}{dX_t}.$$ 

Note that raising $\beta_t$ will make it more costly for the agent to deviate from $a = 0$ and reduce output, thus increasing the agent’s incentives. On the other hand, raising $\beta_t$ also increases the risk borne by the agent. Given the agent’s limited liability, raising the agent’s risk increases the probability of inefficient termination. This tension is the fundamental tradeoff the optimal contract must resolve.

In order to determine the minimal power of incentives that is required for incentive compatibility, consider the agent’s potential gains from shirking. In our setting, if the agent shirks he may gain in two distinct ways. First, he receives an immediate private benefit from the deviation. Second, he distorts the principal’s beliefs about his productivity. This second effect creates the opportunity for the agent to earn an additional information rent based on his superior information. In order to characterize the incentive-compatibility conditions in our setting, we must calculate the magnitude of this information rent.

Given the contract $(c, \tau)$, the agent’s payoff (on or off the equilibrium path), as a function of his private information, can be written as

$$\hat{W}_t(\hat{\delta}_t) \equiv \max_a E_t^a \left[ \int_t^\tau e^{-r(t-s)} (\lambda \hat{a}_s + c_s) ds + e^{-r(t-\tau)} \hat{R}(\hat{\delta}_\tau, \hat{\delta}_t) \bigg| \hat{\delta}_t \right].$$
The agent’s information rent is then defined as his marginal benefit from having incrementally better private information $\hat{\delta}$ relative to the principal’s belief $\delta$:

$$\xi_t \equiv \frac{d}{d\delta} \hat{W}_t(\hat{\delta}) \bigg|_{\delta=\delta_t}.$$

To see how the information rent impacts incentives, suppose the agent shirks and reduces the rate of cash flows by $\$1$ for an instant ($dt$). Doing so provides an immediate private benefit of $\lambda dt$. At the same time, the reduction in firm output causes his “on-equilibrium path” continuation payoff $W_t$ to fall by $\beta_t dt$. But because the reduction in reported cash flows lowers the principal’s beliefs regarding productivity, after the deviation the agent’s beliefs exceed those of the principal’s by $\nu dt$ (see (5)). Thus, his true continuation payoff $\hat{W}_t(\hat{\delta}_t)$ (taking into account the belief discrepancy) goes down by only $(\beta_t - \xi_t \nu) dt$. Hence, in order for the agent not to shirk, we must have $\lambda dt \leq (\beta_t - \xi_t \nu) dt$, or equivalently

$$\beta_t \geq \lambda + \xi_t \nu.$$

The constraint (15) is necessary for incentive compatibility, and thus puts a lower bound on the sensitivity $\beta_t$ of any incentive compatible contract.

Our model differs from standard principal-agent setting due to the absence of common knowledge about the future distribution of output. In standard settings, the agent does not have any information rents, and thus $\xi_t = 0$ and (15) becomes $\beta_t \geq \lambda$ (see, for example, DeMarzo and Sannikov (2006)). Another difference is that while in the standard setting this constraint guarantees incentive compatibility, in our model (15) is necessary but need not be sufficient in general. Nonetheless, we will follow a “first-order approach” and replace the full set of incentive constraints (11) with the necessary first-order condition (15). We will conjecture the optimal contract based on just the necessary incentive constraints, and then verify full incentive compatibility.

The existing literature utilizes a similar first-order approach – see e.g. Werning (2001), Farhi and Werning (2013), Kapicka (2013). There as well, the validity of the approach requires conditions on the agent’s off equilibrium payoff function, such as static monotonicity (Theorem 3 in Pavan, Segal and Toikka (2013) and Theorem 5 in Kapicka
Nevertheless, the typical approach to check these properties is through a numerical procedure, as in Kapicka (2013) or Farhi and Werning (2013). Williams (2013) finds a condition on the contract using a quadratic bound on the agent’s payoff off the equilibrium path. In our setting, we are able to verify the validity of the first-order approach through conditions on the primitives in special cases (pure private ability benchmark or contracts that attain first best). In general, we identify a natural set of conditions on equilibrium payoffs (that are easily checked) which establish global incentive compatibility for a broad class of contracts.

Of course, in order to use the IC constraint (15), we must calculate the agent’s information rent $\xi_t$. The following proposition formalizes (15), and both characterizes and provides a useful lower bound for the agent’s information rent. The proof follows the approach of Williams (2013). The derivation of the first-order incentive conditions, as well as the laws of motion of continuation value and information rents, are based on the stochastic maximum principle.

**Proposition 1 (Necessary IC Condition and Recursive Relaxed Problem).** Without loss of generality, we can restrict attention to incentive compatible contracts with efficient actions ($a_t = 0$). The incentives provided by the contract can be represented by the sensitivity $\beta_i$ of the agent’s continuation value to reported output shocks, and the agent’s continuation value has the recursive representation

$$dW_t = rW_t dt - c_t dt + \beta_i \left( dX_t - \delta_i dt \right)$$

(16)

under strategy $\{a_t = 0\}$. The agent’s information rent $\xi_t$ can be expressed as

$$\xi_t = E_t \left[ \int_t^\tau e^{-(r+\nu)(\tau-s)} \beta_s ds + e^{-(r+\nu)(\tau-t)} R_i(\delta_s, \delta_t) \right]$$

(17)

and has the recursive representation under strategy $\{a_t = 0\}$ of

$$d\xi_t = (r + \nu)\xi_t dt - \beta_t dt + \gamma_t \sigma dZ_t$$

(18)
for some process $\chi_t$. Finally, the IC constraint (15) is a necessary condition for the strategy $\{a_t = 0\}$ to be optimal, and we have the following lower bound for the agent’s information rents:

$$\xi_t \geq \zeta_t \equiv \frac{\lambda}{r} - E_t \left[ e^{-r(T-t)} \psi \right]$$

which holds with equality if the IC constraint (15) is binding at all future dates.

Proof. See Appendix.

Equation (16) is a standard representation which states that the promised payoff $W_t$ grows at the rate of interest less any direct payments to the agent, and is adjusted for the surprise in output according to $\beta_t$.

Equation (17) characterizes the agent’s information rent. To understand it, note that from (16) the agent gains $\beta_t(\hat{\delta}_t - \delta_t)$ from being more profitable than expected in period $t$. Equation (17) then follows from (5), which states that the agent’s informational advantage decays at rate $\nu$, and therefore we should discount these gains at rate $r + \nu$. Equation (18) then follows as a standard representation.

Proposition 1 provides the recursive structure for what the literature on the first-order approach calls the principal’s relaxed problem (see Werning (2001) or Kapicka (2013)). In this problem, the full set of incentive compatibility constraints (11) is replaced by just the first-order conditions (15) which can be expressed in terms of the recursive state variables $(\delta_t, W_t, \xi_t)$. The relaxed problem can be treated as a control problem with the motion of $(\delta_t, W_t, \xi_t)$ characterized by recursive equations (2), (16) and (18). The drifts of $W_t$ and $\xi_t$ reflect the promise-keeping constraints, which perform accounting for the promised amounts that are yet to be delivered. Intuitively, an optimal solution to the relaxed problem provides, at each date, the highest possible expected profit to the principal subject to current fundamentals $\delta_t$, the agent’s promised payoff $W_t$, and the constraint that information rents not exceed $\xi_t$. This solution will coincide with the optimal contract as long as the first-order approach is valid; i.e. as long as global incentive compatibility holds.
Finally, (19) provides a natural lower bound for the information rent. To see why it must hold, suppose the agent knows his true profitability exceeds the principal’s expectation by \( \varepsilon \). If the agent deviates by choosing \( \hat{a}_t = \varepsilon \) from then on, his effective profitability will match the principal’s expectation, and by (5), the belief discrepancy will be preserved. The agent will gain \( \lambda \varepsilon \) from this deviation each period until termination, at which point he will gain \( (\lambda - \psi)\varepsilon / r \), which has present value given by (19). Because this payoff is for a particular deviation, the agent’s true payoff, and thus his information rent, will exceed it (or equal it if the IC constraint is always binding).

Equation (19) reveals that when \( \psi > 0 \), it is possible to lower the agent’s information rents by terminating the contract earlier, as the agent will forfeit the firm specific component of productivity. We will show in Section 6 that an optimal contract will exploit this possibility by accelerating termination in order to control the agent’s information rent in earlier periods.

In contrast, in the pure private ability setting with \( \psi = 0 \), the agent is able to exploit his private information even after termination. In that case, as we will show in Section 5, the information rent will be independent of the termination rule and equal to \( \lambda / r \). Because the agent’s information rents are fixed, the problem simplifies to just two state variables \((\delta_t, W_t)\), which will allow for a very simple characterization of the optimal contract.

4. Fixed Liquidation Contracts

In our setting, a key characteristic of any contract is the set of states for which the contract is terminated. Recall that in the first best, termination optimally occurs when \( \delta_t \) falls to \( \delta_L \). In this section, we consider a family of incentive compatible contracts with a fixed liquidation boundary – that is, contracts which terminate as soon as \( \delta_t \) drops to some fixed threshold \( \delta^* \). While not optimal in general, these simple contracts are interesting because, as we will show, they maximize the principal’s profit given a target agent payoff \( W \) and information rent \( \xi \). As a result, they will be part of a fully optimal contract for any initial
Finally, they show how, for large enough \( W_0 \), the first best can be obtained in our setting despite agency costs.\(^{16}\)

Consider a contract with a fixed termination boundary \( \delta^c \). In order for the contract not to terminate before \( \delta_t \) reaches \( \delta^c \), the agent’s continuation payoff must remain above his outside option until that time. In addition, the agent’s incentive constraint (15) together with the dynamics in (2) imply that \( W_t \) must fall with \( \delta_t \) according to

\[
\frac{dW_t}{d\delta_t} = \frac{dW_t}{dX_t} \cdot \frac{d\delta_t}{dX_t} \geq \frac{\lambda + \xi_t}{\nu} = \bar{\lambda} / \nu + \bar{\xi}.
\]

(20)

Now, using the bound on the agent’s information rent (19), we have

\[
\bar{\xi}_t \geq \frac{\lambda}{r} - E_t [e^{-r(t-t^*)}] = \frac{\lambda}{r} - \exp\left(-\sqrt{2r\left(\frac{\delta_t - \delta^c}{\nu \sigma}\right)}\right) \frac{\psi}{r}
\]

where we compute the terminal discount factor as in (12). Combining these observations, we see that to prevent termination prior to \( \delta^c \), the agent’s current continuation payoff must exceed the following bound:

\[
W_t = R(\delta^c) + \int_{\delta_t}^{\delta^c} \frac{dW}{d\delta} d\delta
\]

\[
\geq R(\delta^c) + (\delta_t - \delta^c) \frac{\lambda}{\nu} + \int_{\delta_t}^{\delta^c} \bar{\xi}_t d\delta
\]

\[
\geq R(\delta^c) + (\delta_t - \delta^c) \left( \frac{\lambda}{\nu} + \frac{\lambda}{r} \right) - \frac{\nu \sigma}{r \sqrt{2r}} \left( 1 - \exp\left(-\frac{\sqrt{2r}}{\nu \sigma} (\delta_t - \delta^c)\right)\right)
\]

\[
\equiv W^c(\delta_t, \delta^c)
\]

---

\(^{16}\) The possibility of attaining the first best when the agent’s rents are sufficiently high has been shown in settings without learning; see for example Clementi and Hopenhayn (2006). The result generally depends on the principal and agent sharing a common discount rate and the firm’s cash flow shocks being bounded below. Biais, Mariotti, and Rochet (2011) discuss the importance of the common discount rate and also why it is insufficient in settings with Brownian cash flow shocks. Here we show that with learning and the option to abandon the project, the first best is attainable after sufficiently positive histories despite having Brownian cash flow shocks.
While the above argument provides a necessary restriction on the agent’s payoff in a fixed liquidation contract, the following proposition establishes its sufficiency:

**Proposition 2 (Feasibility of Fixed Liquidation).** If \( W < W^c(\delta, \delta^c) \), there is no incentive-compatible contract that gives the agent a payoff of \( W \) and delays termination until profitability falls below \( \delta^c \). Otherwise, under the condition that

\[
R(\delta^c) \geq \frac{\psi \nu \sigma}{r \sqrt{2r}},
\]

for any \( W \geq W^c(\delta, \delta^c) \) there exists an incentive compatible contract in which the agent receives \( W \) and termination occurs at \( \delta^c \). In this contract, the fastest possible payouts to the agent are a lump sum of \( W - W^c(\delta, \delta^c) \) initially and thereafter continuous payments until termination at rate

\[
c_i = rW^c(\delta, \delta^c) - \frac{\sigma^2 \nu^2}{2}W_{ii}(\delta, \delta^c) = rR(\delta^c) - \frac{\psi \nu \sigma}{\sqrt{2r}} + \lambda \left( 1 + \frac{r}{\nu} \right)(\delta_i - \delta^c) \geq 0.
\]

With this contract, the agent’s information rents are given by

\[
\xi^c(\delta, \delta^c) = \frac{\lambda}{r} - \frac{\psi \nu}{r} \exp \left( -\sqrt{2r} \left( \frac{\delta - \delta^c}{\nu \sigma} \right) \right)
\]

and the principal’s profit, by

\[
b^c(\delta, W, \xi^c) = \frac{\delta}{r} + \exp \left( -\sqrt{2r} \left( \frac{\delta - \delta^c}{\nu \sigma} \right) \right) \left[ L(\delta^c) + R(\delta^c) - \frac{\delta^c}{r} \right] - W.
\]

Moreover, for any \( \delta^* \geq \delta^c \geq \delta_L \) and \( W \geq W^c(\delta, \delta^c) \), this contract maximizes the principal’s profit subject to the constraint that the agent’s information rents must not exceed \( \xi^c(\delta, \delta^c) \).

---

17 Note that we can write the principal’s payoff as a function of \( \xi^c \) rather than \( \delta^c \) by virtue of (24).
Proof. See Appendix. ■

Condition (22) guarantees that the agent’s payments (23) are nonnegative; these payments are set to keep \( W_t = W^c(\delta, \delta') \) at all future dates. Total surplus is calculated as in Section 2 with \( \delta' \) in place of \( \delta_L \), and information rents follow from (19). The most important element of the proof, though, is establishing global incentive compatibility. We do so by showing that

\[
\hat{W}_t(\hat{\delta}, \hat{\delta}) \leq W^c(\delta, \delta') + \int_{\delta_L}^{\delta'} \xi^c(\delta, \delta') d\delta.
\]

That is, the agent’s maximal expected payoff following any strategy is bounded by integrating his local information rents on the interval \((\delta, \hat{\delta})\), immediately implying global incentive compatibility since \( \hat{W}_t(\hat{\delta}, \hat{\delta}) \leq W^c(\delta, \delta') = W_t \). Remarkably, this natural bound allows us to establish global incentive compatibility without fully solving the agent’s problem. A similar bound will be used again in Sections 5 and 6 to establish global incentive compatibility of the optimal contract, even though there we cannot even express the optimal contract in closed form.

Because total surplus depends only on termination, we can attain first-best efficiency by choosing the termination boundary for the fixed liquidation contract to equal the first best boundary \( \delta_L \). We define the corresponding minimal agent rents required as

\[
W^1(\delta) \equiv W^c(\delta, \delta_L).
\]

We then have the following immediate corollary to Proposition 2:

**Corollary (Attaining First Best).** The first-best outcome can be obtained by an incentive compatible contract if and only \( W \geq W^1(\delta) \).

---

\(^{18}\) For example, suppose the agent deviates to induce immediate termination, reducing cash flows by \((\delta, -\delta')/\nu\). Then the agent’s payoff is \( R(\delta', \delta_L) + \lambda(\delta - \delta')/\nu = R(\delta') + (\lambda/\nu + (\lambda - \psi)/r)(\delta_1 - \delta') \leq W^c(\delta, \delta') \), where the inequality is strict if \( \psi > 0 \).
We finish this section by illustrating the boundary $W^1(\delta)$. Consider the setting of Figure 1 with $\lambda = 25\%$. Figure 2 illustrates the “firm-specific profitability” benchmark $W_{fs}^1$ with $\hat{R}(\delta, \delta) = R_{fs} = 7.5$, $L = 30 + \lambda \delta / r$, as well as the “pure private ability” benchmark $W_{pa}^1$ with $\hat{R}(\delta, \delta) = \lambda \delta / r$ and $L = 37.5$. In both cases, $\delta_L = \$1.35$ million.

![First Best Agent Rents](image)

**Figure 2: Minimum First-Best Agent Rents**

(Pure firm-specific profitability $R_{fs}(\delta) = 7.5$, Pure private ability $R_{pa}(\delta) = \lambda \delta / r$)

The agent’s information rents are higher in the pure private ability benchmark, as the principal must give the agent a greater fraction of total surplus to attain first best. When profitability is purely firm specific, on the other hand, the agent loses any information rents in termination, reducing the slope of $W^1$ to $\lambda / \nu$ when $\delta$ is low. For both settings, as $\delta$ rises, the slope of $W^1(\delta)$ converges to $\lambda / \nu + \lambda / r$ and the volatility of the agent’s continuation value converges to $\lambda (1 + \nu / r) \sigma$.

Without learning, the volatility of the agent’s continuation value would be only $\lambda \sigma$. Learning raises the required power of incentives necessary to offset the agent’s information rents. In the case of pure private ability, this information rent is constant and independent of termination, leading to a simple characterization of the optimal contract as we show in the next section. With firm-specific profitability, the optimal contract is more complex, as
the principal can limit the agent’s information rent by accelerating termination. We analyze this case, and the permanent distortions it induces, in Section 6.

5. Pure Private Ability \((\psi = 0)\)

In this section we determine the optimal contract for the case of pure private ability, \(\psi = 0\). In this setting, because the agent is fully able to exploit his private information after leaving the firm, his information rents are unaffected by termination. We will characterize the optimal contract and show that payments will be delayed until either the first best is attained or the agent is fired. Early termination (prior to payouts) is inefficient, but the degree of inefficiency declines monotonically with the agent’s tenure. Finally, we show how the optimal contract can be implemented via the firm’s payout policy, and show that in this implementation, dividends are smooth relative to the firm’s cash flows.

5.1. Optimal Contract

We consider the setting with \(\psi = 0\). In the simplest case, \(R(\delta) = \lambda \delta / r\) (which implies \(\hat{R}(\delta, \hat{\delta}) = \lambda \hat{\delta} / r\)), but our analysis applies for general linear \(R\). We know from Proposition 2 that if

\[
W_t \geq W^1(\delta_t) = R(\delta_t) + (\delta_t - \hat{\delta}_t) \left( \frac{\lambda}{\nu} + \frac{\lambda}{r} \right),
\]

then the first best can be attained and thus it must be the optimal contract. If \(W_t = R(\delta_t)\) then the contract terminates immediately and the agent will take his outside option. It remains to determine the optimal contract providing the agent with an expected payoff interior to these two extremes, \(W_t \in (R(\delta_t), W^1(\delta_t))\), as shown in Figure 3.
Figure 3: Optimal Contract Dynamics

An optimal contract should minimize the potential inefficiency from “early” termination, that is, termination with $\delta_t > \delta_L$. Note, however, that equation (19) from Proposition 1 implies that the agent’s information rents are at least $\lambda/r$ for any incentive compatible contract. Then, using (20), the agent’s sensitivity to the firm’s profitability satisfies

$$\frac{dW_t}{d\delta_t} \geq \lambda / \nu + \xi_t \geq \lambda / \nu + \lambda / r.$$ 

(28)

As a result, a negative shock to fundamentals will cause $W_t$ to fall, and could lead to termination as early as $\delta(W_t, \delta_t)$, defined graphically in Figure 3, and implicitly by

$$W_t = R(\delta_t) + (\lambda / \nu + \lambda / r)(\delta_t - \delta_t).$$

(29)

Starting from any $(W_t, \delta_t)$, it is possible to implement the fixed liquidation contract for any $\delta^c \in [\delta_t, \delta_t]$. But as long as $\delta_t > \delta_L$ this contract will not be optimal – rather than initiating cash payments to the agent, we can do better by withholding payouts and increasing $W_t$ instead. This increase will cause $\delta_t$ to fall, permitting a more efficient termination policy.

In other words, rewriting the law of motion (16) for $W_t$ as
\[ dW_t = (rW_t - c_t)dt + \beta_t (dX_t - \delta_t dt) = (rW_t - c_t)dt + \frac{dW_t}{d\delta_t} d\delta_t \]  (30)

suggests that the chance of inefficient liquidation can be minimized by

- reducing the volatility of \( W_t \) until the incentive constraint binds
- withholding payments \( c_t \) until \( W_t \) grows to \( W^I(\delta_t) \) and first best is attained

We refer to a contract that satisfies the above as a first-best targeting contract. We show that the optimal contract in the pure private ability case has exactly this form. Along the path, \( W_t \) responds to profitability shocks by moving along the diagonal line illustrated in Figure 3, while at the same time drifting upward at the rate of interest. This upward drift implies that the termination threshold \( \hat{\delta}_t = \hat{\delta}(W_t, \delta_t) \) declines monotonically, until either \( \delta_t = \hat{\delta}_t \) and thus \( W_t = R(\delta_t) \) and the contract is terminated, or \( \delta_t = \delta_L \) and thus \( W_t = W^I(\delta_t) \) and the first best is attained.

To verify this conjecture, we show that this contract, first, is globally incentive compatible and, second, maximizes the principal’s payoff over all incentive compatible contracts.

**Proposition 3 (Full Incentive Compatibility).** Suppose \( W_0 \in (R(\delta_0), W^I(\delta_0)) \). Consider a contract in which \( W_t \) follows (16) with \( c_t = 0 \) and \( \beta_t = \lambda + \nu \lambda / r \) (so that the incentive constraint always binds) until \( W_t = R(\delta_t) \) and the contract is terminated, or \( W_t = W^I(\delta_t) \) and the first best is attained. This contract is fully incentive compatible and at each date \( t \) the agent’s expected future payoff is \( W_t \).

**Proof.** See Appendix. ■

The proof of Proposition 3 shows that the agent’s off-equilibrium payoff is given by

\[ \hat{W}(\hat{\delta}_t) = W_t + \frac{\lambda}{r} (\hat{\delta}_t - \delta_t) \]  (31)

regardless of the strategy he follows. Intuitively, the agent is indifferent among all strategies because his incentive constraint always binds, and \( \beta_t \) and \( \xi_t \) are constant, on and
off the equilibrium path. Notice that the form of the off-equilibrium payoff \( \tilde{W}(\tilde{\delta}_t) \) coincides with the bound (26). Here the bound is obtained exactly, because with pure private ability the agent’s information rents remain constant (independent of termination).

Next we must show that this contract is optimal, in the sense that no other incentive compatible contract with the same expected payoff for the agent achieves a higher expected payoff for the principal (investors). We can write the payoff under our proposed contract as

\[
b(\delta_0, W_0) = \frac{\delta_0}{r} + E\left[ e^{-\tau} \left( R(\delta_\tau) + L(\delta_\tau) - \frac{\delta_\tau}{r} \right) \right] - W_0,
\]

where the (possibly inefficient) termination time \( \tau \) is now determined by the condition \( W_\tau = R(\delta_\tau) \). Recall from (30) that in the interior region where \( W_\tau \in (R(\delta_\tau), W^1(\delta_\tau)) \), because \( c = 0 \) and the incentive constraint binds, the agent’s continuation value evolves according to

\[
dW_i = rW_i dt + \frac{dW_i}{d\delta_i} d\delta_i = rW_i dt + (\lambda / \nu + \lambda / r) d\delta_i
\]

(32)

Because \( d\delta_i \) has volatility \( \nu \sigma \) (see (3)), the function \( b(\delta, W) \) can be found by solving the following parabolic partial differential (HJB) equation

\[
r b(\delta, W) = \delta + rWb_{\nu W}(\delta, W) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b(\delta + \nu z, W + (\lambda / \nu + \lambda / r)z)\bigg|_{z=0} \tag{33}
\]

starting from the first-best boundary,

\[
b(\delta, W) = V(\delta) - W \quad \text{for } W \geq W^1(\delta),
\]

up to the termination boundary,

\[19\] For example, if the agent were to deviate and divert cash flows immediately, he could divert the amount \( (\delta_i - \tilde{\delta}_i) / \nu \) before termination would be triggered, for a total payoff of \( \hat{R}(\tilde{\delta}_i, \delta_i) + \lambda (\delta_i - \tilde{\delta}_i) / \nu = R(\delta_i) + (\lambda / \nu + \lambda / r)(\delta_i - \tilde{\delta}_i) = W_i \).
\[ b(\delta, W) = L(\delta) \quad \text{for} \quad W = R(\delta). \]

Our next result verifies that the function \( b \) does indeed characterize the investors’ payoff under the benchmark contract, and that no other contract can do better.

Proposition 4 (Optimality of the Benchmark Contract). The benchmark contract attains profit \( b(\delta_0, W_0) \) and is the optimal contract when \( \psi = 0 \); that is, any alternative individually rational and incentive compatible contract attains at most \( b(W_0, \delta_0) \).

Proof. See Appendix. ■

The core of the proof is verification that the local incentive constraint must bind, i.e. the value function cannot be improved upon by choosing \( \beta > \lambda(1 + v / r) \). The verification argument makes use of two key properties of \( b(\delta, W) \), specifically

\[
\frac{\partial}{\partial z} b_w(\delta + vz, W + (\lambda / v + \lambda / r) vz) \leq 0 \quad \text{and} \quad b_{ww} \leq 0. \tag{34}
\]

The latter property is the usual concavity of the value function (in \( W \)). The former, somewhat less intuitive, property says that the marginal cost of compensating the agent, which is \(-b_w\), increases after positive cash flow surprises. Together, these properties imply that choosing \( \beta > \lambda(1 + v / r) \) would be suboptimal in the HJB equation. We develop a method based on the stochastic representation of the partial differential equation for \( b_w \) to prove these properties. Conditions analogous to these will also play a role in the general setting of Section 6, where we use them not only to verify that the incentive constraint must bind, but also to show that distortions related to information rents accrue monotonically in general over the life of the contract.

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20 The marginal cost of raising the agent’s payoff is at most 1 because the agent can be paid directly. Before attaining the first best it is strictly less than 1 because surplus is improved by increasing the agent’s rent and avoiding inefficient termination. The condition therefore states that the marginal inefficiency from the moral hazard problem increases with negative cash flow shocks.

21 The methodology developed in the proof of Proposition 4 has already proved useful elsewhere: Faingold and Vasama (2014), in a related setting that also combines moral hazard with real options, adapt our approach to verify their optimal contract.
Finally, we conclude with the following formalization of the dynamics illustrated in Figure 3: that over time, the efficiency of the contract improves, with $W_t$ approaching $W^1(\delta_t)$ and $\delta_t$ approaching $\delta_L$ monotonically.

**Proposition 5 (Dynamics of the Benchmark Contract).** Under the benchmark contract, $W^1(\delta_t) - W_t \rightarrow 0$ and $\delta_t \rightarrow \delta_L$ monotonically. Specifically, we have

$$d\left[W^1(\delta_t) - W_t\right] = -r W_t \, dt \leq 0,$$

and

$$d\delta_t = \frac{-r W_t}{\lambda / \nu + \lambda / r - R'} dt \leq 0.$$

Finally, if $R(\delta_L) > 0$, then given $(W_0, \delta_0)$ there exists a finite time $T$ such that if the contract survives until date $T$ then the first best must be attained.

**Proof.** See Appendix. □

To summarize, the optimal contract in the benchmark case does not pay the agent until the first-best can be attained, which occurs within finite time. Prior to that date, there is a risk of inefficient termination, but the degree of inefficiency – as measured by the level of profitability in excess of $\delta_L$ at the time of termination – declines monotonically over time.

### 5.2. Implementation

To provide additional intuition regarding the benchmark contract, we develop in this section a simple implementation of the optimal contract in which the agent is compensated via an appropriate equity share.\(^{22}\) Our goal in doing so is both to provide additional intuition regarding the benchmark contract, as well as to demonstrate that it can be supported via a natural, empirically consistent payout policy for the firm. In our implementation, the firm maintains an internal cash reserve. If the cash balance is ever depleted, the agent is terminated. If the cash balance grows to exceed a target based on the

\(^{22}\) As in any contracting setting, the implementation is not unique, as different structures that lead to the same ultimate payments to the agent will provide equivalent incentives and payoffs.
firm’s profitability, then the first best is obtained and the firm begins paying dividends from which the agent receives a share.

We develop the implementation for the specific outside option for the agent in which after termination, the agent simply receives private benefits based on his ability:

\[ R(\delta) = \lambda \frac{\delta}{r} \text{ and } \psi = 0, \text{ or equivalently, } \hat{R}(\delta, \hat{\delta}) = \lambda \frac{\hat{\delta}}{r}. \]  

(35)

We can motivate this case as one in which the agent shirks by spending time in an alternative activity, to which he reverts full time if terminated.\(^2\)

Consider a firm that holds cash \( M_t \) and pays dividends at rate \( dD_t \). Then its cash balance will evolve according to

\[ dM_t = rM_t dt - dD_t + dX_t. \]  

(36)

Suppose in addition that the agent receives a share \( \lambda \) of any dividends paid, by holding that fraction of the firm’s equity. Finally, suppose that the agent forfeits this equity in the event of his termination, which occurs whenever the cash balance is depleted, \( M_t = 0 \). We show below that this inside equity provides the agent with appropriate incentives both not to shirk and to choose a payout policy that implements the optimal contract.

**Proposition 6 (Implementing the Benchmark Contract).** Suppose the agent holds a fraction \( \lambda \) of the firm’s equity, forfeited upon termination, which occurs when the firm’s cash balance falls to 0. Then it is incentive compatible for the agent to choose \( a_t = 0 \) and to adopt the payout policy

\[
 dD_t = \begin{cases} 
 0 & \text{if } t < T_t \\
 (rM_t + \delta_t) dt & \text{if } t \geq T_t 
\end{cases}
\]  

(37)

where \( T_t \) is the first date such that

\[ M_t = M^1(\delta_t) \equiv \frac{(\delta_t - \delta_L)}{\nu}. \]  

(38)

\(^2\) In an early working paper version we extend the implementation to a general function \( R \); the results are qualitatively similar with a modification to the maximal and minimal level of cash reserves.
Doing so implements the optimal benchmark contract with \( W_t = \lambda (M_t + \delta_t / r) \).

Proof: See Appendix. ■

The dividend payout policy in Proposition 6 has a natural interpretation. The firm refrains from paying dividends before it accumulates a target level of cash \( M^1(\delta_t) \) which increases with the firm’s productivity. In this state, the firm is financially constrained, as it may be liquidated inefficiently (i.e. the firm will exhaust its cash balance with \( \delta_t > \delta^t \)). Once the firm reaches the target level of cash, it becomes mature. Thereafter, the firm pays out dividends at the rate of its expected earnings,

\[
dD_t = rM_t + \delta_t,
\]

where the first term represents interest on cash reserves, and the second term represents expected cash flows. The firm’s cash balance changes only in response to earnings surprises, i.e. when market beliefs about the firm’s fundamentals change. A positive surprise leads to an improvement in expected fundamentals, and also a corresponding rise in the firm’s cash balance, so that the firm’s cash balance remains sufficient (i.e. \( M_t = M^1(\delta_t) \)) to avoid inefficient termination in the future.

While we have not required it, in our implementation the firm is self-financing using its own internal cash reserves. This property is convenient, and reminiscent to the implementation in Biais et al. (2007) in a pure moral hazard context with bounded shocks. Alternatively, the firm could implement this same financial slack through debt or a credit line, for example.24

In addition, while we have chosen to focus on the fastest possible payout policy, the firm could delay dividend payments relative to payout policy in (37) at no cost (the firm earns interest \( r \) on its cash balances). Nonetheless, there may be other reasons (not modeled here)

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24 One might wonder whether our payout policy is robust to the agent borrowing from a third party to avoid termination when the cash balance approaches zero. We assume the agent cannot pledge the firm’s fixed assets or reported cash flows to such a third party, but he could pledge his future compensation or offer to divert cash flows. Nonetheless, it is straightforward to show that because the agent only recoups or diverts \( \lambda \) for each dollar contributed, a risk neutral third party with discount rate \( r \) could not profit by entering into such an arrangement with the agent.
for the firm not to maintain cash balances in excess of what is required (see e.g. Jensen (1986)).

A remarkable feature of this implementation is the distinction between the agent’s sensitivity to output versus firm value. While his sensitivity to output depends on his potential information rents – and thus also on the learning rate $\nu$ as shown in the necessary IC constraint – his sensitivity to firm value depends only on the agency parameter $\lambda$, which determines his equity share. The reason for this difference is that firm value is also sensitive to current output due to learning about future profitability. Specifically, a $1 increase in output today increases both the firm’s current cash by $1 and its expected future profitability by $\nu$. Because the agent owns a share of the firm, his expected payoff increases as a result of both effects, and the combined impact is sufficient to provide just the right sensitivity to overall output to provide appropriate incentives.

### 5.3. Dividend Smoothing

Figure 4 shows the contract dynamics for a parameterized example. Until time 1.5 the firm has cash balances below the efficient level, and it stands the risk of being liquidated inefficiently. However, in this example inefficient liquidation does not happen. At time 1.5 the firm’s cash level reaches the efficient target, and the firm initiates dividends. Dividends continue until the firm’s profitability falls sufficiently and it is liquidated (optimally) at date 5. Because the firm’s cash flow surprises are absorbed by its cash reserves, total quarterly dividends are significantly smoother than earnings, as shown in the right panel of Figure 4. This “smoothing” of firm payouts follows from the fact that a positive cash flow surprise increases the perceived profitability of the firm, making it desirable to increase financial slack and postpone termination, whereas a negative surprise makes abandonment more likely and reduces the need for financial slack. Because earnings surprises are absorbed by financial slack, payouts depend only on the firm’s expected earnings, as shown in (23).
While we illustrate the dividend smoothing result here for the case of pure private ability, we will see shortly the identical result holds more generally -- it arises fundamentally from the nature of the underlying real option problem, rather than the specific details of the agency problem. The agency problem creates the friction which makes the timing of payouts meaningful. See Section 7 for additional empirical implications and comparative statics.

6. Firm-Specific Profitability ($\psi > 0$)

In this section we analyze the optimal contract when $\delta$ represents, at least in part, firm-specific profitability, so that $\psi > 0$ and the agent has a diminished benefit from his private information outside the firm. In that case, as shown in equation (19) of Proposition 1, the principal can reduce the agent’s information rents through early termination. We show that, in contrast to the setting with pure private ability, the optimal contract is characterized by the accumulation of distortions. Specifically, when the agent underperforms in the short run, the principal commits to raise the minimum termination threshold permanently, reducing the agent’s information rents, and thus relaxing the incentive constraint. This distortion gives rise to an irreversibility property of the optimal contract: early underperformance has permanent effects on the agent’s future that cannot be undone by performing well later.
This section is organized as follows. We start by explaining how permanently raising the minimal termination threshold lowers information rents, and how this can improve the contract relative to one which targets the first-best termination boundary, as in Section 5. After that, we discuss the principal’s value function and explain how the properties of the optimal contract can be derived. Then we illustrate contract dynamics using an example. Having established our main qualitative results, the section ends by elaborating the more technical details of how we can prove key properties of the principal’s value function analytically, and how we verify the validity of the first-order approach.

6.1. Constraining Information Rents

As we have seen, inefficiencies arise in our setting when the principal seeks to keep a greater share of the total surplus. Providing the agent with an expected payoff below \( W^1(\bar{\delta}) \) necessitates the possibility of inefficient termination.

When \( \psi > 0 \), the benchmark contract of Section 5 remains fully incentive compatible: deviations increase the likelihood of termination, and termination is less attractive to the agent. It is no longer optimal, however, because incentives are now stronger than necessary. By relaxing incentives (until the incentive constraints bind), we can reduce the volatility of the agent’s continuation payoff and therefore lower the risk of inefficient termination.

What is the optimal way to relax incentives? The relationship between information rents and termination – see (19) of Proposition 1 – leads to a subtle trade-off in the optimal contract. As we will see below, inefficient termination in the short run can be alleviated by controlling the agent’s information rents through a commitment to less costly termination distortions in the long run.

We begin to illustrate this tradeoff in Figure 5, where we show, starting from \((\delta_0, W_0)\), trajectories for \( W_t \) in response to shocks to \( \delta \) for three alternative contracts. Under the benchmark contract for Section 5, the trajectory is linear with slope \((\lambda/\nu + \lambda/r)\). While this contract is incentive compatible, the incentive constraint (15) is slack because with \( \psi > 0 \) the agent’s information rents are strictly below \( \lambda/r \). In the first-best targeting
contract, we make the incentive constraint bind, but otherwise follow the logic of the benchmark contract – we delay payouts until the first best payoff $W^1(\delta)$ is attained. There we can see the immediate improvement as the trajectory becomes curved, with initial slope $(\lambda / \nu + (\lambda - \psi) / r)$ at the termination boundary, and rising asymptotically to match previous contract. Efficiency is clearly improved, as the volatility of $W_t$ falls and the potential for inefficient termination is reduced.

\[ W^1(\delta) \]

---

**Figure 5: Contract Dynamics for Alternative Contracts**

Finally, the third trajectory represents the fixed-liquidation contract of Proposition 2, in which the termination threshold is raised to $\delta^c$ such that $W^c_0 = W^c(\delta_0, \delta^c)$ and payouts begin immediately. Again, the trajectory has initial slope $(\lambda / \nu + (\lambda - \psi) / r)$, but the slope rises more slowly as delta increases because termination is accelerated relative to the first-best targeting contract, lowering information rents and relaxing the incentive constraint. The figure reveals that by committing to a termination threshold $\delta^c > \delta_L$, we can reduce the volatility of $W_t$ that is required to maintain incentives. This benefit, however, comes at the cost of less efficient termination in the long run. But for $\delta^c$ close to $\delta_L$, the loss will
be second order, and so we should expect that when \( \psi > 0 \), the optimal contract will involve some distortion of the long-run termination threshold.

Figure 6 illustrates these effects in terms of the principal’s profits. At the bottom is the benchmark contract from Section 5, with the incentive constraint slack. We can improve upon this by tightening the incentive constraints. For example, even the crude fixed-liquidation contracts of Section 4, in which payouts begin immediately, do better than leaving the incentive constraint slack. The first-best targeting contract does even better, as it defers payments to allow for more efficient termination as the agent’s continuation value improves. Finally, we will see that the optimal contract does even better by combining all three mechanisms: (i) maintaining tight incentive constraints, (ii) deferring payments to improve the efficiency of termination in the short run, and (iii) converging ultimately to a fixed-liquidation contract with \( \delta^* > \delta_L \) (with \( \delta^* \) determined by past-performance in a way we will describe shortly) as a means to control information rents.

![Figure 6: Principal’s Profit for Alternative Contracts](image-url)
In the analysis that follows, we will see how we can characterize precisely this optimal contract. We will show that, as in our contract from Section 5, there is an initial “young” phase, in which there are no payouts, and binding incentive constraints determine the highest profitability \( \delta_t \) at which termination may occur. During the young phase, this threshold \( \delta_t \) drifts downward over time, as the deferral of the agent’s payment increases the expected total surplus. At the same time, however, there is a second process \( \delta^c_t \) which determines the minimum profitability at which termination will occur. Thus cutoff starts at \( \delta_L \) and rises over time, capturing accumulated permanent distortions that serve to lower the agent’s information rent. Payouts begin, and the firm matures, when \( W_t = W^c(\delta_t, \delta^c_t) \) and thus \( \delta^c_t = \delta_t \), at which point the contract is a stationary, fixed liquidation contract from that point onward. Payouts from then on follow (23) in Proposition 2:

\[
c_t = rR(\delta^c) - \psi \frac{\nu \sigma}{\sqrt{2r}} + \lambda \left( 1 + \frac{r}{\nu} \right) (\delta_t - \delta^c).
\]

Note that only the fixed component of payouts is affected by the presence of firm specific profitability, with the level declining with both \( \psi \) and the accumulated distortions \( \delta^c \). Payouts are still smooth relative to earnings, linear in expected profitability with the same variability as in Section 5.3.

### 6.2. The Principal’s Value Function

We write the principal’s profit function as \( h(\delta, W, \xi) \). Note that unlike in Section 5, we must also include as an argument the agent’s information rent. When \( \psi = 0 \), this argument was unnecessary because the agent’s information rent was constant at \( \lambda / r \), independent of the contract. With \( \psi > 0 \), however, controlling the agent’s information rent becomes important. While the principal’s problem (8) does not constraint the agent’s information rent upfront, in the recursive formulation, at any future date the contract must be optimal subject to the promised continuation payoff and information rent of the agent.

Given the laws of motion of the state variables in Proposition 1, the principal’s profit function \( h(\delta, W, \xi) \) must satisfy the HJB equation
\[ rb(\delta, W, \xi) = \max_{c \geq 0, \beta, \lambda, \nu, \xi} \delta - c + (rW - c)b_2(\delta, W, \xi) + ((r + \nu)\xi - \beta)b_1(\delta, W, \xi) \]
\[ + \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b(\delta + \nu z, W + \beta z, \xi + \chi z) \bigg|_{z = 0}, \]

(39)

where the last term is the second derivative along the direction of the volatility trajectories of the state variables \((\delta, W, \xi)\). As in (34) in Section 5, \(b(\delta, W, \xi)\) satisfies several natural properties: Concavity in the promised states \(W\) and \(\xi\) (see Lemma D in the appendix), and an increasing marginal cost of compensation after positive cash flow surprises (proved in Section 6.5):

\[-\frac{\partial}{\partial z} b_2(\delta + \nu z, W + (\lambda + \nu\xi)z, \xi + \chi z) \bigg|_{z = 0} \geq 0 \]

(40)

with strict inequality as long as \(-b_2 < 1\). 25

Property (40) will be key to deriving important characteristics of the optimal contract, and verifying that the incentive compatibility constraint must always bind. We derive these characteristics in the next subsection.

### 6.3. Properties of the Optimal Contract

The main properties of the optimal contract are as follows. Distortions imposed to reduce the agent’s information rents accrue monotonically, but only until the contract reaches the payout regime. At that point, the contract takes the form of a fixed liquidation contract, with threshold \(\delta^*\) determined by the accumulated distortions. The monotonic buildup of distortions leads to irreversibility: distortions caused by poor initial performance cannot be undone by later good performance.

Distortions are captured by the multiplier \(\phi = b_2(\delta, W, \xi)\) on information rents, with an increase in the multiplier corresponding to a tighter restriction on the agent’s future information rents. To give the multiplier a concrete representation, we can associate it with

---

25 Recall that the marginal cost of compensation is at most 1, because the agent can be paid cash immediately. It is strictly less than 1 whenever raising the agent’s continuation payoff will help to avoid inefficient termination. Thus, (40) says that the marginal inefficiency from lowering the agent’s surplus increases as we approach termination.
the fixed termination boundary which would be optimal with that multiplier; that is, we can define $\delta^\phi(\phi)$ as the solution to

$$\phi = b^c_3(\delta, W, \xi^c(\delta, \delta^\phi))$$

(41)

Given our closed form solution to $b^c$ in (25), equation (41) is equivalent to

$$\delta^\phi(\phi) \equiv \delta_L + \phi \frac{\psi}{1 - r(L' + R')}.$$  

(42)

Because the fixed-liquidation contract $b^c$ is the best possible contract given information rent $\xi^c$ and $W > W^c$ by Proposition 2, once $W^c(\delta, \delta^c(\phi))$ the termination boundary remains fixed and payouts begin.

We consider next the evolution of this multiplier.

1. Initially there are no distortions. Indeed, the principal chooses initial $\xi_0$ to solve

$$\max_{\xi_0} b(\delta_0, W_0, \xi_0).$$

Thus, $\phi_0 = 0$ and $\delta_0^c = \delta_L$.

2. The multiplier $\phi_t$ has no volatility, only drift. Indeed, the first-order condition of (39) with respect to $\chi$ is

$$0 = \frac{\partial}{\partial \chi} \frac{\partial^2}{\partial z^2} b(\delta + v_z, W + \beta z, \xi + \chi z)\bigg|_{z=0} = \frac{\partial^2}{\partial z^2} z b_3(\delta + v_z, W + \beta z, \xi + \chi z)\bigg|_{z=0}$$

$$= 2 \frac{\partial}{\partial \xi} b_3(\delta + v_z, W + \beta z, \xi + \chi z)\bigg|_{z=0}.$$  

(43)

Thus $\phi_t$ does not change along the trajectories of the volatilities of $(\delta_t, W_t, \xi_t)$, and so by Ito’s lemma the volatility of $\phi_t$ is 0.

3. When $\beta_t = \lambda + \nu \xi_t$, then the drift of $\phi_t$ can be evaluated by differentiating the HJB equation with respect to the state $\xi$, substituting $\beta_t = \lambda + \nu \xi_t$ and using the Envelope theorem to obtain
drift of \( b_1(\delta, W, z) = \text{drift of } \phi \)

\[
(rW - c)b_{23}(\delta, W, \xi) + (r\xi - \lambda)b_{33}(\delta, W, \xi)
\]

\[
+ \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b_3(\delta + vz, W + (\lambda + v\xi) z, \xi + \chi z)_{z=0}
\]

\[
= -\frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} \left( vz \right)_{z=0}
\]

\[
= -\sigma^2 v \frac{\partial}{\partial z} b_2(\delta + vz, W + (\lambda + v\xi) z, \xi + \chi z)_{z=0}.
\] (44)

Therefore, given (40), the drift of \( \phi \), is non-negative (and strictly positive if \( -b_2 < 1 \)) and hence the distortions to the contract to control the agent’s information rents accumulate monotonically over time.

4. The incentive constraint \( \beta = \lambda + v\xi \) binds while \( -b_2 < 1 \). Indeed, differentiating the HJB equation with respect to \( \beta \), we get

\[
-b_1(\delta, W, z) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b_2(\delta + vz, W + \beta z, \xi + \chi z)_{z=0}
\]

\[
= -\phi + \sigma^2 \frac{\partial}{\partial z} b_2(\delta + vz, W + \beta z, \xi + \chi z)_{z=0}.
\] (45)

Because \( \phi \geq 0 \), given (40), this derivative is negative and so the incentive constraint binds.

5. Combining points 3 and 4 above, we see that the incentive constraint binds and \( \dot{\phi} > 0 \) until \( -b_2 = 1 \), which occurs exactly when \( W_t \) reaches \( W^c(\delta, \delta^c(\phi_t)) \). From then on, the principal’s profit equals \( b^c(\delta, W_t, \xi_t) \), the termination threshold becomes permanently fixed at \( \delta^c(\phi_t) \), and the continuation contract with the fastest payout path is given by the fixed liquidation contract.26

We formally summarize these properties in the following proposition. The proof in the appendix fills in the details omitted in the discussion above.

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26 There is, obviously, non-uniqueness since payments to the agent can be postponed at no cost.
Proposition 7 (*Evolution of Information Rents*). Suppose that condition (40) holds for the value $\chi$ that solves (43). Then in the optimal contract the incentive constraint must bind as long as $-b_2(\delta, W, \xi) < 1$. The multiplier $\phi_t \equiv b_3(\delta, W, \xi)$ starts at 0 and increases monotonically (but not deterministically) at rate

$$\frac{d\phi_t}{dt} = -\sigma^2\nu\frac{\partial}{\partial z} b_2(\delta + \nu z, W + (\lambda + \nu \xi) z, \xi + \chi z)|_{z=0} > 0,$$

and the agent receives no payments until either termination occurs when $W_t$ reaches $R(\delta_t)$ or payouts begin when $W_t$ reaches $W^c(\delta_t, \delta^c(\phi_t))$ and $-b_2(\delta_t, W_t, \xi_t)$ reaches 1. Once $W_t$ reaches $W^c(\delta_t, \delta^c(\phi_t))$, the multiplier $\phi_t$ as well as the termination threshold $\delta^c(\phi_t)$ become fixed from then on, and the continuation contract with the fastest payouts is given by fixed liquidation contract of Proposition 2.

**Proof.** See Appendix. ■

### 6.4. Illustrating Contract Dynamics

The above results suggest that it is convenient to transform the state space from $(\delta, W, \xi)$ to $(\delta, \delta^c, \delta^c)$. We can do so because information rent $\xi$ can be mapped to the multiplier $\phi$ (by concavity), which maps to the long-run termination threshold $\delta^c$. Finally, $\delta^c$ corresponds, as in Figure 3, to the temporary termination threshold, which is the highest profitability at which termination may occur in the contract, that is, it is the point at which the current trajectory of $(\delta_t, W_t)$ hits the termination boundary in the event of a sudden cash flow drop. The advantage of this alternative parameterization of the state space is that $\delta^c$ and $\delta^c$ have monotone drift and no volatility, whereas $\delta_t$ is a martingale with volatility and no drift. To summarize,\(^{27}\)

(1) $\delta_t$ is the exogenous fundamental representing productivity;

---

\(^{27}\) When we use this parameterization, we denote by $W(\delta, \delta^c, \delta^c)$, the continuation value, $\xi(\delta, \delta^c, \delta^c)$, his information rent, $\dot{\delta}(\delta, \delta^c, \delta^c)$ and $\dot{\delta}^c(\delta, \delta^c, \delta^c)$ the rates of changes of $\delta$ and $\delta^c$ as depicted in Figure 7.
(2) $\delta_\tau \in [\delta_L, \delta_r]$ captures moral hazard distortions as in Section 5: termination is triggered when $\delta_\tau$ falls to $\delta_r$. Under mild conditions, $\delta_\tau$ decreases monotonically over the course of the contract due to the growth in $W_t$.

(3) $\delta_\tau^c \in [\delta_L, \delta_r^c]$ represents distortions due to the control of information rents: $\delta_\tau^c$ starts at $\delta_L$ and increases monotonically until $\delta_\tau^c = \delta_r$, at which point payouts are initiated and the contract reaches steady state.

While moral hazard distortions represented by $\delta_\tau$ weaken over time (as in the contract of Section 5), the information rent distortions represented by $\delta_\tau^c$ tighten over time. When $\delta_\tau^c$ and $\delta_\tau$ meet, the termination threshold becomes permanent and the contract takes the form of Proposition 2.

The left panel of Figure 7 shows how the trajectories of $(\delta, W)$ given $\delta$ vary with $\delta^c$. Higher $\delta^c$ reduces information rents and so flattens the trajectory (which lowers the volatility of $W$). As shown in the right panel, moral hazard distortions (measured by $\delta$) weaken faster after good performance, when $(\delta, W)$ is away from the termination boundary and so the drift of the continuation value $W$ is higher. They also weaken faster when $\delta^c$ is higher, as information rents are reduced. In contrast, information rent distortions (measured by $\delta^c$) are tightened sharply only when termination becomes imminent ($\delta$ near $\delta_\tau$), especially when $\delta^c$ is low, as illustrated in the center panel of Figure 7.28

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28 Recall that the cost of raising $\delta^c$ is second-order at $\delta_L$. Also, while not illustrated, the rate of change of $\delta_\tau^c$ falls as $\delta_\tau$ decreases because the slope $dW / d\delta$ becomes greater (even though $W$ and its drift rise slightly).
The termination threshold in the payout regime depends on performance during the initial stage. Unlike in the pure private ability case, the contract now displays a strong form of hysteresis: poor outcomes at early stages in the contract life have a permanent impact on its long-run efficiency. This result holds because it is optimal to commit to a permanent sacrifice of efficiency in order to reduce information rents after bad outcomes in early stages. Interestingly however, and unlike the pure private ability case, the sample paths in Figure 8 demonstrate that in this setting poor early performance (as shown in the right hand panel versus the left) may trigger a faster initiation of payouts, albeit at a higher termination threshold and therefore permanently reduced cash flows.
Figure 8: Sample Paths for the Three State Variables \((\delta, \delta_l, \delta_r)\).

Given two exogenous paths for profitability \(\delta_t\), the panels show the corresponding evolution of the termination boundary \(\tilde{\delta}_t\) and the permanent distortion \(\delta_c^e\).

\((r = 5\%, \nu = 1, \sigma = 1, L = 100, \hat{R}(\delta) = 100 + 0.2\delta / r, \delta_l = 9.33)\)

6.5. Proving Key Properties of the Value Function

In this section we prove (40), the key property of the principal’s value function, which states that the marginal cost of compensating the agent increases after positive cash flow shocks. Our method of proof is illustrated in Figure 9. First, we define \(h(\delta, \delta, \delta^c)\) to equal the marginal cost of compensation, \(-b_2(\delta, W, \xi)\), under our alternative representation of the state space. Recall that payouts are initiated when \(\delta = \delta^c\). Because the marginal cost to the principal of a $1 payout to the agent is $1, \(h(\delta, \delta, \delta^c) = 1\).
Next consider the termination boundary, $\delta = \delta$. Intuitively, we expect the marginal cost of compensation $h$ to fall along this boundary as $\delta$ rises, as the gain in surplus from increasing the agent’s compensation will be higher with higher profitability. We establish this result below:

**Lemma (Boundary Condition).** The marginal cost of compensation $h(\delta, \delta^c, \delta^r)$ is decreasing in $\delta$ for $\delta \geq \delta^r$.

*Proof.* See Appendix.

Now that we have established that the marginal cost of compensation $h$ is constant along the payout boundary, and decreasing along the termination boundary, we next demonstrate...
(40): \( h \) must be increasing along any trajectory. To do so, we use a new (to our knowledge) proof technique that may be useful in other multidimensional control problems.

We begin with the observation that the marginal cost of compensating the agent, \( h \), must be a martingale; this result is equivalent to the usual “inverse Euler equation” (see e.g. Rogerson (1985) and Spear & Srivastava (1987)) and can be derived by differentiating the HJB equation (39) with respect to \( W \) (see Lemma C in the appendix). Because beliefs \( \delta \) are a martingale, and \( \delta \) and \( \delta' \) have no volatility, the martingale property for \( h \) implies (by Ito’s lemma) the following partial differential equation:

\[
0 = \delta h_2(\delta, \delta, \delta') + \delta' h_3(\delta, \delta, \delta') + \frac{1}{2} \sigma^2 v^2 h_{11}(\delta, \delta, \delta').
\]  

(47)

We know from (42) and (44) that the drift of \( \delta' \) is

\[
\delta' = \Phi \frac{\psi}{1-r(L'+R')} = -\frac{\psi}{1-r(L'+R')} \sigma^2 v \frac{\partial}{\partial Z} b_2(\delta+vz, W + (\lambda + \nu \xi_z) z, \xi + \chi Z) \bigg|_{z=0},
\]

\[
= \psi \frac{\sigma^2 v^2}{1-r(L'+R')} h_1(\delta, \delta, \Phi)
\]

and (40) is equivalent to showing that \( h_1 \) is positive. Using the fact that \( \delta'(\delta, \delta, \delta') < 0 \) (the positive drift of \( W \) causes the threshold \( \delta_\ast \) to move down), and substituting for \( \delta' \), we can rewrite the PDE (47) as

\[
h_2 = -\frac{h_3}{\delta} \delta' = -\frac{1}{2} \frac{\sigma^2 v^2}{\delta} h_{11} = \left( -\frac{\psi \sigma^2 v^2}{1-r(L'+R')} \frac{h_3}{\delta} \right) h_1 - \frac{1}{2} \frac{\sigma^2 v^2}{\delta} h_{11}
\]

(48)

where \( \hat{\mu}(\delta, \delta, \delta') \equiv -\frac{\psi \sigma^2 v^2}{1-r(L'+R')} \frac{h_3}{\delta} \) and \( \hat{\sigma}^2(\delta, \delta, \delta') \equiv -\frac{\sigma^2 v^2}{\delta} > 0 \).

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29 For additional intuition, note that we should also expect \( h(\delta, \delta, \delta') \) to converge to 1 as \( \delta \to \infty \) and the likelihood of early termination disappears.
Equation (48) can be thought of as a PDE in two variables \((\delta, \delta^c)\), for a fixed value of \(\delta^c\), on \(\delta \geq \delta^c \geq \delta^*\). The boundary conditions are

\[
h(\delta, \delta^c, \delta^*) = 1 \quad \text{and} \quad h(\delta, \delta^c, \delta^c) \leq 1 \quad \text{and decreasing in} \quad \delta \quad \text{for} \quad \delta \geq \delta^*.
\]

(49)

The following proposition implies condition (40).

**Proposition 8 (Marginal Cost of Compensation Increases with Cash Flows).** With boundary conditions (49), the solution to equation (48) satisfies \(h(\delta, \delta^c, \delta^c) > 0 \) for \(\delta > \delta^c\). Thus, the marginal cost of compensation increases with positive cash flow shocks and (40) holds.

**Proof.** If process \(x_t\) follows

\[
dx_t = \mu(x_t, k - t, \delta^c) dt + \sigma(x_t, k - t, \delta^c) dZ_t,
\]

(50)

then \(h(x_t, k - t, \delta^c)\) is a martingale by (48). Given the stopping time \(\tau\) when either \(x_t\) hits \(k - t\) or \(k - t\) reaches \(\delta^c\), then

\[
h(x_0, k, \delta^c) = E\left[h(x_\tau, k - \tau, \delta^c)\right].
\]

Now, if \(x^1_t\) is another process that follows (50) with \(x_0 < x^1_0\), then obviously \(x_t < x^1_t\) for all \(t > 0\). Let \(\tau^1\) be the corresponding stopping time for process \(x^1\). Then, \(\tau \leq \tau^1\). Therefore, as long as \(h(x_0, k, \delta^c) < 1\), given boundary condition (49),

\[
h(x_0, k, \delta^c) = E\left[h(x_\tau, k - \tau, \delta^c)\right] < E\left[h(x^1_\tau, k - \tau^1, \delta^c)\right] = h(x^1_0, k, \delta^c)
\]

as required. \(\blacksquare\)

---

\(^{30}\) We have ordered our results for ease of presentation, but the following summary may be prove useful. We started with \(b\) defined as the solution of (39) with \(\beta_i\) set to \(\lambda + \nu \xi_i\). Effectively, we impose the incentive constraint to be binding – and we use this fact when we invoke (44) to get the drift of \(\delta^*\). All the properties of the value function \(b\) are proved under this definition, i.e. the proofs of both Lemma (Boundary Condition) and Lemma D in the appendix (the joint concavity of \(b\) in \(W\) and \(\xi\)) impose the restriction \(\beta_i = \lambda + \nu \xi_i\). The proof of Proposition 7 verifies that the contract cannot be improved by setting \(\beta_i > \lambda + \nu \xi_i\), by showing that the derivative of the right-hand side of the HJB equation (39) is non-positive for any \(\beta_i \geq \lambda + \nu \xi_i\), so function \(b\) also satisfies (39) without the restriction on \(\beta\).
6.6. Verifying Full Incentive Compatibility

In order to verify that the optimal contract is fully incentive compatible, we have to evaluate the agent’s benefits from off-equilibrium path deviations, and show that those are unprofitable. The verification has to identify an upper bound on the agent’s value function off the equilibrium path – an upper bound that cannot exceed the agent’s continuation value on path.

To identify a candidate upper bound, starting from $\delta_t = \hat{\delta}_t$, consider a hypothetical extreme deviation by the agent: diverting an immediate cash amount $A$ so that the principal’s belief drops to $\hat{\delta}_t = \delta_t - \nu A$. The agent gets direct benefits of $\lambda A$ from the diverted amount, plus the off-equilibrium path continuation value of $\hat{W}(\hat{\delta}_t, \tilde{\delta}_t, \hat{\delta}_t^c)$, where $\hat{W}$ is defined as the best payoff that the agent can obtain by acting optimally from then on.

Note that $\tilde{\delta}_t$ and $\delta_t^c$ are unchanged by the deviation as they are unaffected by immediate cash flow shocks (the deviation will only impact their drift going forward). Therefore, global incentive compatibility requires

$$\lambda A + \hat{W}(\hat{\delta}_t, \delta_t - \nu A, \tilde{\delta}_t, \delta_t^c) \leq W(\hat{\delta}_t, \tilde{\delta}_t, \delta_t^c)$$

$$= W(\hat{\delta}_t - \nu A, \tilde{\delta}_t, \delta_t^c) + \int_{\delta_t - \nu A}^{\hat{\delta}_t} \frac{dW_t}{d\delta_t} d\delta_t$$

Recall also (see (20)) that if local incentive constraints bind, then along the path of the deviation in the optimal contract

$$dW_t / d\delta_t = \lambda / \nu + \xi_t.$$

Therefore, global incentive compatibility requires the following bound on the agent’s off-equilibrium payoff:

$$\hat{W}(\hat{\delta}_t, \delta_t - \nu A, \tilde{\delta}_t, \delta_t^c) \leq W(\hat{\delta}_t, \tilde{\delta}_t, \delta_t^c) + \int_{\delta_t - \nu A}^{\hat{\delta}_t} \hat{\xi}_t(\tilde{\delta}_t, \hat{\delta}_t, \delta_t^c) d\tilde{\delta}_t. \quad (51)$$

Proposition 9 below shows that (51) is indeed an upper bound. The proof extends the martingale verification argument of Proposition 2, which demonstrates the first-best
contract is fully incentive compatible using the same bound (51) on the agent’s off-equilibrium path payoffs. Proposition 9 implies that global deviations are not profitable since on the equilibrium path, the bound (51) is the agent’s continuation value of $W_t$.

**Proposition 9 (Verification of Incentive Compatibility).** The optimal contract under the first-order approach is fully incentive compatible and (51) is a global upper bound on the agent’s expected payoff as long as information rents are weakly decreasing in $\delta$ and $\delta^c$, and $\hat{\delta}$ and $\hat{\delta}^c$ are weakly decreasing in $\delta$ (i.e. $\xi_2(\delta,\hat{\delta},\delta^c)$, $\xi_3(\delta,\hat{\delta},\delta^c)$, $\hat{\delta}_t(\delta,\hat{\delta},\delta^c)$, and $\hat{\delta}^c_t(\delta,\hat{\delta},\delta^c) \leq 0$).

We do not provide a formal proof of these properties, but they are intuitive and are straightforward to check numerically (note that these are conditions that need only be checked along the equilibrium path). First, $\xi_2 < 0$ and $\xi_3 < 0$ since the agent’s information rents fall as either the temporary or permanent termination threshold tightens. Second, as Figure 7 demonstrates, both $\hat{\delta}_t(\delta,\hat{\delta},\delta^c)$ and $\hat{\delta}(\delta,\hat{\delta},\delta^c)$ are decreasing in $\delta$: away from termination, information rents tighten less fast and the boundary $\delta$ is relaxed faster.

From these properties, we get the intuition as to why global deviations are unprofitable. The further the agent deviates – the lower the principal’s belief $\delta_t$ – the faster the tightening of information rent distortions ($\hat{\delta}_t$ rises faster) and the slower the relaxation of moral hazard distortions ($\hat{\delta}$ falls slower). Both of these reduce the agent’s information rents. Thus, if on the margin the agent is just indifferent, the further he deviates from the equilibrium path, the worse off he is.

Finally, while primitive conditions which assure these properties might be ideal, the approach here allows us to establish global incentive compatibility for a broader class of contracts than just the optimal one; for example, we can apply the same argument to verify global incentive compatibility of the first-best targeting contract.31

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31 In the first-best targeting contract, $\delta^c$ is permanently fixed at $\delta_t$, $\xi_2(\delta,\hat{\delta},\delta_t) < 0$, i.e. the agent’s information rents fall as the temporary termination threshold tightens, and $\hat{\delta}(\delta,\hat{\delta},\delta_t)$ is becoming more negative as $\delta$ and the agent’s continuation value rise, along any trajectory.
Proof of Proposition 9. Here we provide a martingale verification argument to prove that (51) is an upper bound on the agent’s off equilibrium payoff under the following more general conditions (1) the contract space is parameterized by three state variables \((\delta, \delta, \delta^c)\), with \(\delta\) and \(\delta^c\) having no volatility, (2) if \(\beta(\delta, \delta, \delta^c) = dW / dX\) is the sensitivity of the agent’s continuation value to cash flows, then \(\beta(\delta, \delta, \delta^c) - \nu \xi(\delta, \delta, \delta^c)\) is nondecreasing in \(\delta\), (3) \(\xi_2 \leq 0, \xi_3 \leq 0\) and both \(\dot{\delta}^c(\delta, \delta, \delta^c)\) and \(\dot{\delta}^c(\delta, \delta, \delta^c)\) are weakly decreasing in \(\delta\).

Suppose \(\delta_0 \leq \hat{\delta}_0\) and consider the process

\[
V_t = \int_0^t e^{-r\tau}(c_s + \lambda \hat{\lambda}_s)ds + e^{-r\tau}\left(W_t + \int_{\delta^c} \hat{\delta}(\delta', \delta, \delta^c)d\delta'\right).
\]

It is sufficient to show that \(V_t\) is a supermartingale under any strategy until the stopping time \(\tau\) when \(\delta_t\) hits \(\hat{\delta}_t\), resulting in termination. At termination, (51) is an upper bound on the agent’s off equilibrium outside option, since \(\xi(\delta', \delta, \delta^c) \geq (\lambda - \psi) / r\), and so if \(V_t\) is a supermartingale it follows immediately that (51) is an upper bound also before time \(\tau\), since

\[
V_t = \int_0^t e^{-r\tau}(c_s + \lambda \hat{\lambda}_s)ds + e^{-r\tau}\left(W_t + \int_{\delta^c} \hat{\delta}(\delta', \delta, \delta^c)d\delta'\right) \geq E_t[V_t] \geq \int_0^t e^{-r\tau}(c_s + \lambda \hat{\lambda}_s)ds + e^{-r\tau}E_t\left[\int_0^\tau e^{-r(\tau-s)}(c_s + \lambda \hat{\lambda}_s)ds + e^{-r(\tau-s)}\hat{R}(\hat{\delta}_s, \hat{\delta}_s)\right].
\]

Before time \(\tau\), the drift of \(V_t\) equals \(e^{-r\tau}\) times

\[
c_t + \lambda \hat{\lambda}_t - r\left(W_t + \int_{\delta^c} \hat{\delta}(\delta')d\delta'\right) + \nu(\hat{\delta}_t - \delta_t - \hat{\lambda}_t)(dW_t / d\delta_t - \xi(\delta_t)) + rW_t - c_t
\]

\[
+ \int_{\delta^c} \left(\frac{\nu^2 \sigma^2}{2} \xi_{11}(\delta', \delta, \delta^c) + \xi_2(\delta', \delta, \delta^c)\hat{\delta}(\delta, \delta, \delta^c) + \xi_3(\delta', \delta, \delta^c)\hat{\delta}(\delta, \delta, \delta^c) + \xi_4(\delta', \delta, \delta^c)\hat{\delta}(\delta, \delta, \delta^c)\right)d\delta'
\]

Using the partial differential equation for \(\xi(\delta, \delta, \delta^c)\),

32 If the incentive constraint is always binding, then \(\beta - \nu \xi = \lambda\).
\[
\beta -(r + \nu) \xi + \frac{\nu^2 \sigma^2}{2} \xi_{11} + \xi_{2} \hat{\delta} + \xi_{3} \hat{\delta}' = 0
\]

and subtracting the integral of this from \( \delta_i \) to \( \hat{\delta}_i \), we obtain

\[
\hat{a}_i (\beta(\delta_i, \hat{\delta}_i, \hat{\delta}')) - \lambda - \nu \xi(\delta_i)) + \int_{\delta_i}^{\hat{\delta}_i} \left( \beta(\delta, \hat{\delta}_i, \hat{\delta}') - \nu \xi(\delta) - (\beta(\delta', \hat{\delta}_i, \hat{\delta}')) \right) d\delta'
\]

\[
+ \int_{\delta_i}^{\hat{\delta}_i} \left( \xi_2(\delta', \hat{\delta}_i, \hat{\delta}')(\hat{\delta}(\delta, \hat{\delta}_i, \hat{\delta}') - \hat{\delta}(\delta', \hat{\delta}_i, \hat{\delta})) + \xi_3(\delta', \hat{\delta}_i, \hat{\delta}')(\hat{\delta}'(\delta, \hat{\delta}_i, \hat{\delta}') - \hat{\delta}'(\delta', \hat{\delta}_i, \hat{\delta}')) \right) d\delta'
\]

This is nonpositive when \( \delta_i \leq \hat{\delta}_i \) under the three assumptions presented at the beginning of the proof. Thus, \( V_t \) is a supermartingale. \( \blacksquare \)

Note that, according to Proposition 9, we do not need to perform a separate numerical procedure to verify the first-order approach in addition to computing the contract itself. Computing the agent’s value function off the equilibrium path is a standard method of verification. For example, Farhi and Werning (2011) suggest computing this function and then verifying that it coincides with the one that the planner intended. Pavan, Segal and Toikka (2014) develop integral monotonicity conditions, which can be checked in general dynamic mechanism design settings using the agent’s off-path value function. Here we can avoid computing the agent’s off-path value function numerically – a large simplification – because we have a closed-form upper bound on the agent’s payoffs off path. We reduce the issue of global incentive compatibility to simple properties of the contract summarized in Proposition 9. While we do not give an analytical proof of the properties, they hold numerically, and there are clear economic reasons behind these properties.

7. Empirical Implications

Our model embeds an agency model in the context of a standard real-option environment. The first-best value of the project to the principal and agent is determined by the present value of expected cash flows of \( \delta_o / r \) plus the value of the put option to abandon the
project for \( R(\delta) + L(\delta) \). The presence of the agency problem implies that the value of the project may be less than first best due to premature exercise of the option to abandon, which occurs if the agent’s share of the surplus is too low. We would like to understand how the model’s parameters affect total surplus, the termination threshold and the division of surplus between the principal and agent.

The value of the put option is a primary determinant of project value: option value increases in the volatility of fundamentals \( \delta \), given by \( \sigma' = \nu \sigma \) as well as the “strike” value of \( R(\delta) + L(\delta) \). At the same time, while higher volatility of fundamentals results in termination at a lower level of fundamentals, a higher strike value results in faster exercise. These are some of the key classic implications of option theory, which is applied widely in practice.

The two key agency parameters of our model are \( \lambda \), the benefits the agent is able to extract and \( \psi \), which determines the loss of the agent’s information rents after termination. In addition, the volatility of cash flows \( \sigma \) affects observability of the agent’s actions and interferes with the agency problem as well. Cash flow volatility \( \sigma \) is the transient volatility which can be separated from the volatility of fundamentals: if \( \sigma \) is higher but \( \sigma' = \nu \sigma \) is kept fixed, then cash flows are noisier and serve as a poorer signal of both the agent’s performance and the permanent component of profitability. For clarity, until Section 7.3 we explore the effects of the agency parameters on the optimal contract and the division of surplus between principal and agent, while keeping parameters that define the value of the option to abandon fixed; that is, we fix \( \nu \sigma \), \( R(\delta) + L(\delta) \) and \( \delta_0 \).

We start with an invariance result, which shows that certain parameters have the same effect on the agency problem. Specifically, a rise in \( \sigma \) (while keeping the volatility of fundamentals \( \nu \sigma \) fixed) has the same effect as a smaller rise in \( \lambda \).

**Proposition 10 (Payoff Invariance).** Consider any change of parameters \( (\lambda, \psi, \sigma, \nu) \) that keeps fixed the firm-specific component of profitability \( \psi \) as well as the volatility of fundamentals \( \sigma' = \nu \sigma \). If \( \lambda / \nu + \lambda / r \) also stays the same, then the principal’s payoff

\[
F(W_0 | \delta_0) \equiv \max_{\xi} b(\delta_0, W_0, \xi)
\]
remains unchanged.

**Sketch of Proof.** The result follows from the fact that the principal faces equivalent control problems under any parameter change of this form (with an appropriate adjustment to information rents). Consider the law of motion of the agent’s continuation value under the optimal contract

\[ dW_t = rW_t dt - dC_t + (\lambda / \nu + \xi) d\delta_t = rW_t dt - dC_t + (\lambda / \nu + \lambda / r - E[e^{-(r+\nu)}] \psi / r) d\delta_t. \]

It is easy to see that parameter changes outlined in the proposition leave this equation unchanged for any value of \( E[e^{-(r+\nu)}] \). Thus, given a change of parameters, keeping payments and termination time as functions of fundamentals the same, we obtain an equivalent contract with the same surplus, with the agent’s information rents changing only so far as \( \lambda / r \) changes.  

### 7.1. Agency and Efficiency

The consequence of Proposition 10 is that it reduces the agency problem to just two parameters, \( \lambda \) which indicates the agency problem within the firm, and \( \psi \) which indicates the reduction in information rents in termination.

**Corollary (Comparative Statics).** Given \( (\delta_0, W_0) \), and holding fixed \( \sigma' = \nu \sigma \), the principal’s payoff increases with

- A decrease in the agent’s private benefit rate \( \lambda \)
- An increase in the firm specific component of profitability \( \psi \)
- A decrease in the noise of cash flows \( \sigma \), or an increase in the rate of learning \( \nu \)

**Proof.** Consider an optimal contract for a given set of parameters – we know that the agent’s recommended strategy is \( \{a_t = 0\} \). Now, let us keep the contract the same but reduce \( \lambda \) or raise \( \psi \). Then the contract is still incentive compatible, since the agent gets the same payoff under \( \{a_t = 0\} \), but lower payoff under any deviation. The principal’s profit

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33 It is also easy to see that the global incentive compatibility conditions of Proposition 9 remain unchanged.
is still the same, but the contract is no longer optimal (since the agent’s incentive constraint does not bind). Hence under the optimal contract, profit must be strictly higher.

Now, if we reduce $\sigma$ while keeping $\nu \sigma$ fixed, so that $\nu$ rises, then by Proposition 10 this change is equivalent to an appropriate reduction in $\lambda$, so the principal’s profit must rise as well.

We finish this section by illustrating these results via numerical examples. We start with project parameters $r = 4.5\%$, $\sigma' = \nu \sigma = 0.27$, $R(\delta) = 10$, $L(\delta) = 35 + 7\delta$ and $\delta_0 = 3.15$ (all in $\text{\$ millions}$). Then, the value of the project without the put option is $3.15/r = \text{\$70 million}$, and the value of the put option is

$$
(R(\delta_L) + L(\delta_L) - \delta_L / r)e^{-\sqrt{\sigma'(\delta_0 - \delta_L) / \sigma'}} = 3.36, \text{ where } \delta_L = 1.8.
$$

Thus the first best value of the project is $\text{\$73.36 million}$.

For the agency parameters, consider $\lambda = 25\%$ and $\psi = \lambda$, i.e. profitability is purely firm-specific and the agent receives no information rents at termination. Suppose also that $\nu = 0.3$ so that $\sigma = 0.9$. The solid line in the left panel of Figure 10 shows the principal’s profit as a function of the agent’s expected payoff, which is maximized with $W_0 = \text{\$11.55 million}$ (i.e. $\text{\$1.55 million}$ more than his outside option) for a profit of $\text{\$61.28}$. The solid trajectories in the middle panel show the first best threshold $W^1(\delta)$ as well as the trajectory of $(\delta, W)$ that passes through the initial pair $(\delta_0, W_0)$ in the optimal contract; in this case, termination might occur in the short run as early as $\delta_0 = 2.61$. Finally, the solid curves in the right panel illustrates the drifts of $\delta_0$ and $\delta_1$ along this trajectory in the optimal contract.
Next, consider what happens if $\sigma$ rises to 2.7 and $\nu$ falls to 0.1 (keeping $\nu \sigma = .27$ fixed). Then the agency problem gets worse (indeed, according to Proposition 10, this rise in $\sigma$ is equivalent to an increase instead in $\lambda$ to 0.3152). The dashed line in the left panel of Figure 10 shows that the principal’s profit falls for any level of $W$. The agent’s payoff $W_0$ rises to $12.25$ if it is chosen to maximize the principal’s profit, which falls to $60.26$. Even though $W_0$ rises, as the dashed trajectory in the middle panel shows, the short-run termination threshold is slightly higher than before, at $\delta_0 = 2.63$ due to greater volatility of cash flows. In addition, the right panel shows that the short-run termination threshold improves more slowly than in the prior case. On the other hand, the long-run distortion from an increase in $\delta^c$ is less severe.

Finally, suppose the agency problem worsens instead due to information rents outside the firm; that is, suppose $\sigma = 0.9$ as in the first scenario, but profitability corresponds to pure
private ability with $\lambda = 25\%$ and $\psi = 0$. This change is shown by the dash-dot curves in Figure 10. Optimal $W_0$ rises further to 12.87, and the worst-case liquidation threshold also rises, causing the principal’s profit falls to $59.60$. Note that the trajectories in the middle panel become linear, as termination no longer reduces information rents, and in the right panel the permanent termination threshold $\delta^c$ is driftless – there are no permanent distortions in this case.

7.2. Market-Based Incentives

In the last setting, in which we increased the agent’s external information rents, we kept $R(\delta) = 10$. In other words, the agent’s termination value rises with his private information, but not with the public perception of his ability. This setting might correspond for example, to an extreme “ratchet” scenario in which the agent can only capture rents based on private information. If instead the slope of $R(\delta)$ rises, so that the agent is externally rewarded for his perceived ability, agency costs will decline. Intuitively, such market-based external rewards provide an additional, and more efficient, source of incentives for the agent.\footnote{See Grochulski and Zhang (2014) for a setting in which market-based incentives are stronger than those required by incentive compatibility.}

**Proposition 11 (Market-Based Incentives).** Consider a change in agent and principal’s outside options, holding fixed the aggregate payoff $R(\delta) + L(\delta)$ so that the first best is unchanged. Then

- Holding fixed $R(\delta_0)$, an increase in $R'$ will increase total surplus, and
- Holding fixed $R'$, an increase in $R(\delta_0)$ will decrease total surplus.

*Sketch of Proof.* For the first result, note that $R$ will fall for $\delta < \delta_0$. We can replicate the original contract by simply paying the agent a lump sum at termination to offset this shortfall. Doing so preserves all incentives and payoffs, so the optimal contract must do even better. The second result follows similarly: lowering the level of $R$ overall will increase surplus, and thus raising it will cause surplus to fall. \blacksquare
7.3. Effect of the Learning Rate

Finally, let us consider the effect of changing only the learning parameter $\nu$ on both the principal’s profit and the overall efficiency of the contract. (Recall that in our early examples we changed $\nu$ while holding $\nu \sigma$ fixed.) Intuitively, raising $\nu$ raises the “speed” of learning, which has two impacts: it clearly improves the first best by raising the value of the abandonment option, and it increases the informational consequence of a deviation by the agent (see e.g. (15)), which may increase the magnitude of the agency problem.

While the real option effect is clear, the incentive effect is more subtle. As $\nu$ increases, the slope of the contract trajectories declines; Figure 11 illustrates this effect in the case of pure private ability. As a result, from any initial $(\delta_0, W_0)$, the worst-case termination threshold $\delta_0$ declines – i.e. termination becomes more efficient. The increase in the volatility of $\delta$, however, has the potential to increase the likelihood of termination and thereby reduce efficiency. The later effect crucially depends on the volatility of the distance to the agent’s outside option $W_t - R(\delta_t)$.

![Figure 11: Contract Dynamics as the Rate of Learning Increases](image)

(Pure Private Ability Case)
Consider first the pure private ability case with \( \hat{R}(\delta, \hat{\delta}) = \lambda \hat{\delta} / r \). There, the volatility of \( W_t - R(\delta_t) \) is \( \lambda \sigma \), independent of \( \nu \). Thus, the only impact of an increase in \( \nu \) is to improve efficiency. Figure 12 illustrates this outcome when \( r = 4.5\% \), \( \lambda = 25\% \), \( \psi = 0 \), \( \sigma = 0.9 \), \( \delta_0 = 3.15 \), \( L = 45 \) and \( R(\delta_t) = \lambda \delta_t / r \). As we increase \( \nu \) from 2\% to 15\%, we see that the principal’s profit function rises (left panel) and, more importantly, the loss relative to the first best declines (right panel).

**Figure 12: Impact of Learning Rate on Profits and Efficiency - Pure Private Ability**

Again, as shown in Figure 11, the intuition for the increase in efficiency is that when termination occurs at lower levels of profitability when the rate of learning increases. Further, in region A in Figure 11, the first best can be obtained for lower levels of \( W \). And in region B, though the first best cannot be obtained, termination is still happening at a lower profitability level when \( \nu \) is higher. In contrast, in the limiting case without learning (\( \nu = 0 \)), termination would never be efficient and the first-best outcome would not be attainable.\(^{35}\)

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\(^{35}\) When \( \nu = 0 \), our model reduces to that of DeMarzo and Sannikov (2006), but with the agent’s discount rate the same as that of the principal. Indeed, without learning, because the agent and principal share the same discount rate, it is optimal to postpone payments to the agent forever and let the firm continue to accumulate cash reserves in order to avoid early termination for as long as possible. In order to obtain positive payouts, DeMarzo and Sannikov (2006) therefore assume the agent faces a higher discount rate. Note, however, that dividends in their model are not smooth. Smoothness arises here because of learning: the firm pays its expected level of earnings as a dividend, but absorbs any earnings surprises in its reserves, as a positive
In contrast, if profitability is firm specific and the agent has a fixed outside option \( \hat{R}(\delta, \hat{\delta}) = R_0 \), then the volatility of \( W - R(\delta) \) ranges between \( (\lambda + (\lambda - \psi)\nu / r)\sigma \) and \( (\lambda + \lambda\nu / r)\sigma \), both of which are increasing in \( \sigma \). In that case, an increase in \( \nu \) will also increase the likelihood of termination. To illustrate the impact of this effect consider the example above but with \( \psi = \lambda \), \( R(\delta) = 10 \), \( L(\delta) = 35 + 7\delta \). The right panel of Figure 13 shows that in this case, efficiency relative to the first best actually declines with \( \nu \). As a result, the principal’s profit is non-monotonic in \( \nu \): even though the first-best surplus rises in \( \nu \), the agency problem worsens enough to offset these gains when \( W \) is low. The effect is particularly large for small \( \nu \), where the effect on first-best surplus is second-order, while the deterioration of the agency problem is first-order.

![Figure 13: Impact of Learning Rate on Profits and Efficiency – Firm Specific Profitability](image)

8. **Concluding Remarks**

In this paper we have considered a continuous-time dynamic agency problem in which instantaneous output is affected by the agent’s hidden action choice, project fundamentals, earnings surprise raises the perceived level of the firm’s fundamentals and so increases the magnitude of losses that should be sustained before termination is warranted.
and idiosyncratic (Brownian) noise. Because fundamentals are stochastic and unobserved, both the agent and the principal will learn about the project’s profitability over time by observing output. While the principal and the agent have symmetric information initially, their beliefs will diverge if the agent deviates and confounds the principal’s interpretation of the output signal. This possibility introduces asymmetric information off the equilibrium path, and the opportunity for the agent to earn information rents. We explore the consequences of these information rents for incentive provision via optimal long-term contracts, and characterize the dynamics of the agent’s compensation, firm payout and liquidation / termination policies.

The agent’s potential to earn information rents, as well as the principal’s ability to control them, depend upon the agent’s ability to exploit his private information after leaving the firm. In the pure private ability case, we consider a setting in which the agent can continue to fully exploit his information after termination, as might be the case if the project’s profitability derives from the agent’s unknown skill. In this case, the timing of the termination decision has no impact on the agent’s information rents. Instead, his minimal information rents are fixed based on his ability to substitute effort for skill; that is, an agent who knows his skill is higher than expected can shirk on effort without disappointing the principal (or his next employer).

Because in the pure private ability case the minimal information rents are independent of the timing of termination, we are able to derive a relatively simple form for the optimal contract. In particular, we show that the agent’s compensation is deferred until his promised payoff reaches the point that the first best can be attained. From that point onward the agent is paid continuously until the project’s profitability falls sufficiently low that it is optimal to terminate it. We also show that this contract has a natural implementation in which the agent holds equity in the project, and the firm accumulates cash flows as reserves until either its reserves are depleted (and the agent is fired) or they reach the point at which the firm has sufficient reserves that it can fund its operations and sustain any interim losses until the first best termination time is reached. In this latter case, the firm pays its excess reserves as dividends.
We highlight several special features of this contract and its implementation. First, we show that prior to attaining the first best, the efficiency of the firm’s termination decision steadily improves (in the sense that the level of fundamentals at which termination will occur declines toward the first best) because of the interest earned on the firm’s reserves. We also show that, although the sensitivity of the agent’s payoff to output must be increased to account for his information rents, these information rents do not impact his share of equity in the firm. The intuition is that the sensitivity of the value of equity to output also increases in the presence of learning because high current cash flows raise the expected level of cash flows in the future. Finally, we demonstrate that the firm’s optimal dividend payouts are smoothed relative to earnings – the firm pays its expected level of earnings as a dividend, but absorbs any earnings surprises in its reserves. Again, the intuition is that a positive earnings surprise raises the perceived level of the firm’s fundamentals and so increases the magnitude of losses that should be sustained before termination is warranted.

In the firm-specific profitability case the agent’s information rents are smaller outside the firm than within it. This distinction would arise if the fundamentals that drive the project’s profitability are partially specific to the firm (rather than fully attributable to the agent). In that case, the agent’s information rents can be reduced by terminating the contract. As a result, in the optimal contract a new distortion is introduced in which the principal commits to terminating the contract earlier (after some histories) in order to reduce the agent’s information rents and thereby lower incentive costs up front.

We show that the optimal contract can be characterized by transforming the state space to focus on three state variables: current level of fundamentals, the termination threshold (the level of fundamentals at which termination will occur), and a payout threshold (the level of the termination threshold at which payouts commence and contract is in steady state). The third state variable is new and captures the distortions that arise to control information rents. (In the pure private ability case the payout threshold is fixed and equal to the first best termination level.)

Importantly, we show that the payout threshold increases monotonically over time, at a rate that accelerates when performance is poor and the contract is near termination. The payout
threshold thus reflects an irreversible distortion that affects the long run behavior of the contract – both the long run level of payouts to the agent and the likelihood of termination are affected by the agent’s performance in the early stages of the contract.

9. Appendix

We divide the appendix into three sections. The first section provides proofs for the setting with pure private ability (through Section 5). The second section shows some properties of the value function that are used in earlier proofs. The third section provides proofs for the firm-specific profitability in Section 6.

9.1. Proofs for Sections 3-5

Proposition 1 provides a recursive structure of the contracting environment and an incentive-compatibility condition under the first-order approach. We start with a lemma which justifies why it is sufficient to restrict attention to contracts in which \( a_t = 0 \).

**Lemma A (Sufficiency of Efficient Actions).** Given any incentive-compatible contract, there is another contract with the same payoff to the agent and a weakly higher payoff to the principal with the incentive-compatible effort recommendation \( \{\hat{a}_t = 0\} \).

**Proof.** Denote by \( c_t \) and \( \tau \) the maps from firm cash flows \( dX_t \) to the agent’s compensation and termination time under the original contract, and denote by \( \{a_t \geq 0\} \) the agent’s incentive-compatible recommended strategy. Note that the firm output is observed post-diversion, so that

\[
dX_t = (\hat{\delta}_t - a_t)dt + \sigma \ dZ_t
\]

Consider the new contract

\[
\tilde{c}_t \equiv c_t \left( X_t - \int_0^s a_j \ ds, s \in [0,t] \right) + \int_0^s \lambda a_j \ ds, \quad \tilde{\tau} \equiv \tau \left( X_s - \int_0^s a_j \ ds, s \in [0,t] \right)
\]

which compensates the agent as the original contract but based on firm cash flows adjusted for the amount the agent would have diverted, and pays the agent directly the private
benefits he would have earned. Then, if the agent chooses any strategy \( \{\tilde{a}_t \geq 0\} \) in the new contract, he gets the same payoff flow as if he had chosen the strategy \( \{a_t + \hat{a}_t\} \) under the original contract, given any path of the Brownian motion \( \{Z_t, t \geq 0\} \). In this comparison, actions are defined as functions of paths of the Brownian motion, i.e. given the same path \( \{Z_t, t \geq 0\} \) – and thus the same path \( \{\hat{Z}_t, t \geq 0\} \) of true fundamentals – the path of

\[
X_t = X_0 + \int_0^t (\hat{\delta}_s - \hat{a}_s - a_s) ds + \sigma dZ_s
\]

under the original contract under strategy \( \{\hat{a}_t + a_t\} \) is the same as the path of

\[
X_t - \int_0^t a_s ds = X_0 + \int_0^t (\hat{\delta}_s - \hat{a}_s) ds + \sigma dZ_s - \int_0^t a_s ds
\]

under the new contract under strategy \( \{\hat{a}_t\} \). Hence, the agent’s compensation under strategy \( \{\hat{a}_t\} \) under the new contract is greater by \( \lambda a_t dt \), and so his payoff flows under both strategies under both contracts are the same. Same goes to the termination time and payoff at termination.

Thus, the incentive compatibility of the original contract implies that under the new contract the strategy \( \{\tilde{a}_t = 0\} \) is optimal for the agent. Indeed, if there were a superior strategy \( \{\hat{a}_t \geq 0\} \), then strategy \( \{\hat{a}_t + a_t\} \) would be superior to strategy \( \{a_t \geq 0\} \), a contradiction. Moreover, while the agent gets the same payoff as under the original contract, the principal gets a higher payoff as he efficiently pays the agent’s benefits directly, instead of letting the agent inefficiently divert them.

**Proof of Proposition 1.** In order to start the analysis of the agent’s problem, first, let us discuss the effect of the agent’s actions on the probability measure over output paths as well as the agent’s payoff. Denote by \( P^0 \) the probability measure that arises when the agent refrains from cash diversion, i.e. cash flows follow

\[
dX_t = \delta_t dt + \sigma dZ_t.
\]
If the agent follows another off-equilibrium strategy \( \{ \hat{a}_t \} \), then belief discrepancy between him and the principal \( \alpha_t \equiv \hat{\delta}_t - \delta_t \) follows

\[
d\alpha_t = \nu(\hat{a}_t - \alpha_t) \, dt,
\]

and cash flows follow

\[
dX_t = (\hat{\alpha}_t + \hat{\delta}_t - \hat{a}_t) dt + \sigma \, dZ_t.
\] (52)

Then by Girsanov’s Theorem, an alternative strategy of the agent \( \{ \hat{a}_t \} \), changes the probability measure over cash flow paths from \( P^0 \) by the relative density process

\[
d\Gamma_t = \Gamma_t \frac{\alpha_t - \hat{\alpha}_t}{\sigma} dX_t - \frac{\delta_t - \hat{\delta}_t}{\sigma} dt, \quad \Gamma_0 = 1.
\]

Taking into account the change of measure, the agent’s expected payoff under the strategy \( \{ \hat{a}_t \} \) can be written as

\[
E^\hat{\alpha} \left[ \int_0^1 e^{-rt}(\lambda \hat{\alpha}_t + c_t) dt + e^{-rT}\hat{R}(\hat{\delta}_T, \hat{\delta}_T) \right] = E^0 \left[ \int_0^1 e^{-rt}\Gamma_t(\lambda \hat{a}_t + c_t) dt + \Gamma_T e^{-rT}\hat{R}(\hat{\delta}_T, \hat{\delta}_T) \right].
\]

We can derive first-order conditions for the agent’s strategy \( \{ a_t = 0 \} \) using the stochastic maximum principle, following the approach of Williams (2011).

Placing multipliers \( (p^r_t, p^a_t) \) on the variables \( \{ \Gamma_t, \alpha_t \} \) that the agent controls, with volatilities of multipliers \( (q^r_t, q^a_t) \), the Hamiltonian for the agent’s problem is

\[
H(t, \Gamma, \alpha, \hat{\alpha}, p^r, p^a, q^r, q^a) = \Gamma(\lambda \hat{\alpha} + c) + p^a \nu(\hat{a} - \alpha) + q^a \Gamma \frac{\alpha - \hat{\alpha}}{\sigma}
\]

Differentiating the Hamiltonian, we find that the first-order condition for \( \{ \hat{a}_t = 0 \} \) to be optimal is

\[
\Gamma \lambda + p^a \nu - \frac{q^a \Gamma}{\sigma} \leq 0.
\] (53)

The laws of motion of the multipliers can be found by differentiating the Hamiltonian with respect to the states at \( \{ \hat{a}_t = 0 \} \), i.e.
\[ dp_t^\Gamma = rp_t^\Gamma dt - H_t dt + q_t^\Gamma dZ_t = rp_t^\Gamma dt - c_t dt + q_t^\Gamma dZ_t \]  \hspace{1cm} (54)

and

\[ dp_t^\alpha = rp_t^\alpha dt - H_t dt + q_t^\alpha dZ_t = (r + \nu) p_t^\alpha dt - \frac{q_t^\Gamma}{\sigma} \beta_t dt + q_t^\alpha dZ_t. \]  \hspace{1cm} (55)

The values of the multipliers at time \( \tau \) are

\[ p_t^\Gamma = \frac{\partial}{\partial \Gamma} R(\delta_t) = R(\delta_t) \]  \[ \text{and} \]

\[ p_t^\alpha = \frac{\partial}{\partial \alpha} \hat{R}(\delta_t, \delta_t + \alpha) \bigg|_{\alpha=0} = \frac{\partial}{\partial \delta} \hat{R}(\delta_t, \delta) \bigg|_{\delta=\delta_t}, \]

since \( \Gamma_t = 1 \) under the strategy \( \{\hat{a}_t = 0\} \). Given those terminal conditions, the solution of the backward stochastic differential equation (54) is

\[ p_t^\Gamma = E^0 \left[ \int_t^\tau e^{-r(s-t)} c_s ds + e^{-r(\tau-t)} R(\delta_t) \right]. \]

To see this, note that

\[ \hat{p}_t^\Gamma = \int_0^t e^{-r_s} c_s ds + e^{-r_\tau} p_t^\Gamma \]

is a martingale. Differentiating with respect to \( t \), we find that the drift of \( \hat{p}_t^\Gamma \) is

\[ e^{-r_\tau} c_\tau - r e^{-r_\tau} p_t^\Gamma + e^{-r_\tau} (\text{drift of } p_t^\Gamma) = 0. \]

Hence, \( p_t^\Gamma \) satisfies (54). It follows that \( W_t = p_t^\Gamma \) has representation (16) with \( \beta_t = q_t^\Gamma / \sigma \).

The multiplier on \( \alpha_t \) is the derivative of the agent’s payoff with respect to information discrepancy, or his information rent by definition, i.e. \( \xi_t = p_t^\alpha \). Equation (55) for \( p_t^\alpha \) is the representation (18) (with \( \Gamma_t = 1 \), under strategy \( \{\hat{a}_t = 0\} \) ) and its solution, analogously, is

\[ \xi_t = p_t^\alpha = E^0 \left[ \int_t^\tau e^{-(r + \nu)(s-t)} \beta_s ds + e^{-(r + \nu)(\tau-t)} \frac{\partial}{\partial \delta} \hat{R}(\delta_t, \delta) \bigg|_{\delta=\delta_t} \right], \]

which yields (17).

With \( \xi_t = p_t^\alpha \), \( \beta_t = q_t^\Gamma / \sigma \) and \( \Gamma_t = 1 \), under strategy \( \{\hat{a}_t = 0\} \), the incentive constraint (53) becomes \( \lambda + \nu \xi_t - \beta_t \leq 0 \): this is condition (15).
Finally, we need to show that $\xi_t \geq \zeta_t$, with equality if condition (15) binds at all times. If (15) binds, then (18) becomes
\[
d\xi_t = r\xi_t dt - \lambda_t dt + \chi_t \sigma dZ_t
\]
with terminal condition $\xi_\tau = \frac{\partial}{\partial \delta} \hat{R}(\delta, \hat{\delta}) \bigg|_{\hat{\delta} = \delta}$, and so
\[
\xi_t = \zeta_t = E^0_t \left[ \int_t^\tau e^{-r(s-t)} \lambda_s ds + e^{-r(\tau-t)} \frac{\partial}{\partial \delta} \hat{R}(\delta, \hat{\delta}) \bigg|_{\hat{\delta} = \delta} \right] = \frac{\lambda_t}{r} - E^0_t \left[ e^{-r(\tau-t)} \right] \frac{\psi}{r}.
\]
If (15) holds but may not bind then $d\xi_t = r\xi_t dt - \hat{\lambda}_t dt + \chi_t \sigma dZ_t$, for some $\hat{\lambda}_t \geq \lambda_t$, and so
\[
\xi_t = E^0_t \left[ \int_t^\tau e^{-r(s-t)} \hat{\lambda}_s ds + e^{-r(\tau-t)} \frac{\partial}{\partial \delta} \hat{R}(\delta, \hat{\delta}) \bigg|_{\hat{\delta} = \delta} \right] \geq \zeta_t,
\]
with strict inequality if (15) does not bind on a set of positive measure. ■

**Proof of Proposition 2.** The argument prior to the statement of Proposition 2 shows that if $W < W^c(\delta, \delta^c)$, termination above $\delta^c$ has to be possible in any incentive-compatible contract. Now suppose $W \geq W^c(\delta, \delta^c)$, and consider the proposed contract. Given compensation (23), after time 0, $W_t = W^c(\delta, \delta^c)$ solves the backward stochastic differential equation
\[
dW_t = (rW_t - c_t) dt + \beta_t (dX_t - \delta_t dt)
\]
with $\beta_t = \lambda + v\xi^c(\delta, \delta^c)$ and terminal condition $W_\tau = R(\delta^c)$. Hence $W_t$ is the agent’s continuation value under the strategy $\{a_t = 0\}$, and with the initial payment the agent gets the payoff of $W$.

Let us show that the contract is globally incentive compatible, by bounding from above the agent’s payoff after any deviation $\{\hat{a}_t\}$. Note that the process
\[
V_t = \int_0^t e^{-r(s)} (c_s + \lambda \hat{a}_s) ds + e^{-r(\tau-t)} \left( W^c(\delta, \delta^c) + \int_{\delta}^{\hat{\delta}} \xi(\delta) d\delta \right)
\]
is a martingale under any strategy $\{\hat{a}_t\}$. Indeed, by Ito’s lemma, the drift of $V_t$ is
\[ e^{-rt}(c_t + \lambda \hat{a}_t) - re^{-rt}\left( W^c(\delta_t, \delta^c) + \int_{\delta} \xi(\delta)d\delta \right) + e^{-rt}\frac{\nu^2 \sigma^2}{2} \left( W^c_{\nu}(\delta_t, \delta^c) + \int_{\delta} \xi(\delta)d\delta \right) + \]

\[ e^{-rt}v(\hat{\delta}_t - \delta_t - \hat{a}_t) \left( W^c(\delta_t, \delta^c) - \xi(\delta) \right) = e^{-rt} \int_{\delta} \left( \frac{\nu^2 \sigma^2}{2} \xi(\delta)^r + \lambda - r\xi(\delta) \right) d\delta = 0. \]

We used the fact that the drift of information rents at any level of fundamentals is \( \frac{\nu^2 \sigma^2}{2} \xi(\delta)^r \) by Ito’s lemma, which is equal to \( r\xi(\delta) - \lambda \) by (18) since the first-order incentive constraint binds.

It follows that under any \( \{\hat{a}_t\} \),

\[ E_t \left[ \int_0^T e^{-r(t-s)}(c_s + \lambda \hat{a}_s)ds + e^{-rT} \hat{R}(\delta^c, \hat{\delta}_T) \right] \leq E_t[V_t] = V_t = \]

\[ \int_0^T e^{-r(t-s)}(c_s + \lambda \hat{a}_s)ds + e^{-rT} \left( W^c(\delta_t, \delta^c) + \int_{\delta} \xi(\delta)d\delta \right) \]

since

\[ \hat{R}(\delta^c, \hat{\delta}_T) = R(\delta^c) + \frac{\lambda - \psi}{r} (\hat{\delta}_T - \delta^c) < W^c(\delta^c, \delta^c) + \int_{\delta} \xi(\delta)d\delta \]

whenever \( \hat{\delta}_T > \delta^c \). Hence, the agent’s payoff off equilibrium is bounded from above by

\[ W^c(\delta_t, \delta^c) + \int_{\delta} \xi(\delta)d\delta. \]

On equilibrium, the agent cannot get more than his promised payoff \( W^c(\delta_t, \delta^c) \) through any global deviation, so the contract is fully incentive compatible.

The expressions for information rents and profit given in the proposition are correct for this contract because (1) the agent’s incentive constraint (15) is always binding so \( \xi_0 = \zeta_0 \) and (2) total surplus depends only on the termination threshold.
What is left to show is that if \( \delta^c \geq \delta_L \), then this contract maximizes total surplus subject to the information rents not exceeding \( \xi^c(\delta, \delta^c) \). (Given that the sum of profit and \( W \) is surplus, the contract also maximizes profit subject to a specific value of \( W \geq W^c(\delta, \delta^c) \)).

For a multiplier \( \phi \geq 0 \), consider the maximization problem

\[
\max_{\tau} \mathbb{E} \left[ \int_0^{\tau} e^{-r\tau} \delta_t dt + e^{-r\tau} (L(\delta_t) + R(\delta_t)) \right] - \phi \left( \frac{\lambda}{r} E[e^{-r\tau}] \frac{\psi}{r} \right).
\]

This problem is analogous to finding first best, except with the termination payoff replaced by \( L(\delta) + R(\delta) + \phi \frac{\psi}{r} \). The optimal termination threshold uniquely maximizes

\[
\exp \left( \frac{\sqrt{2r}}{\nu\sigma} \right) \left( L(\delta) + R(\delta) + \phi \frac{\psi}{r} - \frac{\delta}{r} \right).
\]

The optimal termination threshold is \( \delta^c \) given \( \phi \) if the first-order condition

\[
\phi \frac{\psi}{r} = \frac{\delta^c}{r} - L(\delta^c) - R(\delta^c) - \frac{\nu\sigma}{\sqrt{2r}} \left( \frac{1}{r} - L'(\delta^c) - R'(\delta^c) \right)
\]

holds, and \( \phi \geq 0 \) when \( \delta^c \geq \delta_L \).

The solution attains surplus and \( \zeta_0 \) of

\[
\frac{\delta}{r} + \exp \left( -\frac{\sqrt{2r}}{\nu\sigma} \left( \frac{\delta - \delta^c}{r} \right) \right) \left( L(\delta^c) + R(\delta^c) - \frac{\delta^c}{r} \right) \quad \text{and}
\]

\[
\zeta_0 = \frac{\lambda}{r} - \exp \left( -\frac{\sqrt{2r}}{\nu\sigma} \left( \frac{\delta - \delta^c}{r} \right) \right) \frac{\psi}{r},
\]

as in our contract.

Now, if there were a contract that attained higher surplus with information rents \( \hat{\xi}_0 \leq \xi^c(\delta, \delta^c) \) (and thus \( \hat{\zeta}_0 \leq \zeta_0 \) by Proposition 1), then the termination time in this contract would yield a strictly higher value of the objective function in (56), a contradiction. We conclude that the contract described in Proposition 2 indeed maximizes surplus subject to the information rents not exceeding \( \xi^c \). ■

**Proof of Proposition 3.** Let us show that the agent’s off-equilibrium value function is
\[ \hat{W}'(\hat{\delta}_t) = W_t + \frac{\lambda}{r}(\hat{\delta}_t - \delta_t), \] (57)

and that it is attained under an arbitrary strategy \( \{\hat{a}_t\} \). Note that

\[ dW_t = rW_t dt - dC_t + \lambda(1 + \nu / r)(dX_t - \delta_t dt) \]

until the termination time \( \tau \), where \( dC_t = 0 \) until \( W_t = W^t(\delta_t) \).

Consider the process

\[ V_t = \int_0^t e^{-rs} (\lambda \hat{a}_s ds + dC_s) + e^{-rT} \hat{W}'(\hat{\delta}_t) . \]

Differentiating \( V_t \) with respect to \( t \), we have

\[ e^{-rT} dV_t = \lambda \hat{a}_t dt + dC_t - r\left( W_t + \frac{\lambda}{r}(\hat{\delta}_t - \delta_t) \right) dt + dW_t + \frac{\lambda}{r}d(\hat{\delta}_t - \delta_t) \]

\[ = \lambda \hat{a}_t dt - \lambda(\hat{\delta}_t - \delta_t) dt + \lambda(1 + \nu / r) \left( dX_t - \delta_t dt \right) + \frac{\lambda}{r} \nu(\hat{a}_t - (\hat{\delta}_t - \delta_t)) dt \]

\[ = \lambda(1 + \nu / r) \sigma dZ_t \]

Thus, \( E[dV_t] = 0 \). Since \( V_t \) is a martingale, it follows that

\[ V_t = E_0[V_t] \]

\[ = \int_0^t e^{-rs} (\lambda \hat{a}_s ds + dC_s) + e^{-rT} E_t\left[ \int_t^R e^{-r(s-t)} (\lambda \hat{a}_s ds + dC_s) + e^{-r(t-t)} \left( R(\delta_t) + \frac{\lambda}{r}(\hat{\delta}_t - \delta_t) \right) \right] \]

since \( \hat{W}'(\hat{\delta}_t) = R(\delta_t) + (\lambda / r)(\hat{\delta}_t - \delta_t) \). Subtracting the definition of \( V_t \), we obtain

\[ \hat{W}'(\hat{\delta}_t) = E_0\left[ \int_t^R e^{-r(s-t)} (\lambda \hat{a}_s ds + dC_s) + e^{-r(t-t)} \left( R(\delta_t) + \frac{\lambda}{r}(\hat{\delta}_t - \delta_t) \right) \right] . \]

i.e. \( \hat{W}'(\hat{\delta}_t) \) is the off-equilibrium continuation payoff continuation, which is attained by any strategy.

The proposition makes two claims, (1) that \( W_t \) is the agent’s future expected payoff on the equilibrium path – this is indeed the case because \( W_t = \hat{W}(\hat{\delta}_t) \) is the agent’s value function.
on path – and (2) that the contract is fully incentive compatible – this is true because the strategy $a_t = 0$ (as well as any other) maximizes the agent’s payoff.

**Proof of Proposition 4.** The proof is a standard verification argument. We provide it here for the sake of completeness. The nontrivial part are the relevant properties of function $b(\delta, W)$, which we prove in Appendix 9.2. Define the process

$$G_t \equiv \int_{0}^{t} e^{-\tau s} (\delta_s ds - dC_s) + e^{-\tau s} b(\delta_s, W_s)$$

In order to verify that the benchmark contract attains profit $b(\delta_0, W_0)$, and any alternative incentive-compatible contract attains profit at most $b(\delta_0, W_0)$, we need to prove that $G_t$ is a martingale under the benchmark contract, and always a supermartingale, until time $\tau$. The proof relies on several properties of the function $b(\delta, W)$, which are proved analytically in Appendix 9.2.

Using Ito’s lemma, whenever $C_t$ is a continuous process,

$$\frac{dG_t}{e^{-\tau t}} = \delta_t dt - dC_t - rb(\delta_t, W_t) dt + b_w(\delta_t, W_t)(rW_t dt - dC_t) +$$

$$\frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b(\delta_t + vz, W_t + \beta_t z) \big|_{z=0} dt + \frac{\partial}{\partial z} b(\delta_t + vz, W_t + \beta_t z) \big|_{z=0} \sigma dZ_t.$$

Whenever $C_t$ jumps from $C_t + \Delta$ to $C_t$, $W_t$ jumps from $W_t - \Delta$ to $W_t$, and $G_t$ jumps by

$$dG_t = e^{-\tau t} (b(\delta_t, W_t - \Delta) - b(\delta_t, W_t) - \Delta).$$

Now, consider the benchmark contract. In the region $W \in [R(\delta), W^1(\delta))$, $E[dG_t] = 0$ because $b(\delta, W)$ satisfies (33), and since $\beta_t = \lambda(1+\nu/r)$ and $dC_t = 0$. In the region $W \geq W^1(\delta)$, it is easy to check that function $b(\delta, W) = V(\delta) - W$ also satisfies (33), since first-best surplus $V(\delta)$ satisfies the standard option-pricing differential equation

$$rV(\delta) = \delta + \frac{\nu^2 \sigma^2}{2} V''(\delta)$$

and also $b_w = -1$. Therefore, $E[dG_t] = 0$ as long as $\beta_t = \lambda(1+\nu/r)$, for any choice of $dC_t$.

The process $G_t$ is a martingale under the benchmark contract, as desired.
Now, consider alternative incentive-compatible contracts. First, since

\[ b_W(\delta, W) \geq -1, \]

any choice of \( dC_t \) other than those of the benchmark contract lowers \( E[dG_t] \) (strictly in region \( W \in [R(\delta), W^1(\delta)] \)) and no effect in region \( W \geq W^1(\delta) \). Second, consider the effect of the choice of \( \beta_t \geq \lambda(1 + \nu/r) \) on \( E[dG_t] \). Differentiating \( E[dG_t]/dt \) with respect to \( \beta \), we find

\[
\frac{\sigma^2}{2} \frac{\partial}{\partial \beta} \frac{\partial^2}{\partial z^2} b(\delta + \nu z, W + \beta z) \bigg|_{z=0} = \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} (z b'_w(\delta + \nu z, W + \beta z)) \bigg|_{z=0} = \\
\sigma^2 \frac{\partial}{\partial z} b_w(\delta + \nu z, W + \beta z) \bigg|_{z=0} dt = \sigma^2 \left( (\beta - \lambda(1 + \frac{\nu}{r})) b_{ww} + \lambda(1 + \frac{\nu}{r}) b_{ww} + \nu b_{wa} \right).
\]

This expression is non-positive because, as we show in Appendix 9.2,

\[ b_w \geq -1, \quad b_{ww} \leq 0 \quad \text{and} \quad \lambda(1 + \frac{\nu}{r}) b_{ww} + \nu b_{wa} \leq 0 \]

for \( W \in [R(\delta), W^1(\delta)] \), and all three properties hold with equality if \( W \geq W^1(\delta) \). Therefore, the drift of \( G_t \) is maximized by choosing \( \beta_t \geq \lambda(1 + \nu/r) \), as in the optimal contract, so under any arbitrary contract the drift of \( G \) is non-positive. Moreover, the property \( b_{ww} \leq 0 \) implies that the principal’s value function is weakly concave and the optimal contract cannot be strictly improved by public randomization.

At this point, we can complete the proof of the proposition with a standard argument. The conjectured optimal contract achieves profit \( b(\delta_0, W_0) \) because

\[
E \left[ \int_0^T e^{-rt} (\delta ds - dC_s) + e^{-rt} L(\delta_t) \right] = \lim_{t \to \infty} E[G_{\min(t, \tau)}] = G_0 = b(\delta_0, W_0).
\]

At the same time, any alternative contract that satisfies the necessary incentive-compatibility condition attains profit

\[
E \left[ \int_0^T e^{-rt} (\delta ds - dC_s) + e^{-rt} L(\delta_t) \right] = \lim_{t \to \infty} E[G_{\min(t, \tau)}] \leq G_0 = b(\delta_0, W_0).
\]

In either case, we can take the limit \( t \to \infty \) because function \( b(\delta, W) \) grows linearly in \( W \), and since the transversality condition \( \lim_{t \to \infty} E[1_{t<\tau} e^{-rt} W_t] \to 0 \) as \( t \to \infty \) holds. \( \blacksquare \)
Proof of Proposition 5. Note from the definition of $W^1(\delta_i)$ in (27), and the dynamics of the benchmark contract (32),

$$dW^1(\delta_i) = (\lambda / \nu + \lambda / r)d\delta_i = dW'_i - rW'_i dt$$

Next, (29) implies

$$R(\delta_i) - (\lambda / \nu + \lambda / r)\delta_i = W_i - (\lambda / \nu + \lambda / r)\delta_i.$$ 

Because

$$d \left[ W_i - (\lambda / \nu + \lambda / r)\delta_i \right] = rW'_i dt$$

has no volatility, there is no stochastic term in $d\delta_i(W_i)$ and the calculation of $d\delta_i$ follows by simple differentiation. Then $d\delta_i < 0$ given the bound on $R'$ in (7). Finally, since $W_i \geq R(\delta_i)$, then $\int_0^T rW'_i dt \geq rR(\delta_i)T$ and thus $W_i$ reaches $W^1(\delta_i)$ no later than at time,

$$T = \frac{W^1(\delta_i) - W_0}{rR(\delta_i)}$$

if the contract survives.  

Proof of Proposition 6: Define the process $W_i = \lambda(M_i + \delta_i / r)$. Then termination occurs when $M_i = 0$ and thus $W_i = \lambda\delta_i / r = R(\delta_i)$. Moreover,

$$dW'_i = \lambda dM_i - \lambda / rd\delta_i$$

$$= \lambda(rM_i + \delta_i)dt - \lambda dD_i + \lambda \sigma dZ + (\lambda / r)\nu \sigma dZ$$

$$= rW'_i dt - \lambda dD_i + \lambda (1 + \nu / r)(dX_i - \delta_i dt)$$

Because the agent’s compensation is $c_i dt = \lambda dD_i$, Proposition 3 implies the agent’s continuation value given any action strategy $\{a_i\}$ is

$$\hat{W}_i(\delta_i) = W_i + \frac{\lambda}{r}(\delta_i - \delta_i) = \lambda(M_i + \delta_i / r).$$

To see that it is incentive compatible for the agent to adopt the payout policy (37) if he has discretion over dividends, note that he is indifferent among all payout policies. Indeed, a
payout of \( dD_t \) lowers \( M_t \) by \( dD_t \), and thus \( \hat{W}_t(\hat{\delta}_t) \) by \( \lambda dD_t \), but adds \( \lambda dD_t \) to the agent’s immediate utility of consumption.

Having established incentive compatibility, we now verify that the payout policy (37) implements the optimal contract. First note that \( M_t = M^1(\delta_t) \) is equivalent to

\[
W_t = \lambda(M^1(\delta_t) + \delta_t / r) = \lambda((\delta_t - \delta_t) / \nu + \delta_t / r)
\]

\[
= \lambda(1 / \nu + 1 / r)(\delta_t - \delta_t) + \lambda \delta_t / r
\]

\[
= W^1(\delta_t)
\]

Thus, dividends and consumption are zero until the first best is attained at date \( T_1 \). After that, \( c_t dt = \lambda dD_t = \lambda(rM_t + \delta_t) = rW_t \), matching the first best. ■

9.2. Properties of the Benchmark Value Function

We will show that for \( W \in [R(\delta), W^d(\delta)] \), the function \( b(\delta, W) \) satisfies

\[
b_{Wy} \leq 0 \quad \text{and} \quad \lambda(1 + \frac{\nu}{r})b_{Wy} + \nu b_{W} \leq 0.
\]

(58)

Then \( b_{Wy}(\delta, W) \leq 0 \) together with \( b_{W}(\delta, W^d(\delta)) = -1 \) imply that \( b_{W}(\delta, W) \geq -1 \) for all \( W \in [R(\delta), W^d(\delta)] \). We prove analogous properties in the more general setting of Section 6 in Appendix 9.3, but here we provide alternative proofs that are more elementary.

It is useful to understand the dynamics of the pair \((\delta_t, W_t)\) under a conjectured optimal contract first. Figure A1 illustrates dynamics of the pair \((\delta_t, W_t)\) when it follows

\[
dW_t = (rW_t - c_t)dt + \lambda(1 + \nu / r)\sigma dZ_t \quad \text{and} \quad d\delta_t = \nu \sigma dZ_t.
\]

(59)

The lines parallel to \( W^d(\delta) \) are the paths of the joint volatilities of \((\delta_t, W_t)\). Due to the positive drift of \( W_t \), the pair \((\delta_t, W_t)\) moves across these lines in the direction of increasing \( W_t \). When \( W_t \) reaches the level \( R(\delta_t) \), termination results.
The phase diagram of \((\delta_t, W_t)\) provides two important directions: the direction of joint volatilities, in which \(dW/d\delta = \lambda(1/
u + 1/r)\) and the direction of drifts, in which \(W\) increases but \(\delta\) stays the same. Condition (58) requires that \(b_W(\delta, W)\) weakly decreases in both of these directions.

To study how \(b_W(\delta, W)\) depends on \((\delta, W)\), it is useful to know that \(b_W(\delta_t, W_t)\) is a martingale (Lemma B-I) and that \(b_W(\delta, R(\delta))\) increases in \(\delta\) (Lemma B-II).

**Lemma B-I (Inverse Euler Equation).** When the evolution of \((\delta_t, W_t)\) is given by (59), then \(b_W(\delta_t, W_t)\) is a martingale.

**Proof.** Differentiating the partial differential equation for \(b(\delta, W)\) with respect to \(W\), we obtain

\[
0 = rWb_{WW}(\delta, W) + \frac{1}{2} \frac{\partial^2}{\partial \sigma^2} b_W(\delta + \nu \sigma z, W + \lambda(1 + \frac{\nu}{r}) \sigma z)|_{z=0}.
\]

The right hand side of this equation is the drift of \(b_W(\delta, W)\) when \(W_t < W^1(\delta_t)\) by Ito’s lemma. Therefore, \(b_W(\delta_t, W_t)\) is a martingale until the time \(\tau'\) when \(W\) hits either \(R(\delta)\) or \(W^A(\delta)\). 

---

**Figure A1.** Dynamics of \(\delta_t\) and \(W_t\)**

---

First Best
Lemma B-II. \( b_W(\delta_R(\delta)) \) weakly increases in \( \delta \).

Proof. Note that

\[
b(\delta_0, W_0) = V(\delta_0) - W_0 - E[e^{-\tau}(V(\delta_\tau) - R(\delta_\tau) - L(\delta_\tau))]
\]

That is, the loss of profit relative to first best can be quantified by adding up the losses from inefficient termination across terminal histories. The gain from an improvement in the initial condition from \((R(\delta_0), \delta_0)\) to \((R(\delta_0) + \epsilon, \delta_0)\) arises from the fact that now each history of shocks \(\{Z_t\}\) leads to termination at a future time \(\tau\), at a level of \(\delta_\tau < \delta_0\), instead of immediately.

We want to show that \( b_W(\delta, R(\delta)) \) weakly increases in \( \delta \), and to that end we will compare two initial conditions \((\delta_0, R(\delta_0) + \epsilon)\) and \((\delta_0', R(\delta_0') + \epsilon)\), where \(\delta_0' > \delta_0\). Suppose that an improvement the initial condition from \((\delta_0', R(\delta_0'))\) to \((\delta_0', R(\delta_0') + \epsilon)\) leads to termination at time \(\tau'\) at point \(\delta_\tau'\), while an improvement in initial conditions from \((\delta_0, R(\delta_0))\) to \((\delta_0, R(\delta_0) + \epsilon)\) leads to termination at time \(\tau\) at point \(\delta_\tau\). We want to argue that for any history of shocks \(\{Z_t\}\), \(\tau' > \tau\) and \(\delta_\tau' - \delta_\tau > \delta_0 - \delta_\tau\). Then this, in combination with the property that the difference

\[
V(\delta) - R(\delta) - L(\delta)
\]

is increasing and convex in \( \delta \) (since \( V(\delta) \) is convex and \( R(\delta) \) and \( L(\delta) \) are linear), would imply that

\[
b(\delta_0', R(\delta_0') + \epsilon) > b(\delta_0', R(\delta_0') + \epsilon) - b(\delta_0', R(\delta_0)) \quad \text{when } \delta_0' > \delta_0.
\]

Then it follows that that \( b_W(\delta, R(\delta)) \) weakly increases in \( \delta \).

Note that for any history \(\{Z_t\}\), \(\delta_{\tau'} - \delta_{\tau} = \delta_0' - \delta_0\) and \(W_{\tau'} - W_{\tau} \geq W_0' - W_0 \geq 0\) as long as \( R'(\delta) \geq 0 \). It follows immediately that \( \tau \leq \tau' \).

We can use Lemmas B-I and B-II to reach conclusions about how \( b_W(\delta, W) \) changes as \( W \) increases or as \( \delta \) and \( W \) increase in the direction \( dW / d\delta = \lambda(1/\nu +1/r) \).

Lemma B-III. \( b_W(\delta, W) \) weakly decreases in \( W \), i.e. \( b_{W_W} \leq 0 \).
Proof. Let us show that for any \( \delta_0 \geq \delta_L \), for any two values \( W^1_0 < W^2_0 \),

\[
b_w(\delta_0,W^1_\tau) \geq b_w(\delta_0,W^2_\tau).
\]

Consider the processes \((\delta^i_t,W^i_t)\) \((i = 1,2)\) that follow (59) starting from values \((\delta_0,W^i_0)\) and \((\delta_0,W^2_0)\) for \( W^1_0 < W^2_0 \). Then for any path of \( Z \), we have \( \delta^1_t = \delta^2_t \) and \( W^2_t - W^1_t \geq 0 \) until time \( \tau_1 \) when \( W^1_t \) reaches the level of \( R(\delta^1_t) \). Therefore, \( W^2_t \) reaches \( R(\delta^2_t) \) at time \( \tau_2 \geq \tau_1 \), and at the level of fundamentals \( \delta^2_{\tau_2} \leq \delta^1_{\tau_1} \). Using Lemma B-I and Lemma B-II,

\[
b_w(\delta_0,W^1_0) = E[b_w(\delta^1_{\tau_1},W^1_{\tau_1})] \geq E[b_w(\delta^2_{\tau_2},W^2_{\tau_2})] = b_w(\delta_0,W^2_0).
\]

Lemma B-IV. \( b_w(\delta,W) \) weakly decreases in the direction, in which \( W \) and \( \delta \) increase according to \( dW / d\delta = \lambda(1+\nu/\tau) / \nu \), i.e. \( \lambda(1+\nu/\tau)b_{ww} + \nu b_{w \delta} \leq 0 \).

Proof. Consider starting values \((\delta^i_0,W^i_0)\) that satisfy

\[
\delta^2_0 - \delta^1_0 = \Delta > 0 \text{ and } W^2_0 - W^1_0 = \lambda(1+\nu/\tau) \Delta.
\]

Starting from these values, let the processes \((\delta^i_t,W^i_t)\) \((i = 1,2)\) follow (59). Then for any path of \( Z \), at all times \( \delta^2_t - \delta^1_t = \Delta \) and \( W^2_t - W^1_t \geq \lambda(1+\nu/\tau) \Delta \) (with equality after time 0 only if \( W^1_t = W^1(\delta^1_t) \)). Therefore, \( W^1_t \) reaches the level of \( R(\delta^1_t) \) at an earlier time \( \tau_1 \) than the time \( \tau_2 \) when \( W^2_t \) reaches the level of \( R(\delta^2_t) \), and at a weakly higher level of fundamentals \( \delta^1_{\tau_1} \geq \delta^2_{\tau_2} \). Using Lemma B-I and Lemma B-II,

\[
b_w(\delta_0,W^1_0) = E[b_w(\delta^1_{\tau_1},W^1_{\tau_1})] \geq E[b_w(\delta^2_{\tau_2},W^2_{\tau_2})] = b_w(\delta_0,W^2_0).
\]
9.3. Proofs for Section 6

First, we introduce a lemma, which generalizes Lemma B-I and provides an analogue of the inverse Euler equation in our setting. This result helps us prove important properties of the value function, such as condition (40).

Lemma C (Inverse Euler Equation). \( b_2(\delta_t, W_t, \xi_t) \) is a martingale under the optimal contract.

**Proof.** Differentiating the HJB equation with respect to \( W \) using the Envelope Theorem, we get

\[
0 = (rW - c)b_{22}(\delta, W, \xi) + ((r + \nu)\xi - \beta)b_{23}(\delta, W, \xi) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b_2(\delta + \nu z, W + \beta z, \xi + \chi z)_{|z=0}
\]

for the optimal choices of \((c, \beta, \chi)\). The right hand side is the drift of \( b_2(\delta, W, \xi) \).

**Proof of Proposition 7:** We have to (1) verify that the agent’s constraint binds while \( b_2(\delta_t, W_t, \xi_t) > -1 \), (2) justify the law of motion of the multiplier \( \phi_t \) and (3) show that once \( b_2(\delta_t, W_t, \xi_t) = -1 \), \( b(\delta_t, W_t, \xi_t) = b^*(\delta_t, W_t, \xi_t) \) and the continuation contract is given by Proposition 2. The arguments are already in Section 6.3 – here we just fill in some details.

The process

\[
G_t = \int_0^t e^{-\gamma s}(\delta_s - c_s) ds + e^{-\gamma t}b(\delta_t, W_t, \xi_t)
\]

has drift equal to \( e^{-\gamma} \) times

\[
\delta - c - rb(\delta, W, \xi) + (rW - c)b_2(\delta, W, \xi) + ((r + \nu)\xi - \beta)b_3(\delta, W, \xi) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} b_2(\delta + \nu z, W + \beta z, \xi + \chi z)_{|z=0}.
\]

Differentiating the right-hand side with respect to \( \beta \) we obtain

---

\(^{36}\) The condition that \( b_2 \) is a martingale under the optimal contract is the Inverse Euler equation of principal-agent models. It states that the drift of the Lagrange multiplier on \( W \) (the marginal cost of increasing the agent’s utility) is 0. In settings with risk aversion, this equals the agent’s inverse marginal utility. See Spear and Srivastava (1987).
\[-b_3(\delta, W, \xi) + \sigma^2 \frac{\partial}{\partial z} b_2(\delta + vz, W + \beta z, \xi + \chi z) \bigg|_{z=0} = \]
\[-b_3(\delta, W, \xi) + \sigma^2 \frac{\partial}{\partial z} b_2(\delta + vz, W + (\lambda + v\xi) z, \xi + \chi z) \bigg|_{z=0} + \sigma^2 (\beta - (\lambda + v\xi)) b_{2z}(\delta, W, \xi).\]

This is negative as long as \( b_3(\delta, W, \xi) > 0 \), condition (40) hold and \( b \) is concave in \( W \). Thus, the HJB equation implies that \( G_t \) is a strict submartingale if \( \beta_t > \lambda + v\xi \) or if \( cd_t > 0 \) while \( b_3(\delta, W_t, \xi_t) > -1 \) which would imply that the contract is suboptimal, i.e. it attains profit

\[ E \left[ \int_0^{\tau} e^{-\gamma t} (\delta_t - c_t) + e^{-\gamma t} b(\delta_t, W_t, \xi_t) \right] = E[G_t] < G_0 = b(\delta_0, W_0, \xi_0), \]

where \( \tau \) is earlier of termination time or the time when \( b_3(\delta, W_t, \xi_t) \) reaches -1. Note that once \( b_3(\delta, W_t, \xi_t) \) reaches -1, \( b(\delta_t, W_t, \xi_t) = b^*(\delta_t, W_t, \xi_t) \) is the profit from the optimal continuation contract – see below.

The law of motion of \( \phi_t = b_3(\delta_t, W_t, \xi_t) \) has been derived using Ito’s lemma in Section 6.3. The drift of \( \phi_t \) is positive by condition (40).

Once \( b_3(\delta_t, W_t, \xi_t) \) reaches -1 for the first time, the concavity of \( b(\delta, W, \xi) \) in \( W \) implies that \( b_3(\delta_t, W', \xi_t) = -1 \) for all \( W' \geq W_t \).\(^{37}\) Since \( b(\delta_t, W', \xi_t) = b^*(\delta_t, W', \xi_t) \) for large enough \( W' \) by Proposition 2, it follows immediately that \( b(\delta_t, W_t, \xi_t) = b^*(\delta_t, W_t, \xi_t) \) also, and so \( W_t = W^c(\delta_0, \delta^c) \) where \( \delta^c \) corresponds to the current level of promised information rents \( \xi_t \), such that

\[ \xi_t = \frac{\lambda}{r} - \frac{\psi}{r} \exp\left(-\sqrt{2r}\left(\frac{\delta_t - \delta^c}{\sqrt{\sigma}}\right)\right). \]

To attain the value function \( b^*(\delta_t, W_t, \xi_t) \), the termination threshold must be fixed at \( \delta^c \) from then on forever, and the fastest payout path is given by Proposition 2. □

---

\(^{37}\) The HJB equation implies that \( b_2 \geq -1 \), since otherwise the right-hand side becomes infinite.
Property (40) of the value function is justified in Section 6.5. Here we provide a proof of a key lemma used there, as well as a proof of concavity of the value function.

**Proof of Lemma (Boundary Condition).** Given initial fundamentals \(\delta\), consider a contract that maximizes profit minus \(\phi\) times information rent for continuation value \(R(\delta) + \varepsilon\).

Suppose this contract attains profit \(f_0^e\) and information rent \(\xi_0^e\). Because information rents are \((\lambda - \psi)/r\) at the boundary (and recall that \(\phi = b_2\)),

\[
h(\delta, \delta, \phi) = \lim_{\varepsilon \to 0} \frac{L(\delta) - f_0^e - \phi \left(\frac{(\lambda - \psi)}{r} - \xi_0^e\right)}{\varepsilon}.
\]

Denote by \((W, \xi)\) the processes for the agent’s continuation value and information rents under this contract.

For a higher level of fundamentals \(\delta' > \delta\), consider the contract characterized by the processes \((W' = W + R(\delta') - R(\delta), \xi)^{38}\). This contract is incentive compatible and results in the same termination time \(\tau\), and therefore the same information rents \(\xi_0^e\). The payment flows in the new contract must exceed those in the old by \(r(R(\delta') - R(\delta))\), in order to keep the drifts of \(W'\) and \(W\) the same. Denote by \(f_0^{e'}\) the principal’s profit under the new contract.

Under the former contract, total surplus is

\[
f_0^e + R(\delta) + \varepsilon = L(\delta) + R(\delta) + E \left[ \int_0^\tau e^{-rt} (\delta_t - rL(\delta_t) - rR(\delta_t)) dt \right],
\]

Likewise, for the new contract,

\[
f_0^{e'} + R(\delta') + \varepsilon = L(\delta') + R(\delta') + E \left[ \int_0^\tau e^{-rt} (\delta'_t - rL(\delta'_t) - rR(\delta'_t)) dt \right],
\]

where \(\delta'_t = \delta_t + \delta' - \delta\). Since

\[\]
\[ \delta'_t - rL(\delta'_t) - rR(\delta'_t) = \delta_t - rL(\delta_t) - rR(\delta_t) + (\delta' - \delta)(1 - rL' - rR'), \]

it follows that (using (19)),

\[ f'_0^\varepsilon - L(\delta') = f_0^\varepsilon - L(\delta) + \frac{1 - E[e^{-r\tau}]}{(r^\varepsilon - \lambda + \psi)/\psi} (\delta' - \delta)(1 - L' - R'). \]

This contract provides a lower bound on the value obtained by maximizing profit minus \( \phi \) times information rent for continuation value \( R(\delta') + \varepsilon \) at fundamentals \( \delta' \). Taking \( \varepsilon \) to 0, we find that

\[
h(\delta', \delta', \phi) \geq \lim_{\varepsilon \to 0} \frac{L(\delta') - f_0^\varepsilon - \phi \left( (\lambda - \psi) / r - \frac{\xi^\varepsilon}{\xi_0} \right)}{\varepsilon}
\]

\[ = \lim_{\varepsilon \to 0} \frac{L(\delta) - f_0^\varepsilon - \phi \left( (\lambda - \psi) / r - \frac{\xi^\varepsilon}{\xi_0} \right) + \frac{(\lambda - \psi)}{r - \xi_0} \delta' - \delta (1 - rL' - rR')}{\varepsilon} \]

\[ = h(\delta, \delta, \phi) + \xi(\delta, R(\delta), \phi) \frac{\delta' - \delta}{\psi} (1 - rL' - rR'), \]

where \( \xi(\delta, W, \phi) \) denotes the information rent that corresponds to the point \((\delta, W, \phi)\) of the state space, which increases with \( W \). ■

Finally, we prove the concavity conditions of the principal’s value function that we utilized above.

**Lemma D (Concavity).** Consider the control problem defined by the laws of motion of state variables

\[
d\delta_t = \nu \sigma \, dZ_t, \quad dW_t = rW_t \, dt - dC_t + \beta_t \sigma \, dZ_t \quad \text{and} \quad d\xi_t = \left( (r + \nu)\xi_t - \beta_t \right) \, dt + \chi_t \sigma \, dZ_t,
\]

controls \((dC_t \geq 0, \beta_t = \lambda + \nu \xi_t, \chi_t)\) and payoff flow \( \delta \) until stopping time \( \tau \) when \( W_t \) hits \( R(\delta) \). There is a requirement that at time \( \tau, \xi_t = (\lambda - \psi)/r \) and continuation payoff is \( R(\delta) + L(\delta) \). Assume that both \( R(\delta) \) and \( RL(\delta) \equiv R(\delta) + L(\delta) \) are linear in \( \delta \). Then the value function for this control problem \( f(\delta, W, \xi) \) (i.e. total surplus) is jointly concave in \( W \) and \( \xi \).

**Corollary.** It follows that the principal’s value function is jointly concave in \( W \) and \( \xi \), since \( b(\delta, W, \xi) = f(\delta, W, \xi) - W \).

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Proof of Lemma D. We start with a couple of simple observations. First, \( f(\delta, W, \xi) \) is nondecreasing in \( W \) since it is always possible to reduce \( W \) through the control \( dC_t \).

Second, denote by \( \overline{W}(\delta, \xi) \) the minimal continuation value, for which maximal surplus can be attained subject to information rents being \( \xi \). That is, if \( \delta^c \) is such that \( \xi = \xi^c(\delta, \delta^c) \), then \( \overline{W}(\delta, \xi) = W^c(\delta, \delta^c) \) (see Proposition 2). Then \( f(\delta, W, \xi) = f(\delta, \overline{W}(\delta, \xi), \xi) \) for all \( W > \overline{W}(\delta, \xi) \).

We prove concavity in \( W \) and \( \xi \) by explicit construction. Given contracts that attain any two initial tuples of states (with the same \( \delta \)), we construct a contract for the linear combination of these tuples that attains at least as high surplus as the corresponding linear combination. In virtue of Step 1, we only need to perform an explicit construction for the case when one of the tuples is on the termination boundary, and we cover this case in Step 2.

Note that we are not proving concavity in \( \delta \): first-best surplus is actually convex in \( \delta \).

**Step 1.** Assume that for all \( \alpha \in (0, 1) \), \( \delta, W > R(\delta) \) and \( \xi > (\lambda - \psi)/r \),

\[
f(\delta, \alpha R(\delta) + (1 - \alpha)W, \alpha \frac{\lambda - \psi}{r} + (1 - \alpha)\xi) \geq \alpha R(\delta) + (1 - \alpha)f(\delta, W, \xi) \tag{60}
\]

Then \( f \) is jointly concave in \( W \) and \( \xi \).

**Proof.** Consider controlled processes \((\delta_t, W^{1}_t, \xi^{1}_t)\) and \((\delta_t, W^{2}_t, \xi^{2}_t)\) with controls \((dC^1_t, \beta^1_t, \chi^1_t)\) and \((dC^2_t, \beta^2_t, \chi^2_t)\). Denote by \( \tau \) the stopping time when \( W^{1}_t \) or \( W^{2}_t \) hits \( R(\delta_t) \), whichever happens sooner. For initial conditions \((\delta_0, W^1_0, \xi^1_0) = \alpha(\delta_0, W^1_0, \xi^1_0) + (1 - \alpha)(\delta_0, W^2_0, \xi^2_0) \), consider the feasible policy \((dC^1_t, \beta^1_t, \chi^1_t) = \alpha(dC^1_t, \beta^1_t, \chi^1_t) + (1 - \alpha)(dC^2_t, \beta^2_t, \chi^2_t) \) until time \( \tau \), followed by the optimal policy thereafter. This policy attains value

\[
E \left[ \int_0^\tau e^{-r\tau} d\tau + e^{-r\tau} f(\delta_t, \alpha W^1_t + (1 - \alpha)W^2_t, \alpha \xi^1_t + (1 - \alpha)\xi^2_t) \right] \geq \alpha E \left[ \int_0^\tau e^{-r\tau} d\tau + e^{-r\tau} f(\delta_t, W^1_t, \xi^1_t) \right] + (1 - \alpha)E \left[ \int_0^\tau e^{-r\tau} d\tau + e^{-r\tau} f(\delta_t, W^2_t, \xi^2_t) \right] = \alpha f(\delta_0, W^{1}_0, \xi^{1}_0) + (1 - \alpha)f(\delta_0, W^{2}_0, \xi^{2}_0) \]
by (60). Since this is a lower bound on $f(\delta_0, W_0, \xi_0)$, the claim holds. This completes the proof of Step 1.

**Step 2.** Equation (60) holds.

*Proof.* It is enough to show that (60) holds for $W \leq \overline{W}(\delta, \xi)$. Indeed, then if $W > \overline{W}(\delta, \xi)$ we have

$$f(\delta, \alpha R(\delta) + (1 - \alpha)W, \alpha \frac{\lambda - \psi}{r} + (1 - \alpha)\xi) \geq f(\delta, \alpha R(\delta) + (1 - \alpha)\overline{W}(\delta, \xi), \alpha \frac{\lambda - \psi}{r} + (1 - \alpha)\xi) \geq \alpha RL(\delta) + (1 - \alpha)f(\delta, \overline{W}(\delta, \xi), \xi) = \alpha RL(\delta) + (1 - \alpha)f(\delta, W, \xi).$$

Now, for $W \leq \overline{W}(\delta, \xi)$, we will explicitly construct a contract with information rents $\alpha \frac{\lambda - \psi}{r} + (1 - \alpha)\xi$ and continuation value $W' \leq \alpha R(\delta) + (1 - \alpha)W$ that attains surplus of not less than

$$\alpha RL(\delta) + (1 - \alpha)f(\delta, W, \xi).$$

Since the value function is weakly increasing in $W$, we get the desired conclusion.

First, we construct the contract for a lower level of $\delta'$, and then we “lift” the contract to $\delta$ (via a procedure analogous to that of Lemma (*Boundary Condition*).)

Denote $\gamma = \sqrt{2r} / (v\sigma)$. In the problem of maximizing surplus subject to information rents of $\xi$, termination occurs at level $\delta^c_0$ such that $\xi = \xi^c(\delta, \delta^c_0)$.

The minimal level of $W$, for which this surplus level is attainable, is $\overline{W}(\delta, \xi) = W^c(\delta, \delta^c_0)$.

We have instead a contract that gives payoff $W \leq \overline{W}(\delta, \xi)$ to the agent. In order to reduce the agent’s payoff, we raise the termination threshold to a new level $\overline{\delta}$ (to be determined), define a function $\xi(\delta')$ to be the solution of the differential equation

$$r\xi(\delta') = \lambda + \frac{v^2\sigma^2}{2} - \xi''(\delta')$$

on $[\overline{\delta}, \delta]$ with boundary conditions $\xi(\overline{\delta}) = (\lambda - \psi) / r$ and $\xi(\delta) = \xi$. That is, if the agent’s information rents are $(\lambda - \psi) / r$ when $\delta_j$ hits $\overline{\delta}$ (due to termination) and $\xi$ when $\delta_j$ hits $\delta$ (e.g. because the hitting time always triggers a continuation contract with information...
rents $\xi$) then the agent’s information rents in the middle are given by $\xi(\delta')$ if also the local incentive constraint (15) is binding.

Define a function $W(\delta')$ on $[\delta, \delta]$ by

$$W(\delta') = R(\delta') + \int_{\delta}^{\delta'} (\lambda / v + \xi(x))dx.$$ 

Now, we determine $\delta$ by the requirement that $W(\delta) = W$, and since $W < W^c(\delta, \delta^c)$, it must be that $\delta > \delta^c$.

For any $\delta' = [\delta, \delta]$, define a contract as follows. Let $\delta_0 = \delta'$, and let $(W_i', \xi_i) = (W(\delta_i), \xi(\delta_i))$ until $\delta_i$ hits either $\delta$ or $\delta$. In this contract the agent has to be paid to keep his continuation value at $W(\delta_i)$, and the constraint $\beta_i \geq \lambda + v\xi_i$ holds with equality by the definition of $W(\delta')$. Once $\delta_i$ hits $\delta$, we have termination, and once $\delta_i$ hits $\delta$, we implement the optimal continuation contract, which has surplus $f(\delta, W, \xi)$. We will use this family of contracts to prove inequality (60), except that the contract will be lifted to level $\delta$ from $\delta'$ (as in Lemma (Boundary Condition)).

Specifically, suppose we want to attain $(\delta, W', \xi') = (\delta, \alpha R(\delta) + (1 - \alpha)W', \alpha(\lambda - \psi) / r + (1 - \alpha)\xi)$ via a contract. Then choose $\delta'$ such that $\xi(\delta') = \xi'$. This contract attains continuation value $W(\delta')$. We then “lift” the contract to the level of fundamentals $\delta$ (via a procedure analogous to that of Lemma (Boundary Condition)). This raises the total surplus relative to termination surplus by

$$\frac{r\xi' - (\lambda - \psi)}{\psi} (\delta - \delta')(\frac{1}{r} - RL'),$$

by the exact same argument as in the proof of Lemma (Boundary Condition). The agent’s information rent remains unchanged. The agent’s continuation value rises from $W(\delta')$ to $W(\delta') + (\delta - \delta')R'$. We claim that surplus in this contract is greater than $\alpha RL(\delta) + (1 - \alpha)f(\delta, W, \xi)$.
and the agent’s continuation value is $W(\delta') + (\delta - \delta') R' \leq W'$. Since the surplus of the optimal contract for the triplet $(\delta, W', \xi')$ is at least as great as the surplus from this contract, we obtain the desired conclusion.

To see why $W(\delta') + (\delta - \delta') R' < W'$, note that $\xi(\delta')$ is an increasing and concave function, since $\xi \leq \lambda / r$, and so $W(\delta') + (\delta - \delta') R'$ is a convex function. It follows, therefore, that

$$\xi(\delta') = \alpha \frac{\lambda - \psi}{r} + (1 - \alpha) \xi > \frac{\delta - \delta'}{\delta - \delta} \left( \frac{\lambda - \psi}{r} \right) + \frac{\delta' - \delta}{\delta - \delta} \xi \Rightarrow \alpha < \frac{\delta - \delta'}{\delta - \delta}$$

(by concavity of $\xi(\delta')$) and so

$$W(\delta') + (\delta - \delta') R' < \frac{\delta - \delta'}{\delta - \delta} (W(\delta) + (\delta - \delta) R') + \frac{\delta' - \delta}{\delta - \delta} W < \alpha R(\delta) + (1 - \alpha) W = W'.$$

To prove that contract surplus is greater than $\alpha RL(\delta) + (1 - \alpha) f(\delta, W, \xi)$, we need to show that surplus is even “more concave” than $\xi(\delta')$. If $\tau$ is the time when $\delta_t$ hits $\delta$ or $\tilde{\delta}$, then

$$\xi' - \frac{\lambda - \psi}{r} = E \left[ \int_0^1 e^{-\tau} (\psi) \ dt + e^{-\tau} 1_{\delta_t = \tilde{\delta}} \left( \xi - \frac{\lambda - \psi}{r} \right) \right]$$

and total surplus relative to termination surplus is

$$f' - RL(\delta') = E \left[ \int_0^1 e^{-\tau} \delta_t dt + e^{-\tau} 1_{\delta_t = \tilde{\delta}} f(\delta, W, \xi) + e^{-\tau} 1_{\delta_t = \tilde{\delta}} RL(\tilde{\delta}') \right] - RL(\delta').$$

For the “lifted” contract, total surplus, relative to termination surplus (i.e. the difference between surplus and termination surplus), is

$$f' - RL(\delta') + \frac{r \xi' - (\lambda - \psi)}{\psi} (\delta - \delta')(\frac{1}{r} - RL').$$

We would like to show that

$$f' - RL(\delta') + \frac{r \xi' - (\lambda - \psi)}{\psi} (\delta - \delta')(\frac{1}{r} - RL') \geq \frac{f(\delta, W, \xi) - RL(\delta)}{\xi' - \frac{\lambda - \psi}{r}} \geq \frac{f(\delta, W, \xi) - RL(\delta)}{\xi - \frac{\lambda - \psi}{r}} \quad (61)$$

for all $\delta'$ - this is what we mean when we say surplus is “more concave” than $\xi(\delta')$ – this implies that
\[ f' - RL(\delta') + \frac{r\psi - (\lambda - \psi)}{\psi} (\delta - \delta')(\xi - RL') \geq (1 - \alpha) (f(\delta, W, \xi) - RL(\delta)) \]

\[ \Rightarrow f' - RL(\delta') + \frac{r\psi - (\lambda - \psi)}{\psi} (\delta - \delta')(\xi - RL') + RL(\delta) = \alpha RL(\delta) + (1 - \alpha) f(\delta, W, \xi) \]

We will show in fact that the left-hand side of (61) is decreasing in \( \delta' \) (hence the inequality must hold, since at \( \delta' = \delta \), both sides are the same).

It is useful to evaluate relevant expressions relative to the case when \( W \) takes a lower value of \( \tilde{W} \equiv W^c(\delta, \tilde{\delta}) \). Then the agent’s information rents \( \xi \) would be

\[ \tilde{\xi} = \xi(\delta, \tilde{\delta}) = \frac{\lambda}{r} - \exp\left(-\gamma(\delta - \tilde{\delta})\right) \frac{\psi}{r} \]

so that \( f(\delta, \tilde{W}, \tilde{\xi}) \) is the maximal surplus subject to information rents being \( \tilde{\xi} \).\(^{39}\) For that case, for the contract we constructed at \( \delta' \), the difference between surplus and termination payoff would be

\[ \frac{\delta'}{r} - RL(\delta') + \exp\left(-\gamma(\delta' - \tilde{\delta})\right) \left(RL(\tilde{\delta}') - \frac{\tilde{\delta}}{r}\right), \]

since the termination threshold is permanently fixed at \( \tilde{\delta} \). However, if \( W > \tilde{W} \) then the difference between surplus and termination payoff is

\[ \frac{\delta'}{r} - RL(\delta') + \exp\left(-\gamma(\delta' - \tilde{\delta})\right) \left(RL(\tilde{\delta}) - \frac{\tilde{\delta}}{r}\right) + E\left[e^{-\tau t} 1_{\delta = \tilde{\delta}} \right] \left(f(\delta, W, \xi) - f^c(\delta, \tilde{\delta})\right) \]

where \( f^c(\delta, \tilde{\delta}) \) is total surplus if termination occurs always at \( \tilde{\delta} \). The only difference between the cases is that once \( \delta_t \) hits \( \delta \), the continuation contract has surplus \( f(\delta, W, \xi) \) rather than \( f^c(\delta, \tilde{\delta}) \).

Likewise, when \( \tilde{W} = W^c(\delta, \tilde{\delta}) \), the contract we constructed at \( \delta' \) would give the agent information rents that satisfy

\[ \xi' - \frac{\lambda - \psi}{r} = \xi^c(\delta', \tilde{\delta}) - \frac{\lambda - \psi}{r} = \frac{\psi}{r} - \exp\left(-\gamma(\delta' - \tilde{\delta})\right) \frac{\psi}{r}. \]

\(^{39}\) In order to avoid confusion with notation, note that \( W^c(\delta, \tilde{\delta}) \leq W \leq \tilde{W}(\delta, \xi) = W^c(\delta, \tilde{\delta}_0) \), with equalities if \( \tilde{\delta} = \tilde{\delta}_0 \).
However if $W > \tilde{W}$, then information rents satisfy
\[
\xi' - \frac{\lambda - \psi}{r} = \frac{\psi}{r} (1 - \exp(-\gamma(\delta' - 0))) + E \left[ e^{-r \tau_1 1_{\delta,=\delta}} \right] (\xi - \xi^c(\delta, \delta)).
\]

The adjustment that lifts the contract increases the difference between surplus and termination payoff by
\[
\frac{r \xi' - (\lambda - \psi)}{\psi} (\delta - \delta')(\frac{1}{r} - RL') =
\]
\[
(1 - \exp(-\gamma(\delta' - 0))) (\delta - \delta')(\frac{1}{r} - RL') + E \left[ e^{-r \tau_1 1_{\delta,=\delta}} \right] (\xi - \xi^c(\delta, \delta))(\delta - \delta')(1 - rRL')
\]
and with the adjustment, total the difference between total surplus and termination payoff (at the lifted contract) becomes
\[
\frac{\delta}{r - RL(\delta)} \left(1 - \exp(-\gamma(\delta' - 0))) + \exp(-\gamma(\delta' - 0))(\delta - \delta')(1/r - RL') +
\right.
\]
\[
E \left[ e^{-r \tau_1 1_{\delta,=\delta}} \right] \left( f(\delta, W, \tilde{\xi}) - f^c(\delta, \tilde{\delta}) + (\xi - \xi^c(\delta, \tilde{\delta}))(\delta - \tilde{\delta})(1 - rRL') \right)
\]
Let us treat the case of $W = \tilde{W} = W^c(\delta, \tilde{\delta})$ first and then make an adjustment. Dividing through by $\xi' - \frac{\lambda - \psi}{r}$, we obtain
\[
\frac{\frac{\delta}{r - RL(\delta)} \left(1 - \exp(-\gamma(\delta' - 0))) + \exp(-\gamma(\delta' - 0))(\delta - \delta')(1/r - RL') \right)}{\frac{\psi}{r} (1 - \exp(-\gamma(\delta' - 0)))}
\]
\[
= \frac{\psi}{r} \frac{\delta - rRL(\delta)}{1 - \exp(-\gamma(\delta' - 0)))} + \frac{1 - rRL'}{\psi} \frac{\exp(-\gamma(\delta' - 0))(\delta' - \tilde{\delta})}{1 - \exp(-\gamma(\delta' - 0)))}.
\]
We claimed earlier that this is decreasing in $x = \delta' - \tilde{\delta}$ (see (61)). Indeed, differentiating, we get
\[
\frac{-\gamma \exp(-\gamma x) x + \exp(-\gamma x)(1 - \exp(-\gamma x))}{(1 - \exp(-\gamma x))^2} - \frac{\exp(-\gamma x) x \gamma \exp(-\gamma x)}{(1 - \exp(-\gamma x))^2} = \frac{\exp(-\gamma x) 1 - \gamma x - \exp(-\gamma x)}{(1 - \exp(-\gamma x))^2} < 0.
\]
Furthermore, since
\[
E \left[ e^{-r \tau} I_{\delta, \delta} \right] \left( f(\delta, W, \xi) - f^c(\delta, \xi) \right) + \left( \xi - \xi^c(\delta, \xi) \right) (\delta - \delta')(1 - r \rho L') = \\
E \left[ e^{-r \tau} I_{\delta, \delta} \right] \left( \xi - \xi^c(\delta, \xi) \right) \\
\frac{f(\delta, W, \xi) - f^c(\delta, \xi)}{\xi - \xi^c(\delta, \xi)} + (\delta - \delta')(1 - r \rho L')
\]

is decreasing in \( \delta' \) as well, the argument extends to the case of \( W \in (W^c(\delta, \xi), \bar{W}(\delta, \xi)) \) and so we get the desired result. ■

10. References.


