Contagion and Uninvadability in Social Networks with Bilingual Option*

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Abstract

We study the long run outcome of local interactions in an infinite population of players, each of whom chooses one of two conventions or adopts both (i.e., chooses the "bilingual option") at an additional cost. In this class of games, we completely characterize when a convention spreads contagiously from a finite subset of players to the entire population in some network, and conversely, when a convention is never invaded by the other convention in any network. Generically, at least one convention spreads contagiously in some network, and for some range of payoff parameters, both conventions each spread contagiously in respective networks. Our proofs for this characterization provide new insights on how the network structure affects contagion. *Journal of Economic Literature* Classification Numbers: C72, C73, D83.

Keywords: equilibrium selection; bilingual game; local interaction; contagion; uninvadability.

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Web page: www.oyama.e.u-tokyo.ac.jp/papers/bilingual.html.

1 Introduction

[Lead]

Consider an infinite population of players who are connected with each other through a graph ("social network"). Suppose that each player uses one of two computer programming languages, or two types of technologies in general, A and B. The payoff from each interaction with his neighbors is given by the following 2×2 coordination game:

	A	B
A	a, a	b, c
B	c, b	d, d

where $a > c$ and $d > b$, so that (A, A) and (B, B) are strict Nash equilibria. We assume that $a > d$, i.e., (A, A) Pareto-dominates (B, B), while $a - c < d - b$, i.e., (B, B) risk-dominates (A, A). We further assume that $d \geq c$ (which together with the above assumptions implies that $a \geq b$), i.e., coordination on some action is always better than miscoordination. It is well known (see, e.g., Morris (2000)) that the risk-dominant action B spreads contagiously from a finite subset of players to the entire population in some network, and that it is never invaded by the other action A in any network. Thus, in 2×2 coordination games, the risk-dominant action is always both contagious and uninvadable. In fact, contagion and uninvadability are equivalent in this class of games.

Now suppose that players can adopt a combination of the two actions, a "bilingual option" AB, with an additional cost $e > 0$. A player who plays AB receives a (gross) payoff a (d, resp.) from an interaction with an A-player (B-player, resp.). When two AB-players interact, they adopt the superior action A and receive a. This situation is described by the following payoff matrix:[1]

	A	AB	B
A	a, a	$a, a - e$	b, c
AB	$a - e, a$	$a - e, a - e$	$d - e, d$
B	c, b	$d, d - e$	d, d

where (A, A) and (B, B) are the only pure-strategy Nash equilibria. One may expect that, when the value of the cost parameter e is large, the action AB is not much relevant so that the situation is close to the previous 2×2

[1] This game has been studied by Galesloot and Goyal (1997), Goyal and Janssen (1997), Immorlica et al. (2007), and Easley and Kleinberg (2010).

case, while as e becomes smaller, AB becomes closer to dominating B so that eventually B will be abandoned and only A will survive.

In this paper, we completely characterize when an action is contagious and when it is uninvadable in this class of 3×3 games. Conforming to the conjecture in the previous paragraph, we show that if e is large, then B is contagious and uninvadable, while if e is small, then A is contagious and uninvadable. Generically, either A or B is contagious, but, in contrast to the 2×2 case, both actions are each contagious if e is in a medium range (which is nonempty and open under an additional condition on parameter values), i.e., A spreads contagiously in some networks while B does in some others. In other words, uninvadability is a strictly stronger property than contagion.

Our proofs for the above characterization provide new insights on how the network structure affects contagion. A class of networks is called *critical* if these networks induce a maximal amount of contagion, i.e., whenever an action can spread contagiously in some network, it does so within this class of networks. We show that the class of all "linear" networks is not critical in determining contagion in the bilingual game, and provide an example of a critical class that includes "non-linear" networks.

In his series of papers, Morris (1997, 1999, 2000) defines general notions of contagion and uninvadability, develops a method using potential functions to provide a sufficient condition for uninvadability (and hence a necessary condition for contagion), and gives an example of a symmetric 4×4 game to demonstrate the multiplicity of contagious actions. In particular, in his 4×4 example, contagion of these actions occurs in "linear" networks. For our class of games, we utilize the potential method to show uninvadability, while we construct a "non-linear" network to obtain contagion of one of the two actions.

[Section 5: what is possible and not in 2×2]

Contagious behavior in the bilingual game is analyzed by Goyal and Janssen (1997) and Immorlica et al. (2007). Both papers, however, focus on specific classes of networks and provide only sufficient (necessary, resp.) conditions for contagion (uninvadability, resp.), which are strictly stronger than the condition we obtain as a full characterization. This implies that their classes of networks are not critical.

As Morris (1997, 1999) argues, local interaction games and incomplete information games have formal connections, and both belong to a more general class of "interaction games". Accordingly, our results on local interaction games can be interpreted in the context of incomplete information games, whereby we provide interesting implications on global games and robustness to incomplete information.

[Immorlica: Appendix]

2 Local Interaction Games

Let $\mathcal{X}$ be a countably infinite set of players, and $P\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ a function such that

1. $P(x,x)=0$ for all $x\in\mathcal{X}$,
2. $P(x,y)=P(y,x)$ for all $x,y\in\mathcal{X}$, and
3. $0<\sum_{y\in\mathcal{X}}P(x,y)<\infty$ for all $x\in\mathcal{X}$.

A *local interaction system*, or *network*, $(\mathcal{X},P)$ defines an undirected graph with vertices $\mathcal{X}$ and edges weighted by P. (We will use the terms "local interaction system" and "network" interchangeably.)[2] We will restrict our attention to *unbounded* local interaction systems; i.e., $\sum_{(x,y)\in\mathcal{X}\times\mathcal{X}}P(x,y)=\infty$. Write $\Gamma(x)=\{y\in\mathcal{X}\mid P(x,y)>0\}$ for the set of neighbors of player $x\in\mathcal{X}$. Denote

$$P(y|x)=\frac{P(x,y)}{\sum_{z\in\Gamma(x)}P(x,z)},$$

which is well defined due to property 3 above.

Players have a (common) finite set of actions S and a (common) payoff function $u\colon S\times S\to\mathbb{R}$. With the action set S fixed, a *local interaction game* is represented by the tuple $(\mathcal{X},P,u)$. Let $\Delta(S)$ denote the set of probability distributions over S. Given payoff function u, write $br(\pi)$ for the set of pure best responses to $\pi\in\Delta(S)$:

$$br(\pi)=\{h\in S\mid u(h,\pi)\geq u(h',\pi)\text{ for all }h'\in S\},\tag{2.1}$$

where $u(h,\pi)=\sum_{k\in S}\pi_k u(h,k)$.

An *action configuration* is a function $\sigma\colon\mathcal{X}\to S$. Given an action configuration σ, we denote by $\pi(\sigma|x)\in\Delta(S)$ the action distribution, weighted by $P(\cdot|x)$, over the actions of player x's neighbors: i.e.,

$$\pi_h(\sigma|x)=\sum_{y\in\Gamma(x)\colon\sigma(y)=h}P(y|x).$$

The payoff for player $x\in\mathcal{X}$ playing action $s\in S$ is given by the weighted sum (with respect to $P(\cdot|x)$) of payoffs from the interactions with his neighbors:

$$U(s,\sigma|x)=\sum_{y\in\Gamma(x)}P(y|x)u(s,\sigma(y)),$$

[2] One could instead focus on local interaction systems with *constant weights*, where $P(x,y)\in\{0,1\}$ for all $x,y\in\mathcal{X}$. All the results in this paper would remain unchanged since any local interaction system with rational weights can be replicated by a local interaction system with constant weights.

which equals $u(s, \pi(\sigma|x))$. Write $BR(\sigma|x)$ for the set of pure best responses for player x to action configuration σ:

$$BR(\sigma|x) = \{s \in S \mid U(s, \sigma|x) \geq U(s', \sigma|x) \text{ for all } s' \in S\}, \tag{2.2}$$

which equals $br(\pi(\sigma|x))$.

We consider the sequential best response dynamics on network $(\mathcal{X}, P)$ as defined below. (There being finitely many actions, for a sequence of actions $(s^t)_{t=0}^{\infty}$, $\lim_{t\to\infty} s^t = s$ if and only if there exists T such that $s^t = s$ for all $t \geq T$.)

Definition 1. A sequence of action configurations $(\sigma^t)_{t=0}^{\infty}$ is a *best response sequence* if it satisfies the following properties: (i) for all $t \geq 1$, there is at most one $x \in \mathcal{X}$ such that $\sigma^t(x) \neq \sigma^{t-1}(x)$; (ii) if $\sigma^t(x) \neq \sigma^{t-1}(x)$, then $\sigma^t(x) \in BR(\sigma^{t-1}|x)$; and (iii) if there exists $T \geq 0$ such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then $\lim_{t\to\infty} \sigma^t(x) \neq s$.

Property (i) requires that in each period at most one player revise his action, while property (ii) requires that the revising player switch to a myopic best response to the current distribution of his neighbors' actions. Property (iii) requires that actions that are never a best response be abandoned eventually. In particular, (ii) and (iii) imply that if there exists T such that $s \notin BR(\sigma^t|x)$ for all $t \geq T$, then there exists T' such that $\sigma^t(x) \neq s$ for all $t \geq T'$. Note that for a given initial action configuration, there are in general multiple best response sequences, as properties (i) and (iii) do not specify which player revises actions in which period.

We are concerned with the following questions. Is it possible in some network and some finite group of players such that if that group initially plays action s^*, then the whole population will eventually play s^*? In this case, s^* is said to be contagious. Or, is it always the case in any network that if s^* is played by almost all players, it continues to be played by almost all players? If so, s^* is said to be uninvadable. Below we formally define the relevant concepts following Morris (1997, 1999).

Definition 2. Given an unbounded local interaction system $(\mathcal{X}, P)$, action s^* is *contagious in* $(\mathcal{X}, P)$ if there exists a finite subset Y of $\mathcal{X}$ such that every best response sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma^0(x) = s^*$ for all $x \in Y$ satisfies $\lim_{t\to\infty} \sigma^t(x) = s^*$ for each $x \in \mathcal{X}$. Action s^* is *contagious* if it is contagious in some unbounded local interaction system.

Note that contagion of s^* in $(\mathcal{X}, P)$ requires that, once the finite set Y of initial s^*-players is chosen, s^* be eventually played by all the players along *any* best response sequence.

For uninvadability, the notion "almost all" is formalized by "except for a set of players whose weight with respect to P is finite".[3] For an action

[3] A finite set has a finite weight, and the converse is true if the neighborhood weights are bounded away from 0: i.e., for some $c > 0$, $\sum_y P(x, y) \geq c$ for all x.

configuration σ and a subset of actions $S' \subset S$, we write

$$\sigma_P(S') = \frac{1}{2} \sum_{(x,y)\colon \sigma(x)\in S' \text{ or } \sigma(y)\in S'} P(x,y).$$

In particular, for an action $s^* \in S$, $\sigma_P(S \setminus \{s^*\}) = (1/2)\sum_{(\sigma(x),\sigma(y))\neq(s^*,s^*)} P(x,y)$, which is the total weight of pairs of players who play action profiles other than (s^*, s^*).

Definition 3. Given an unbounded local interaction system $(\mathcal{X}, P)$, action s^* is *uninvadable in* $(\mathcal{X}, P)$ if there exists no best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$ and $\lim_{t\to\infty} \sigma_P^t(S \setminus \{s^*\}) = \infty$. Action s^* is *uninvadable* if it is uninvadable in any unbounded local interaction system.

By definition, if s^* is contagious, then actions other than s^* are not uninvadable; if s^* is uninvadable, then actions other than s^* are not contagious.

Here, uninvadability as well as contagion are defined for the universal domain of unbounded networks. Our main result (Theorem 1) characterizes this strong (weak, resp.) form of uninvadability (contagion, resp.). In Section 5, we will consider several restricted domains of networks and examine whether an action that is invaded (contagious, resp.) in the universal domain becomes uninvadable (remains contagious, resp.) in restricted domains.

3 The Bilingual Game

Hereafter, we consider the class of 3×3 games described in the Introduction. We denote the actions A, AB, and B by 0, 1, and 2, respectively, so that $S = \{0, 1, 2\}$, and let the payoff function $u\colon S \times S \to \mathbb{R}$ be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{3.1a}$$

where we assume

$$b < c \le d < a, \quad a - c < d - b, \quad e > 0. \tag{3.1b}$$

Action profiles $(0, 0)$ and $(2, 2)$ are the only pure-strategy Nash equilibria. By the assumption that $d < a$, $(0, 0)$ Pareto-dominates $(2, 2)$, while by $a - c < d - b$, $(2, 2)$ pairwise risk-dominates $(0, 0)$.[4] By the additional assumption that $c \le d$, this game is *supermodular* with respect to the order

[4]In Appendix A.6, we analyze the case where $(0, 0)$ is both Pareto-dominant and pairwise risk-dominant.

on actions $0 < 1 < 2$, i.e., $u(h', k) - u(h, k) \leq u(h', k') - u(h, k')$ if $h < h'$ and $k < k'$.

We will exploit the property of supermodular games, that the best response correspondence is nondecreasing in the stochastic dominance order. For $\pi, \pi' \in \Delta(S)$, we write $\pi \precsim \pi'$ (and $\pi' \succsim \pi$) if π' stochastically dominates π, i.e., if

$$\sum_{k \geq h} \pi_k \leq \sum_{k \geq h} \pi'_k$$

for all $h \in S$. If u is supermodular, then

$$\max br(\pi) \leq \max br(\pi')$$
$$\min br(\pi) \leq \min br(\pi')$$

whenever $\pi \precsim \pi'$.

4 Characterization

In this section, we show that the Pareto-dominant action 0 prevails if the bilingual cost e is small, while the pairwise risk-dominant action 2 survives if e is large. The thresholds will be constructed based on two parameters:

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b) + (a-c)(a-d)}.$$

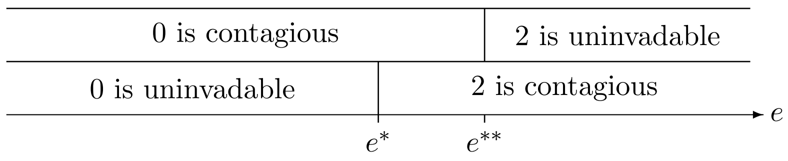
Verify that $e^* \lesseqqgtr e^{**}$ if $c - b \lesseqqgtr a - c$. The following result characterizes contagious and uninvadable actions in the bilingual game, quantifying our argument in the Introduction.

Theorem 1. *Let u be the bilingual game given by* (3.1).

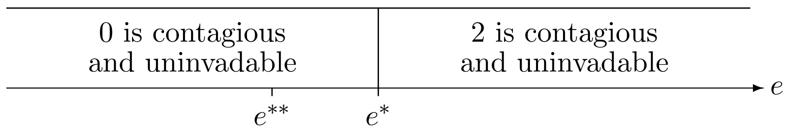
(i) 0 *is contagious if $e < \max\{e^*, e^{**}\}$ and uninvadable if $e < e^*$.* (ii) 2 *is contagious if $e > e^*$ and uninvadable if $e > \max\{e^*, e^{**}\}$.*

Note that for any (generic) value of e, at least one action is contagious and at most one action is uninvadable; when $e \in (e^*, e^{**})$ (which is nonempty if $c - b < a - c$), the two actions 0 and 2 are each contagious (in respective networks) and hence neither action is uninvadable.

One can verify that e^* and e^{**} increase as (i) b increases or c decreases or (ii) a and c increase by the same amount or d and b decrease by the same amount; that is, the contagion and uninvadability regions (in the space of e) of action 0 expand as action 0 becomes (i) less risky (i.e., b increases or c decreases) or (ii) more efficient (i.e., a increases with $a - c$ held fixed or d decreases with $d - b$ held fixed). This comparative statics is in stark



(1) $c - b < a - c$



(2) $c - b \geq a - c$

contrast with that in the 2×2 case, where the risk-dominance and hence the characterizations for contagion and uninvadability are not affected by any payoff change with $a - c$ and $d - b$ held fixed.

In Subsections 4.1 and 4.2, we prove the contagion and the uninvadability parts of Theorem 1, respectively.

Example 1. Let $a = 11$, $b = 0$, $c = 3$, and $d = 10$: the game is represented by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 11 & 11 & 0 \\ 11 - e & 11 - e & 10 - e \\ 3 & 10 & 10 \end{pmatrix} \end{array}.$$

Thus, $c - b = 3 < a - c = 8$, and $e^* = 5/3$ and $e^{**} = 40/19$. By Theorem 1, if $e > 40/19$, 2 is contagious and uninvadable; if $5/3 < e < 40/19$, both 0 and 2 are contagious; and if $e < 5/3$, 0 is contagious and uninvadable.

4.1 Contagion

We restate the contagion part of Theorem 1:

Proposition 1. *Let u be the bilingual game given by* (3.1).
(i) 0 *is contagious if* $e < \max\{e^*, e^{**}\}$. (ii) 2 *is contagious if* $e > e^*$.

We decompose the proof into two lemmas. Lemma 1 provides sufficient conditions for contagion of actions 0 and 2 in general 3×3 supermodular games. Lemma 2 then checks by direct computation when those conditions are satisfied in the bilingual game. Our main theoretical contribution is in the proof of Lemma 1, where we explicitly construct local interaction systems in which contagion occurs as desired.

To better understand how contagion occurs in the bilingual game, consider a population of players indexed by integers $x \in \mathcal{X} = \mathbb{Z}$, where player x interacts with players $x \pm 1$ with equal weights; see Figure 1. Suppose that

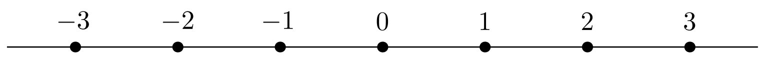


Figure 1: Nearest neighbor linear interaction

at time $t = 0$, all players play B except for players -1, 0, and 1 who play A, and assume that the bilingual cost e is small so that $e < (a - d)/2$ (where $(a-d)/2 \leq e^*$). We demonstrate that A spreads contagiously. (For concreteness, we here consider a particular best response sequence, while one can verify that contagion occurs for all best response sequences as the definition requires.) Note that, since A is pairwise risk-dominated by B, no player is willing to switch from B to A. Suppose that player 2 adjusts his action at $t = 1$. With his two neighbors playing A and B, respectively, he abandons B and switches to AB since $e < (a - d)/2 \leq (a - c)/2$. Suppose next that player 3 revises his action at $t = 2$. Since he has one AB-neighbor and one B-neighbor, by $e < (a - d)/2$ he abandons B and switches to A or AB (depending on the payoff parameter values); let us assume that he chooses AB. Now let player 2 revise back again at $t = 3$. This time his neighbors are playing A and AB (instead of B), and hence he now switches to A. In this way, the region of A-players spreads, together with the "bilingual" region of AB-players between the A- and the B-regions; see Table 1.

	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$t = 0$	$\cdots$	B	A	A	A	B	B	B	$\cdots$
$t = 1$	$\cdots$	B	A	A	A	AB	B	B	$\cdots$
$t = 2$	$\cdots$	B	A	A	A	AB	AB	B	$\cdots$
$t = 3$	$\cdots$	B	A	A	A	A	AB	B	$\cdots$

Table 1: Contagion of action A

The above construction, which works only for $e < (a - d)/2$, is extended to obtain contagion of A for $e < e^*$ (and symmetrically that of B for $e > e^*$) in Lemma 1(a) where we construct a "linear" network with four neighbors (two for each side) with appropriately chosen weights (Figure 2). In order to obtain contagion further for the range $[e^*, e^{**})$ (which is nonempty when $c - b < a - c$), however, such a construction does not work and we need to construct a "non-linear" network in Lemma 1(b), in which different players may have different types of interacting neighborhoods (Figure 3).

For $p \in (0, 1/2)$ and $q, r \in (0, 1)$, $r \leq q$, let

$$\pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \quad \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right),$$

and

$$\pi^c = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \quad \pi^d = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \quad \pi^e = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right),$$
$$\rho^c = \left(\tfrac{1-q}{2}, 0, \tfrac{1+q}{2}\right), \quad \rho^d = \left(\tfrac{1+r}{2}, 0, \tfrac{1-r}{2}\right), \quad \rho^e = \left(\tfrac{q-r}{2q}, \tfrac{q+r}{2q}, 0\right).$$

The conditions for contagion of actions 0 and 2 are stated in terms of best responses to the above mixed actions.

Lemma 1. *Let u be any 3×3 supermodular game.*

(a) (i) *If for some $p \in (0, 1/2)$,*

$$\max br(\pi^a) = 0, \quad \max br(\pi^b) \leq 1, \tag{4.1}$$

then 0 is contagious. (ii) *If for some $p \in (0, 1/2)$,*

$$\min br(\pi^a) \geq 1, \quad \min br(\pi^b) = 2, \tag{4.2}$$

then 2 is contagious.

(b) (i) *If for some $q, r \in (0, 1)$ with $r \leq q$,*

$$\max br(\pi^c) = 0, \quad \max br(\pi^d) \leq 1, \quad \max br(\pi^e) = 0, \tag{4.3}$$

then 0 is contagious. (ii) *If for some $q, r \in (0, 1)$ with $r \leq q$,*

$$\min br(\rho^c) = 2, \quad \min br(\rho^d) \geq 1, \quad \min br(\rho^e) = 2, \tag{4.4}$$

then 2 is contagious.

Proof. (a) Since cases (i) and (ii) are symmetric, we only show case (i). Let $p \in (0, 1/2)$ satisfy (4.1). We construct a local interaction system $(\mathcal{X}, P)$ in which action 0 spreads contagiously from a finite set of players $Y \subset \mathcal{X}$.

Let $\mathcal{X} = \mathbb{Z}$, and P be defined by

$$P(x, y) = \begin{cases} p & \text{if } |x - y| = 1 \\ \frac{1}{2} - p & \text{if } |x - y| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The defined local interaction system is depicted in Figure 2.

Let $Y = \{-3, \dots, 2\}$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t \to \infty} \sigma^t(x) = 0 \tag{$\diamondsuit_x$}$$

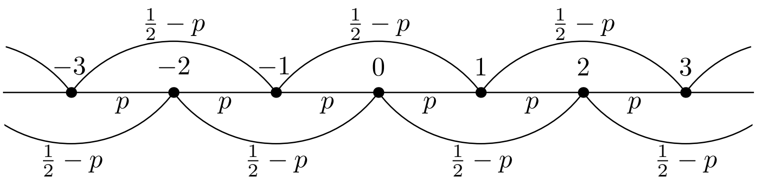


Figure 2: Linear interaction

holds for all $x \in \mathcal{X}$. We only consider players $x \geq 0$; the analogous argument applies to $x < 0$. Recall that, for an action configuration σ, $\pi(\sigma|x)$ denotes the distribution over neighbors' actions. We note that by the assumption (4.1) and the supermodularity of u,

$$\min BR(\sigma|x) = 0 \quad \text{if } \pi(\sigma|x) \precsim \pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \tag{4.5}$$

$$\min BR(\sigma|x) \leq 1 \quad \text{if } \pi(\sigma|x) \precsim \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right). \tag{4.6}$$

We first show ($\diamondsuit_0$) and ($\diamondsuit_1$), or more strongly, that

$$\sigma^t(x) = 0 \text{ for } x = -2, \ldots, 1$$
$$\sigma^t(x) \leq 1 \text{ for } x = -3, 2$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t - 1$, then $\pi(\sigma^{t-1}|x) \precsim \pi^a$ for $x = -2, \ldots, 1$ and $\pi(\sigma^{t-1}|x) \precsim \pi^b$ for $x = -3, 2$, so that we have $\sigma^t(x) = 2$ for $x = -2, \ldots, 1$ and $\sigma^t(x) \leq 1$ for $x = -3, 2$ by (4.5) and (4.6), respectively.

Assume ($\diamondsuit_{x-2}$) and ($\diamondsuit_{x-1}$). Then, there exists T_0 such that $\sigma^t(x-2) = \sigma^t(x-1) = 0$ for all $t \geq T_0$, so that $\pi(\sigma^t|x) \precsim \pi^b$ for all $t \geq T_0$. By (4.6), this implies that there exists T_1 such that $\sigma^t(x) \leq 1$ for all $t \geq T_1$. It follows that $\pi(\sigma^t|x+1) \precsim \pi^b$ for all $t \geq T_1$. By (4.6), this implies that there exists T_2 such that $\sigma^t(x+1) \leq 1$ for all $t \geq T_2$. It thus follows that $\pi(\sigma^t|x) \precsim \pi^a$ for all $t \geq T_2$, which by (4.5) implies that there exists T_3 such that $\sigma^t(x) = 0$ for all $t \geq T_3$, meaning that ($\diamondsuit_x$) holds.

(b) Since cases (i) and (ii) are symmetric, we only show case (i). Let $q, r \in (0, 1)$, $r \leq q$, satisfy (4.3). We construct a local interaction system $(\mathcal{X}, P)$ in which action 0 spreads contagiously from a finite set of players $Y \subset \mathcal{X}$.

Let $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$, and P be defined by

$$P((\alpha,i),(\alpha,j)) = \begin{cases} 1-q & \text{if } |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$P((\alpha,i),(\beta,j)) = P((\beta,j),(\alpha,i)) = \begin{cases} q+r & \text{if } i = j \\ q-r & \text{if } i = j+1 \text{ and } j \geq 0 \\ q-r & \text{if } i = j-1 \text{ and } j \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$P((\beta,i),(\beta,j)) = 0$ for all i, j.

The defined local interaction system is depicted in Figure 3.

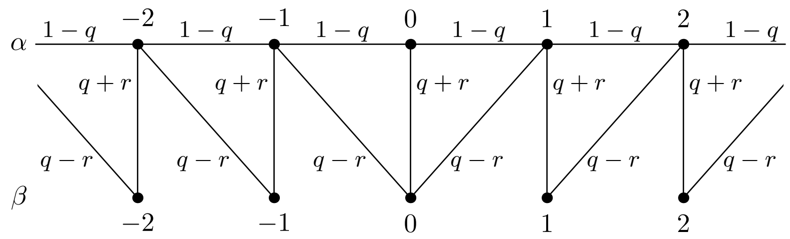


Figure 3: Non-linear interaction

Let $Y = \{(\alpha,i) \mid i = -1,0,1\} \cup \{(\beta,i) \mid i = -1,0,1\}$, and consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. We want to show that

$$\lim_{t\to\infty} \sigma^t(\alpha,i) = 0 \text{ and } \lim_{t\to\infty} \sigma^t(\beta,i) = 0 \tag{$\heartsuit_i$}$$

holds for all $i \in \mathbb{Z}$. We only consider $i \geq 0$; the analogous argument applies to $i < 0$. We note that by the assumption (4.3) and the supermodularity of u,

$$\max BR(\sigma|x) = 0 \text{ if } \pi(\sigma|x) \precsim \pi^c = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \tag{4.7}$$

$$\max BR(\sigma|x) \leq 1 \text{ if } \pi(\sigma|x) \precsim \pi^d = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \tag{4.8}$$

$$\max BR(\sigma|x) = 0 \text{ if } \pi(\sigma|x) \precsim \pi^e = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right). \tag{4.9}$$

We first show $(\heartsuit_1)$, or more strongly, that

$$\sigma^t(\alpha,i) = \sigma^t(\beta,i) = 0 \text{ for } i = -1,0,1$$

for all $t \geq 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then $\pi(\sigma^{t-1}|(\alpha,i)) \precsim \pi^c$ and $\pi(\sigma^{t-1}|(\beta,i)) \precsim \pi^e$, $i = -1,0,1$, so that we have $\sigma^t(\alpha,i) = \sigma^t(\beta,i) = 0$, $i = -1,0,1$, by (4.7) and (4.9).

Assume $(\heartsuit_{i-1})$. Then, there exists T_0 such that $\sigma^t(\alpha,i-1) = \sigma^t(\beta,i-1) = 0$ for all $t \geq T_0$, so that $\pi(\sigma^t|(\alpha,i)) \precsim \pi^d$ for all $t \geq T_0$. By (4.8), this implies that there exists T_1 such that $\sigma^t(\alpha, i) \leq 1$ for all $t \geq T_1$. It follows that $\pi(\sigma^t|(\beta, i)) \precsim \pi^e$ for all $t \geq T_1$. By (4.9), this implies that there exists T_2 such that $\sigma^t(\beta, i) = 0$ for all $t \geq T_2$. It thus follows that $\pi(\sigma^t|(\alpha, i)) \precsim \pi^c$ for all $t \geq T_2$, which by (4.7) implies that there exists T_3 such that $\sigma^t(\alpha, i) = 0$ for all $t \geq T_3$. We thus obtain $(\heartsuit_i)$. ▮

Denote

$$e^\sharp = \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)}.$$

Verify that $e^{**} \lesseqgtr e^\sharp$ if $c - b \lesseqgtr a - c$. The following result characterizes when the hypotheses in Lemma 1 are satisfied in the bilingual game.

Lemma 2. *Let u be the bilingual game given by* (3.1)*.*
(a) (i) *Condition* (4.1) *holds for some $p \in (0, 1/2)$ if $e < e^*$.* (ii) *Condition* (4.2) *holds for some $p \in (0, 1/2)$ if $e > e^*$.*
(b) *Condition* (4.3) *holds for some $0 < r \leq q < 1$ if $e < \min\{e^{**}, e^\sharp\}$.*

Proof. See Appendix A.1. ▮

Proof of Proposition 1. (i) Suppose that $e < \max\{e^*, e^{**}\}$. We consider two cases separately, depending on $c - b \lesseqgtr a - c$. In the case of $c - b \geq a - c$, since $e < \max\{e^*, e^{**}\} = e^*$, condition (4.1) holds for some $p \in (0, 1/2)$ by Lemma 2(a-i), and hence 0 is contagious by Lemma 1(a-i). In the case of $c - b < a - c$, since $e < \max\{e^*, e^{**}\} = e^{**} = \min\{e^{**}, e^\sharp\}$, condition (4.3) holds for some $0 < r \leq q < 1$ by Lemma 2(b-i), and hence 0 is contagious by Lemma 1(b-i). In both cases, 0 is contagious.
(ii) Suppose that $e > e^*$. Then condition (4.2) holds for some $p \in (0, 1/2)$ by Lemma 2(a-ii), and hence 2 is contagious by Lemma 1(a-ii). ▮

Case (a) is consistent with the one-dimensional setup of Goyal and Janssen (1997). They show that under local interaction on a circle, 0 is contagious if $e < e^*$ while 2 is contagious if $e > e^*$.

4.2 Uninvadability

We restate the uninvadability part of Theorem 1:

Proposition 2. *Let u be the bilingual game given by* (3.1)*.*
(i) 0 *is uninvadable if $e < e^*$.* (ii) 2 *is uninvadable if $e > \max\{e^*, e^{**}\}$.*

The condition for uninvadability is stated by using the concept of *monotone potential maximizer* (*MP-maximizer*) due to Morris and Ui (2005). We employ its refinement, strict MP-maximizer, due to Oyama et al. (2008). For our purpose, we define it only for the smallest and the largest actions, which we denote by $\underline{s}$ and $\overline{s}$, respectively.[5] For a function $f\colon S\times S\to\mathbb{R}$ and a probability distribution $\pi\in\Delta(S)$, write $br_f(\pi)=\arg\max_{h\in S} f(h,\pi)$. (Thus the best response correspondence br for the game u as defined in equation (2.1) is now denoted br_u.) Function f is symmetric if $f(h,k)=f(k,h)$ for all $h,k\in S$ (i.e., it is a symmetric $|S|\times|S|$ matrix).

Definition 4. (i) $\underline{s}$ is a *strict MP-maximizer* of u if there exists a symmetric function $v\colon S\times S\to\mathbb{R}$ with $v(\underline{s},\underline{s})>v(h,k)$ for all $(h,k)\neq(\underline{s},\underline{s})$ such that for all $\pi\in\Delta(S)$,

$$\max br_u(\pi)\leq\max br_v(\pi). \tag{4.10}$$

Such a function v is called a *strict MP-function* for $\underline{s}$.

(ii) $\overline{s}$ is a *strict MP-maximizer* of u if there exists a symmetric function $v\colon S\times S\to\mathbb{R}$ with $v(\overline{s},\overline{s})>v(h,k)$ for all $(h,k)\neq(\overline{s},\overline{s})$ such that for all $\pi\in\Delta(S)$,

$$\min br_u(\pi)\geq\min br_v(\pi). \tag{4.11}$$

Such a function v is called a *strict MP-function* for $\overline{s}$.

A strict MP-maximizer is a strict Nash equilibrium and, in supermodular games, is unique if it exists (Oyama et al. (2008)).

Lemma 3. *Let u be any game. If $s^*=\underline{s},\overline{s}$ is a strict MP-maximizer of u with MP-function v and if u or v is supermodular, then s^* is uninvadable.*

Proof. See Appendix A.2. ∎

In the bilingual game, Proposition 1 and Lemma 3 imply that action 0 (2, resp.) is never a strict MP-maximizer if $e>e^*$ ($e<\max\{e^*,e^{**}\}$, resp.), and hence, no strict MP-maximizer exists if $e^*<e<\max\{e^*,e^{**}\}$. The following lemma establishes existence of a strict MP-maximizer for the remaining cases (except for knife-edge values of e).

Lemma 4. *Let u be the bilingual game given by* (3.1).

(i) 0 *is a strict MP-maximizer if* $e<e^*$. (ii) 2 *is a strict MP-maximizer if* $e>\max\{e^*,e^{**}\}$.

Proof. See Appendix A.3. ∎

In 2×2 coordination games, a risk-dominant equilibrium is a strict MP-maximizer. A strict MP-maximizer generically exists in symmetric 3×3 supermodular games such that the symmetric action profiles are all Nash equilibria (Morris (1999), Oyama and Takahashi (2009)). Lemma 4 establishes an analogous result for our bilingual game, where only two symmetric action profiles are Nash equilibria.

[5]Here, we define for actions, rather than action profiles, since we only consider symmetric action profiles of symmetric games.

5 (Non-)Critical Classes of Networks

Thus far, we have allowed for the universal domain of all networks, and in particular, for our contagion result (Lemma 1) we had a maximal freedom to choose a network to obtain contagion. There, the constructed networks have different contagious actions for the same set of payoff parameters. This, in turn, suggests that one can differentiate (subclasses of) networks by analyzing strategic behavior on each network for various payoff parameters of our bilingual game. In this section, we study several subclasses of networks and derive conditions for contagion and uninvadability in these subclasses. This exercise enables us to classify networks in terms of those conditions in the bilingual game, which will provide a finer analysis than the one based on 2×2 coordination games.

In particular, we examine whether a given class of networks is critical for contagion. Formally, for a given game u and for a class $\mathcal{C}$ of unbounded networks, action s^* is *contagious in* $\mathcal{C}$ (*uninvadable in* $\mathcal{C}$, resp.) if it is contagious in some network in $\mathcal{C}$ (uninvadable in every network in $\mathcal{C}$, resp.). We say that a class $\mathcal{C}$ is *critical for contagion* if any action s^* that is contagious in the universal domain is also contagious in $\mathcal{C}$. In that case, one can restrict attention to that class to characterize contagious actions. Conversely, if $\mathcal{C}$ is non-critical for contagion, some action is contagious in no network in $\mathcal{C}$ but in some network outside $\mathcal{C}$. For example, if the game u is a 2×2 coordination game, the risk-dominant equilibrium is contagious in the network in Figure 1, and hence that network forms a (singleton) critical class for contagion. On the other hand, if u is the bilingual game, it follows from our analysis in the previous section that the network in Figure 1 is not critical for some parameter values, while the union of two classes of networks given by Figures 2 and 3 is critical.

In what follows, we consider two classes of "simple" networks, which we call linear and multidimensional lattice networks, and show that these classes of networks are not critical for contagion in the bilingual game. We also compare tree networks and "ladder" networks and demonstrate that the bilingual game distinguishes these networks, which are indistinguishable based on 2×2 games.

5.1 Linear Networks

We first introduce linear networks and analyze contagion and uninvadability in those networks. A local interaction system $(\mathcal{X}, P)$ is *linear* if $\mathcal{X} = \mathbb{Z}$ and interaction weights P are invariant up to translation: $P(x, y) = P(x+z, y+z)$ for $x, y, z \in \mathbb{Z}$. (Note that any linear network is unbounded.) Clearly, both the networks in Figure 1 and in Figure 2 are linear. On the other hand, the network in Figure 3 is not linear.[6]

Due to translation invariance and symmetry of P, we have $P(0, x) = P(-x, 0) = P(0, -x)$ for each $x \in \mathbb{Z}$, hence $P(x|0) = P(-x|0)$. Conversely, conditional weights $P(x|0)$ of player 0 determine translation invariant weights $P(x, y)$ uniquely (up to positive constant multiplication) if $P(0|0) = 0$, and $P(x|0)$ satisfies reflection symmetry, i.e., $P(x|0) = P(-x|0)$ for all $x > 0$.

It follows from the proof of Lemma 1 that in the class of linear networks given in Figure 2, action 0 (2, resp.) is contagious if $e < e^*$ ($e > e^*$, resp.). The following theorem shows that these conditions are also sufficient for uninvadability in the class of all linear networks.

Theorem 2. *Let u be the bilingual game given by* (3.1).

(i) 0 *is contagious and uninvadable in the class of linear networks if* $e < e^*$. (ii) 2 *is contagious and uninvadable in the class of linear networks if* $e > e^*$.

Proof. See Appendix A.4. ▮

The theorem gives a different characterization from the one for the universal domain given in Theorem 1 when $c - b < a - c$ and $e^* < e < e^{**}$, which implies that in this range of parameter values, the class of linear networks is *not* critical for contagion.

This characterization generalizes to a slightly larger class of networks where each node on a line is replicated into finitely many nodes. Formally, $(\mathcal{X}, P)$ is a *replicated linear network* if $\mathcal{X} = \{1, \ldots, m\} \times \mathbb{Z}$ and P is invariant up to translation, i.e., $P(x, y) = P(x + z, y + z)$ for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \{1, \ldots, m\} \times \mathbb{Z}$, where sums in the first coordinate, $x_1 + z_1$ and $y_1 + z_1$, are defined modulo m. For example, Figure 4 in Morris (2000, Example 4), Figure 2 (thick line graphs) in Immorlica et al. (2007), and our Figure??? are replicated linear networks, whereas the network in Figure 3 is not.

Theorem 3. *Let u be the bilingual game given by* (3.1).

(i) 0 *is contagious and uninvadable in the class of replicated linear networks if* $e < e^*$. (ii) 2 *is contagious and uninvadable in the class of replicated linear networks if* $e > e^*$.

The proof is analogous to that of Theorem 2 and thus omitted. This theorem implies that the class of all replicated linear networks is not critical for contagion.

[6] Even if we map $\mathcal{X} = \{\alpha, \beta\} \times \mathbb{Z}$ to $\mathbb{Z}$ by relabeling (α, i) with $2i$ and (β, i) with $2i + 1$, interaction weights do not satisfy translation invariance.

5.2 Multidimensional Lattice Networks

We next show that the characterization in the previous subsection generalizes to multidimensional lattice networks with traslation invariant interaction weights. For the sake of concreteness, we here focus on the m-dimensional lattice with *n-max distance interactions*, where each player interacts with all players within n steps away in each of the m coordinates, i.e., $\mathcal{X} = \mathbb{Z}^m$, and $P(x, y) = 1$ if $1 \leq \max_{i=1,\ldots,m} |x_i - y_i| \leq n$ and $P(x, y) = 0$ otherwise. A more general treatment is relegated to Appendix A.5, where we consider a broader class of networks on $\mathbb{Z}^m$ such that interaction weights $P(x, y)$ are translation invariant and conditional weights $P(x|0)$ are approximated (with an appropriate normalization) by a density function on $\mathbb{R}^m$.

For 2×2 coordination games, Morris (2000) demonstrates that the characterization for contagion and uninvadability in the linear lattice still holds with higher dimensions as long as the interaction radius n is sufficiently large. We obtain an analogous characterization for our bilingual game.

Theorem 4. *Let u be the bilingual game given by* (3.1)*. Fix the dimension m.*

(i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$,* 0 *is contagious and uninvadable in the n-max distance interaction network on $\mathbb{Z}^m$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \geq \bar{n}$,* 2 *is contagious and uninvadable in the n-max distance interaction network on $\mathbb{Z}^m$.*

Proof. See Appendix A.5. ∎

The proof is analogous to that of Lemma 1. In the case of $e < e^*$, for example, we show the contagion of action 0 by an induction argument along a sequence of regions of 0-players surrounded by "bilingual" regions. Here, each 0-player region is the set of lattice (i.e., integer-coordinate) points contained in a large m-dimensional ball with an outer m-dimensional ring of 1-players.

To conclude, the class of n-max distance interaction networks with large n as well as the class of (replicated) linear networks are not critical for contagion, and hence the network in Figure 3 exhibits fundamentally different properties in strategic behavior from those simple networks.

5.3 A Tree and a Ladder

In this subsection, we report another pair of examples in which the analysis of our 3×3 game strictly refines the classification of networks based on 2×2 coordination games.

6 Interpretations in Incomplete Information Games

Local interaction games and incomplete information games, though capturing different economic or social situations, share the same formal structures and thus belong to a more general class of "interaction games" (Morris (1997, 1999), Morris and Shin (2003)): in local interaction games, each node interacts with a set of neighbors and payoffs are given by the weighted sum of those from the interactions; in incomplete information games, each type interacts with a subset of types and payoffs are given by the expectation of those from the interactions. Indeed, Morris (1997, 1999) demostrates, in spite of some technical differences, that several tools and results in the context of incomplete information games can be utilized also in the context of local interaction games, and vice versa.[7] In this section, we interpret our results, in particular the disucssions in the previous section, in the language of incomplete information games, thereby shedding new lights on two existing lines of literature, robustness to incomplete information (Kajii and Morris (1997), Morris and Ui (2005)) and global games (Carlsson and van Damme (1993), Frankel et al. (2003), Oury (2009)).

6.1 Robustness to Incomplete Information

We say that a Nash equilibrium (s_1^*, s_2^*) of a two-player game u is *robust to incomplete information* if every incomplete information game in which the payoffs are given by u with high probability has a Bayesian Nash equilibrium that plays (s_1^*, s_2^*) with high probability (Kajii and Morris (1997)). Robustness to incomplete information corresponds to uninvadability in local interaction systems in that both notions require that a small amount of "crazy types" should not affect the aggregate behavior.

Indeed, they have the same characterizations in many classes of games, including games with an MP-maximizer. In parallel with Lemma 3, an MP-maximizer of a game u with MP-function v is robust to incomplete information if u or v is supermodular (Morris and Ui (2005)). Combining this result with Lemma 4, we obtain a sufficient condition for robustness in the bilingual game.

Conversely, a necessary condition for robustness is obtained by constructing incomplete information games in which a given action profile is never played in equilibrium. Specifically, in any 3×3 supermodular game u, adjusting the proof of Lemma 1, one can construct incomplete information games in which the payoffs are given by u with probability arbitrarily close

[7] A class of dynamic games with Poisson action revisions due to Matsui and Matsuyama (1995) (perfect foresight dynamics) also belong to interaction games, where each revising player interacts with a set of past and future players and payoffs are given the discounted sum of flow payoffs from the interactions (Takahashi (2008)).

to 1 and playing 0 (2, resp.) everywhere is a unique rationalizable strategy if (4.1) ((4.2), resp.) holds for some $p \in (0, 1/2)$, or (4.3) ((4.4), resp.) holds for some $q, r \in (0, 1)$ with $r \leq q$ (Oyama and Takahashi (2011)). The necessary condition thus follows by applying this result to the bilingual game combined with Lemma 2.

These arguments characterize exactly as in Theorem 1 when an equilibrium in the bilingual game is robust to incomplete information.

Proposition 3. *Let u be the bilingual game given by* (3.1).

(i) $(0, 0)$ *is a unique robust equilibrium if* $e < e^*$. (ii) $(2, 2)$ *is a unique robust equilibrium if* $e > \max\{e^*, e^{**}\}$. (iii) *No action profile is robust if* $e^* < e < \max\{e^*, e^{**}\}$.

6.2 Global Games

Global games constitute a subclass of incomplete information games, where the underlying state θ is drawn from the real line, and each player i receives a noisy signal $x_i = \theta + \nu\varepsilon_i$ with ε_i being a noise error independent across players and from θ. Under supermodularity and state-monotonicity in payoffs, it is shown by a contagion argument that an essentially unique equilibrium survives iterative deletion of dominated strategies as $\nu \to 0$, while the limit equilibrium may depend on the distribution of noise terms ε_i (Frankel et al. (2003)).

Global game perturbations in the class of all incomplete information perturbations can be viewed as linear networks in the class of all networks. In global games, the distribution of the opponent's signal x_j conditional on x_i is (approximately) invariant up to translation (for small $\nu > 0$) due to the assumption of state-independent noise errors, which parallels the translation invariance in linear networks. In fact, Basteck and Daniëls (2010) prove that, in any global game of 3×3 supermodular games independently of the noise distribution, action profile $(0, 0)$ ($(2, 0)$, resp.) is played at θ as $\nu \to 0$ if (4.1) ((4.2), resp.) holds for some $p \in (0, 1/2)$ at that state θ. Together with Lemma 2(a), this leads to the following characterization of global-game noise-independent selection in the bilingual game, the same one as in Theorem 2.

Proposition 4. *Let u be the bilingual game given by* (3.1).

(i) $(0, 0)$ *is a noise-independent global game selection if* $e < e^*$. (ii) $(2, 2)$ *is a noise-independent global game selection if* $e > e^*$.

Since this characterization is different from that in Proposition 3, global games are not a critical class of incomplete information games that determines whether or not an action profile is robust to incomplete information.[8]

[8]Oury (2009) shows, for general supermodular games, that the noise-independence characterization in the unidimensional state space extends to multidimensional spaces.

Appendix

A.1 Proof of Lemma 2

Recall

$$\begin{aligned} e^* &= \frac{(a-d)(d-b)}{2(c-b)}, \\ e^{**} &= \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)}, \\ e^\sharp &= \frac{(d-b)\{2(a-c)-(d-b)\}}{2(a-c)}, \end{aligned}$$

where $e^* \lesseqgtr e^{**} \lesseqgtr e^\sharp$ if $c-b \lesseqgtr a-c$.

Proof of Lemma 2. (a) We first note that for all $p \in (0,1/2)$, $u(2,\pi^b) > u(0,\pi^b)$ and hence $0 \notin br(\pi^b)$.

We divide the argument into two cases: (α) $e > (a-c)/2$ and (β) $e \le (a-c)/2$.

(α) $e > (a-c)/2$: In this case, if we let $p = 0$ (hence $\pi^a = \pi^b$), $br(\pi^a) = br(\pi^b) = \{2\}$, and thus condition (4.2) holds for some $p \in (0,1/2)$ close to 0 due to the upper semi-continuity of br.

(β) $e \le (a-c)/2$: In this case, for all $p \in (0,1/2)$, $u(1,\pi^a) > u(2,\pi^a)$ and hence $2 \notin br(\pi^a)$. Therefore, $\max br(\pi^a) = 0 \Leftrightarrow u(0,\pi^a) > u(1,\pi^a)$ and $\max br(\pi^b) \le 1 \Leftrightarrow u(1,\pi^b) > u(2,\pi^b)$, while $\min br(\pi^a) \ge 1 \Leftrightarrow u(1,\pi^a) > u(0,\pi^a)$ and $\min br(\pi^b) = 2 \Leftrightarrow u(2,\pi^b) > u(1,\pi^b)$.

Verify that $e^* \le (a-c)/2$ with the equality holding if and only if $c = d$. Consider first the case where $e^* < (a-c)/2$ (or $c < d$). Then, since

$$\begin{aligned} u(0,\pi^a) - u(1,\pi^a) &= (d-b)\left\{p - \frac{(d-b)-2e}{2(d-b)}\right\}, \\ u(1,\pi^b) - u(2,\pi^b) &= (d-c)\left\{\frac{(a-c)-2e}{2(d-c)} - p\right\}, \end{aligned}$$

it follows that condition (4.1) holds for some $p \in (0,1/2)$ if and only if

$$\frac{(d-b)-2e}{2(d-b)} < \frac{(a-c)-2e}{2(d-c)} \iff e < e^*,$$

while condition (4.2) holds for some $p \in (0,1/2)$ if and only if

$$\frac{(a-c)-2e}{2(d-c)} < \frac{(d-b)-2e}{2(d-b)} \iff e > e^*.$$

Our analysis on multidimensional lattice networks in Subsection 5.2 parallels her analysis applied to the bilingual game.

If $e^* = (a-c)/2$ (or $c = d$), then $u(0, \pi^a) > u(1, \pi^a)$ and $u(1, \pi^b) > u(2, \pi^b)$ for some $p \in (0, 1/2)$ close to 1/2 whenever $e < e^*$. (The condition $e > e^*$ never holds in the current case of $e \leq (a-c)/2$ $(= e^*)$.)

(b) We first note that $u(2, \pi^d) > u(0, \pi^d)$ and hence $0 \notin br(\pi^d)$ for all $r \in (0, 1)$. Therefore,

$$\begin{aligned} \max br(\pi^d) \leq 1 &\iff u(1, \pi^d) > u(2, \pi^d) \\ &\iff r < \frac{(a-c)-2e}{a-c}. \end{aligned} \tag{A.1}$$

For the last inequality to hold, it is necessary that $e < (a-c)/2$.

Under the condition that $e < (a-c)/2$, note that $u(1, \pi^c) > u(2, \pi^c)$ and hence $2 \notin br(\pi^c)$ for all $q \in (0, 1)$. Therefore,

$$\begin{aligned} \max br(\pi^c) = 0 &\iff u(0, \pi^c) > u(1, \pi^c) \\ &\iff q > \frac{(d-b)-2e}{d-b}. \end{aligned} \tag{A.2}$$

Finally,

$$\begin{aligned} &\max br(\pi^e) = 0 \\ &\iff u(0, \pi^e) > u(1, \pi^e) \text{ and } u(0, \pi^e) > u(2, \pi^e) \\ &\iff r > \frac{(d-b)-2e}{d-b} q \text{ and } r > \frac{(d-b)-(a-d)}{a-b} q. \end{aligned} \tag{A.3}$$

From (A.1)–(A.3), it follows that condition (4.3) holds for some $0 < r \leq q < 1$ if and only if

$$\begin{cases} \dfrac{(a-c)-2e}{a-c} > \left\{ \dfrac{(d-b)-2e}{d-b} \right\}^2 \\ \dfrac{(a-c)-2e}{a-c} > \dfrac{(d-b)-(a-d)}{a-b} \cdot \dfrac{(d-b)-2e}{d-b}, \end{cases}$$

which reduces to $e < \min\{e^\sharp, e^{**}\}$. ∎

A.2 Proof of Lemma 3

We show a stronger result, that a strict MP-maximizer is uninvadable with respect to a wider class of best response sequences. We write BR_f for the best correspondence in the local interaction game $(\mathcal{X}, P, f)$:

$$\begin{aligned} BR_f(\sigma|x) = \Big\{ s \in S \;\Big|\; &\textstyle\sum_{y \in \Gamma(x)} P(y|x) f(s, \sigma(y)) \\ &\geq \textstyle\sum_{y \in \Gamma(x)} P(y|x) f(s', \sigma(y)) \text{ for all } s' \in S \Big\}. \end{aligned}$$

(Thus the best response correspondence for u as defined in equation (2.2) is now denoted BR_u.) Recall that $BR_f(\sigma|x) = br_f(\pi(\sigma|x))$. We consider sequences that satisfy the following property.

Definition A.1. Given a local interaction system $(\mathcal{X}, P)$ and for a payoff function $f\colon S \times S \to \mathbb{R}$, a sequence $(\sigma^t)_{t=0}^{\infty}$ satisfies property $\mathbf{B}^*$ in f if for each $t \geq 1$, there exists $x^t \in \mathcal{X}$ such that $\sigma^t(x^t) \in BR_f(\sigma^{t-1}|x^t)$ and $\sigma^t(y) = \sigma^{t-1}(y)$ for all $y \neq x^t$.

Best response sequences as defined in Definition 1 clearly satisfy this property (with $f = u$).

Let s^* be a strict MP-maximizer of u with a strict MP-function v. Recall that v is a symmetric function (i.e., $v(h,k) = v(k,h)$). The game defined by a symmetric function v is called a *potential game*, and given that $\{(s^*, s^*)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, $s^* \in S$ is called a *potential maximizer* of v. The following result is due to Morris (1999, Proposition 6.1). We provide its proof for completeness.

Lemma A.1. *Suppose that s^* is a potential maximizer of a potential game v. For any unbounded local interaction system $(\mathcal{X}, P)$ and for any sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma_P^0(S \setminus \{s^*\}) < \infty$ that satisfies property $\mathbf{B}^*$ in v, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let s^* be a potential maximizer of a potential game v. Let $\bar{\gamma} = \max_{h,k}\big(v(s^*,s^*) - v(h,k)\big) < \infty$ and $\underline{\gamma} = \min_{(h,k)\neq(s^*,s^*)}\big(v(s^*,s^*) - v(h,k)\big) > 0$. Fix any local interaction system $(\mathcal{X}, P)$. Let $(\sigma^t)_{t=0}^{\infty}$ be any sequence such that $\sigma_P^0(S \setminus \{s^*\}) < \infty$, and assume that it satisfies property $\mathbf{B}^*$ in v. Let $(x^t)_{t=1}^{\infty}$ be such that $\sigma^t(x^t) \in BR_v(\sigma^{t-1}|x^t)$ and $\sigma^t(y) = \sigma^{t-1}(y)$ for all $y \neq x^t$.

Let

$$V(t) = \frac{1}{2} \sum_{(x,y)\in\mathcal{X}\times\mathcal{X}} P(x,y)\big(v(\sigma^t(x), \sigma^t(y)) - v(s^*, s^*)\big).$$

Note that

$$-\bar{\gamma}\sigma_P^t(S \setminus \{s^*\}) \leq V(t) \leq -\underline{\gamma}\sigma_P^t(S \setminus \{s^*\}).$$

Since $\sigma_P^0(S \setminus \{s^*\}) < \infty$, we have $V(0) > -\infty$. Also we have

$$\begin{aligned} &V(t) - V(t-1) \\ &\qquad = \sum_{y\in\Gamma(x^t)} P(x^t, y)\big(v(\sigma^t(x^t), \sigma^{t-1}(y)) - v(\sigma^{t-1}(x^t), \sigma^{t-1}(y))\big) \geq 0 \end{aligned}$$

by property $\mathbf{B}^*$. It follows from the induction on t that V is nondecreasing, so that $V(t) \geq V(0)$ for all t.

Then we have $\sigma_P^t(S \setminus \{s^*\}) \leq -V(t)/\underline{\gamma} \leq -V(0)/\underline{\gamma}$ for all t. ▮

Lemma 3 is a direct corollary of the following.

Lemma A.2. *Suppose that s^* is a strict MP-maximizer of u with a strict MP-function v. If u or v is supermodular, then for any unbounded local interaction system $(\mathcal{X}, P)$ and for any sequence $(\sigma^t)_{t=0}^{\infty}$ with $\sigma_P^0(S \setminus \{s^*\}) < \infty$ that satisfies property $\boldsymbol{B}^*$ in u, there exists $M < \infty$ such that $\sigma_P^t(S \setminus \{s^*\}) \leq M$ for all $t \geq 0$.*

Proof. Let $s^* = \underline{s}, \overline{s}$ be a strict MP-maximizer of u with a strict MP-function v. We only consider the case where $s^* = \underline{s}$. Fix any local interaction system $(\mathcal{X}, P)$. Let $(\sigma^t)_{t=0}^{\infty}$ be any sequence such that $\sigma_P^0(S \setminus \{\underline{s}\}) < \infty$, and assume that it satisfies property $\mathbf{B}^*$ in u. Let $(x^t)_{t=1}^{\infty}$ be such that $\sigma^t(x^t) \in BR_u(\sigma^{t-1}|x^t)$ and $\sigma^t(y) = \sigma^{t-1}(y)$ for all $y \neq x^t$.

Now let $(\hat{\sigma}^t)_{t=0}^{\infty}$ be defined by $\hat{\sigma}^0 = \sigma^0$ and for $t \geq 1$,

$$\hat{\sigma}^t(x) = \begin{cases} \max BR_v(\hat{\sigma}^{t-1}|x^t) & \text{if } x = x^t, \\ \hat{\sigma}^{t-1}(x) & \text{otherwise.} \end{cases}$$

Then, $(\hat{\sigma}^t)_{t=0}^{\infty}$ satisfies $\mathbf{B}^*$ in v. Therefore, by Lemma A.1, there exists M such that $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show that if u or v is supermodular, then

$$\sigma^t(x) \leq \hat{\sigma}^t(x) \text{ for all } x \in \mathcal{X}. \tag{$\star_t$}$$

for all $t \geq 0$. Then, $\sigma_P^t(S \setminus \{\underline{s}\}) \leq \hat{\sigma}_P^t(S \setminus \{\underline{s}\})$ for all t, and since $\hat{\sigma}_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t, it follows that $\sigma_P^t(S \setminus \{\underline{s}\}) \leq M$ for all t.

We show by induction that $(\star_t)$ holds for all $t \geq 0$. First, $(\star_0)$ trivially holds by the definition of $\hat{\sigma}^0$. Then, assume $(\star_{t-1})$. It implies that for all $x \in \mathcal{X}$, $\pi(\sigma^{t-1}|x) \precsim \pi(\hat{\sigma}^{t-1}|x)$. By construction, $\sigma^t(x) = \hat{\sigma}^t(x)$ for all $x \neq x^t$. For $x = x^t$, if u is supermodular, then

$$\begin{aligned} \sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_u(\pi(\hat{\sigma}^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t), \end{aligned}$$

where the second inequality follows from the supermodularity of u, and the third inequality follows from the MP condition (4.10). If v is supermodular, then

$$\begin{aligned} \sigma^t(x^t) &\leq \max br_u(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\sigma^{t-1}|x^t)) \\ &\leq \max br_v(\pi(\hat{\sigma}^{t-1}|x^t)) = \hat{\sigma}^t(x^t), \end{aligned}$$

where the second inequality follows from the MP condition (4.10), and the third inequality follows from the supermodularity of v. Therefore, in each case, $(\star_t)$ holds. ∎

A.3 Proof of Lemma 4

For $f\colon S \times S \to \mathbb{R}$ and $h \in S$, let

$$\Pi_h(f) = \{\pi \in \Delta(S) \mid h \in br_f(\pi)\}.$$

Note that 0 is a strict MP-maximizer of u with MP-function v if and only if $\{(0,0)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, and

$$\Pi_2(u) \subset \Pi_2(v) \text{ and } \Pi_1(u) \subset \Pi_1(v) \cup \Pi_2(v),$$

while 2 is a strict MP-maximizer of u with MP-function v if and only if $\{(2,2)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$, and

$$\Pi_0(u) \subset \Pi_0(v) \text{ and } \Pi_1(u) \subset \Pi_0(v) \cup \Pi_1(v).$$

Recall

$$e^* = \frac{(a-d)(d-b)}{2(c-b)},$$
$$e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)},$$

and denote

$$e^\flat = \frac{(a-d)(d-b)}{a-b}.$$

Verify that $e^\flat \lesseqgtr e^* \lesseqgtr e^{**}$ if $c-b \lesseqgtr a-c$.

Lemma 4 is proved by Lemmas A.3–A.5 which follow. Lemma A.3 considers the cases of $e < e^*$ and $e^* < e \leq e^\flat$ (note that $e^* \leq e^\flat$ if and only if $c-b \geq a-c$), and Lemmas A.4 and A.5 cover the cases of $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$ and $e > (a-c)/2$, respectively (observe that $\max\{e^{**}, e^\flat\} < (a-c)/2$); see Figure A.1.

Lemma A.3. (i) *If $e < e^*$, then 0 is a strict MP-maximizer.* (ii) *If $e^* < e \leq e^\flat$, then 2 is a strict MP-maximizer.*

Proof. Let v be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 2\lambda e & \lambda e & \lambda e-(a-c)+e \\ \lambda e & 0 & -(a-d)+e \\ \lambda e-(a-c)+e & -(a-d)+e & -(a-d)+2e \end{pmatrix} \end{array}, \qquad \text{(A.4)}$$

where

$$\lambda = \frac{d-c}{d-b} > 0.$$

We show that this function v works as a strict MP-function if $e \leq \max\{e^*, e^\flat\}$ and $e \neq e^*$.

We first have the following.

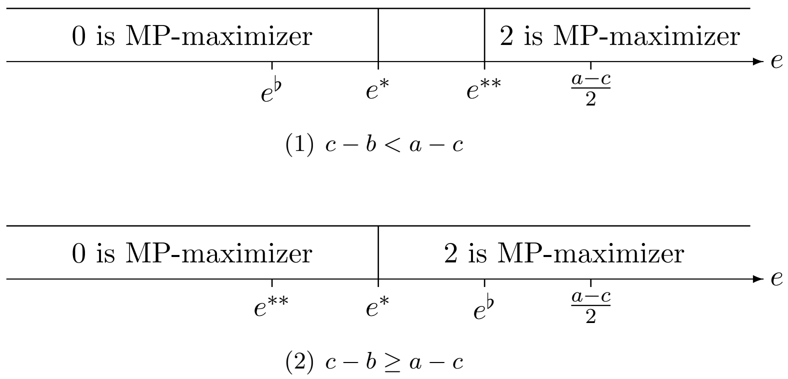


Figure A.1: MP-maximizer

Claim 1. (i) $\{(0,0)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if* $e < e^*$. (ii) $\{(2,2)\} = \arg\max_{(h,k)\in S\times S} v(h,k)$ *if* $e > e^*$.

Verify that

$$v(0,k) - v(1,k) = \lambda\big(u(0,k) - u(1,k)\big) \tag{A.5}$$

$$v(1,k) - v(2,k) = u(1,k) - u(2,k) \tag{A.6}$$

for all $k = 0,1,2$. These immediately imply the following.

Claim 2. $\Pi_1(u) = \Pi_1(v)$.

For $\pi = (\pi_0, \pi_1, \pi_2) \in \Delta(S)$, we have

$$u(0,\pi) - u(1,\pi) = (d-b)\left(\frac{e}{d-b} - \pi_2\right), \tag{A.7}$$

$$u(1,\pi) - u(2,\pi) = (d-c)\left\{\pi_0 - \frac{(a-b)e - (a-d)(d-b)}{(d-b)(d-c)}\right\} + (a-d)\left(\frac{e}{d-b} - \pi_2\right), \tag{A.8}$$

and

$$v(2,\pi) - v(0,\pi) = \big(u(2,\pi) - u(0,\pi)\big) + (c-b)\left(\frac{e}{d-b} - \pi_2\right). \tag{A.9}$$

These imply the following.

Claim 3. $\Pi_2(u) \subset \Pi_2(v)$.

Proof. Assume that $\pi = (\pi_0, \pi_1, \pi_2) \in \Pi_2(u)$ ($\Leftrightarrow$ $u(2,\pi) \geq u(0,\pi)$ and $u(2,\pi) \geq u(1,\pi)$). First, by (A.6), $u(2,\pi) \geq u(1,\pi)$ implies $v(2,\pi) \geq v(1,\pi)$. Second, if $\pi_2 \geq e/(d-b)$, then by (A.5) and (A.7), we have

$v(1,\pi) \geq v(0,\pi)$ and therefore $v(2,\pi) \geq v(0,\pi)$, while if $\pi_2 < e/(d-b)$, then by (A.9), $u(2,\pi) \geq u(0,\pi)$ implies $v(2,\pi) > v(0,\pi)$. We thus have $\pi \in \Pi_2(v)$. ▮

Claim 4. *If $e \leq e^\flat$, then $br_u = br_v$.*

Proof. Suppose that $e \leq e^\flat$. In light of Claim 2, we want to show that $\Pi_0(u) = \Pi_0(v)$ and $\Pi_2(u) = \Pi_2(v)$.

Note in (A.8) that $e \leq e^\flat$ implies $\{(a-b)e - (a-d)(d-b)\}/\{(d-b)(d-c)\} \leq 0$. By (A.7) and (A.8), we therefore have $u(0,\pi) \geq u(1,\pi) \Rightarrow u(1,\pi) \geq u(2,\pi)$ and $u(2,\pi) \geq u(1,\pi) \Rightarrow u(1,\pi) \geq u(0,\pi)$. By (A.5) and (A.6), it thus follows that $\pi \in \Pi_0(u) \Leftrightarrow u(0,\pi) \geq u(1,\pi) \Leftrightarrow v(0,\pi) \geq v(1,\pi) \Leftrightarrow \pi \in \Pi_0(v)$ and $\pi \in \Pi_2(u) \Leftrightarrow u(2,\pi) \geq u(1,\pi) \Leftrightarrow v(2,\pi) \geq v(1,\pi) \Leftrightarrow \pi \in \Pi_2(v)$. ▮

We now complete the proof of Lemma A.3. (i) If $e < e^*$, Claims 1, 2, and 3 imply that 0 is a strict MP-maximizer. (ii) If $e^* < e \leq e^\flat$, Claims 1 and 4 imply that 2 is a strict MP-maximizer. ▮

Lemma A.4. *If $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$, then 2 is a strict MP-maximizer.*

Proof. Suppose that $\max\{e^{**}, e^\flat\} < e \leq (a-c)/2$. Let v be defined by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{pmatrix} 0 & -\lambda e & -\lambda e - \{(a-c)-e\} \\ -\lambda e & -2\lambda e & \lambda\{(d-b)-2e\} - \{(a-c)-e\} \\ -\lambda e - \lambda\{(a-c)-e\} & \lambda\{(d-b)-2e\} - \{(a-c)-e\} & \lambda\{(d-b)-2e\} - \{(a-c)-2e\} \end{pmatrix} \end{array}, \tag{A.10}$$

where $\underline{\lambda} < \lambda \leq \overline{\lambda}$ with

$$\underline{\lambda} = \frac{(a-c)-2e}{(d-b)-2e} \geq 0, \qquad 0 < \overline{\lambda} = \frac{(a-c)(d-b)-(a-b)e}{(a-b)\{(d-b)-e\}} < 1$$

(such λ exists since $\underline{\lambda} < \overline{\lambda}$ by $e > e^{**}$). We show that this function v works as a strict MP-function.

First, the function (A.10) is maximized at $(2,2)$ (by $\lambda > \underline{\lambda}$). Second, one can verify, for all $k = 0,1,2$,

$$v(0,k) - v(1,k) = \lambda\big(u(0,k) - u(1,k)\big) \tag{A.11}$$

$$v(0,k) - v(2,k) \geq \mu\big(u(0,k) - u(2,k)\big) \tag{A.12}$$

for $\mu > 0$ such that

$$\lambda\frac{(d-b)-e}{d-b} + \frac{e}{d-b} \leq \mu \leq -\lambda\frac{(d-b)-e}{a-d} + \frac{(a-c)-e}{a-d}$$

(such μ exists by $\lambda \leq \overline{\lambda}$), and

$$v(1,k) - v(2,k) \geq u(1,k) - u(2,k) \tag{A.13}$$

(since $\lambda \leq \overline{\lambda} < (d-c)/(d-b)$ by $e > e^{\flat}$). Therefore, $\pi \in \Pi_0(u) \Rightarrow \pi \in \Pi_0(v)$ by (A.11)–(A.12) and $\pi \in \Pi_1(u) \Rightarrow \pi \in \Pi_0(v) \cup \Pi_1(v)$ by (A.13). ▮

Lemma A.5. *If $e > (a-c)/2$, then 2 is a strict MP-maximizer.*

Proof. Action 2 is strictly p-dominant with

$$p = \max\left\{\frac{a-c-e}{a-c}, \frac{a-c}{(a-c)+(d-b)}\right\},$$

i.e., $\{2\} = br_u(\pi)$ for any $\pi = (\pi_0, \pi_1, \pi_2) \in \Delta(S)$ such that $\pi_2 > p$ (Morris et al. (1995), Kajii and Morris (1997)). If $e > (a-c)/2$, we have $p < 1/2$. Therefore, the function

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & -p \\ -p & -p & 1-2p \end{pmatrix} \end{array} \tag{A.14}$$

is a strict MP-function for 2 (see Morris and Ui (2005) or Oyama et al. (2008, Lemma 4.1)). ▮

A.4 Proof of Theorem 2

Since the network used in the proof of Lemma 1(a) is linear, combined with Lemma 2(a), it follows that if $e < e^*$ (resp. $e > e^*$), then 0 (resp. 2) is contagious in linear networks. Also, by Proposition 2(i), if $e < e^*$, then 0 is uninvadable, hence uninvadable in linear networks. Thus, we only need to show that 2 is uninvadable in linear networks if $e > e^*$.

By Lemma 2, there exists $p \in (0, 1/2)$ that satisfies (4.2). By the upper semi-continuity of br, there exists $\varepsilon \in (0, 1/2 - p)$ such that $\min br(\tilde{\pi}^a) \geq 1$ and $\min br(\tilde{\pi}^b) = 2$, where

$$\tilde{\pi}^a = \left(\tfrac{1}{2} + \varepsilon, p, \tfrac{1}{2} - p - \varepsilon\right), \quad \tilde{\pi}^b = \left(\tfrac{1}{2} - p + \varepsilon, p, \tfrac{1}{2} - \varepsilon\right).$$

Fix any linear network $(\mathbb{Z}, P)$. Since $P(0|0) = 0$ and $P(x|0) = P(-x|0)$ for any $x > 0$, we have $\sum_{x=1}^{\infty} P(x|0) = 1/2$. Let n_1 be the smallest integer such that $\sum_{x=1}^{n_1} P(x|0) \geq p$. Let n_2 be a sufficiently large integer such that $\sum_{x>n_2} P(x|0) \leq \varepsilon$.

Consider any best response sequence $(\sigma^t)_{t=0}^{\infty}$ such that $\sigma_P^0(\{0,1\}) < \infty$. Then there exists a co-finite subset K of $\mathbb{Z}$ (i.e., $\mathbb{Z} \setminus K$ is finite) such that for each $k \in K$, $\sigma^0(x) = 2$ if $|x - (2n_2+1)k| \leq n_1 + n_2$. (Otherwise, $\sigma^0(x) \neq 2$ for infinitely many x, which contradicts the finiteness of $\sigma_P^0(\{0,1\})$.)

For each $k \in K$, we want to show that

$$\sigma^t(x) = 2 \text{ if } |x - (2n_2+1)k| \le n_2,$$
$$\sigma^t(x) \ge 1 \text{ if } n+1 \le |x - (2n_2+1)k| \le n_1 + n_2$$

for all $t \ge 0$. Indeed, this holds for $t = 0$ by construction, and if it holds for $t-1$, then $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^b$ for $|x - (2n_2+1)k| \le n_2$ and $\pi(\sigma^{t-1}|x) \succsim \tilde{\pi}^a$ for $n_2 + 1 \le |x - (2n_2+1)k| \le n_1 + n_2$, so that we have $\sigma^t(x) = 2$ for $|x - (2n_2+1)k| \le n_2$ and $\sigma^t(x) \ge 1$ for $n_2+1 \le |x-(2n_2+1)k| \le n_1+n_2$. Therefore, $\#\{x \in \mathbb{Z} \mid \sigma^t(x) \ne 2\} \le (2n_2+1)\#(\mathbb{Z} \setminus K)$, where $\#X$ denotes the cardinality of X, and hence $\sigma_P^t(\{0,1\})$ is bounded from above.

A.5 Proof of Theorem 4

We fix the dimension m. A sequence (P_n) of interaction weights on the m-dimensional lattice $\mathbb{Z}^m$ is *well-behaved* if the following conditions are satisfied.

- For each n, P_n is invariant up to translation, i.e., $P_n(x,y) = P_n(x+z, y+z)$ for $x, y, z \in \mathbb{Z}^m$.
- There exist a pair of nonnegative integrable functions $f, g \colon \mathbb{R}^m \to \mathbb{R}_+$ such that for almost every $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{R}^m$,

  $$n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0) \to f(\nu)$$

  as $n \to \infty$ (pointwise convergence), and

  $$n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0) \le g(\nu)$$

  for every n.[9]

For example, the sequence of n-max distance interactions is well-behaved since $n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0)$ converges to 2^{-m} times the indicator function of $\{\nu \in \mathbb{R}^m \mid \max_i \nu_i \le 1\}$.

The next result characterizes contagious and uninvadable actions in any well-behaved sequence of multidimensional lattice networks. Note that Theorem 4 follows as an immediate corollary.

Theorem A.5. *Let u be the bilingual game given by* (3.1)*. Fix the dimension m and a well-behaved sequence (P_n) of interaction weights on $\mathbb{Z}^m$.*

(i) *If $e < e^*$, then there exists $\bar{n}$ such that for any $n \ge \bar{n}$,* 0 *is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.* (ii) *If $e > e^*$, then there exists $\bar{n}$ such that for any $n \ge \bar{n}$,* 2 *is contagious and uninvadable in $(\mathbb{Z}^m, P_n)$.*

[9]For $\eta \in \mathbb{R}$, $[\eta]$ denotes the largest integer that does not exceed η.

Proof. We will show (i) only. The proof for (ii) is similar.

By Lemma 2, there exists $p \in (0, 1/2)$ that satisfies (4.1). By the upper semi-continuity of br, there exists $\varepsilon \in (0, 1/2 - p)$ such that $\max br(\hat{\pi}^a) = 0$ and $\max br(\hat{\pi}^b) \leq 1$, where

$$\hat{\pi}^a = \left(\tfrac{1}{2} - \varepsilon, p, \tfrac{1}{2} - p + \varepsilon\right), \quad \hat{\pi}^b = \left(\tfrac{1}{2} - p - \varepsilon, p, \tfrac{1}{2} + \varepsilon\right).$$

Let $f(\nu)$ be the pointwise limit of $n^m P_n(([n\nu_1], \ldots, [n\nu_m])|0)$ as $n \to \infty$. Since P_n is symmetric and translation invariant, f is symmetric, i.e., $f(\nu) = f(-\nu)$ for almost any ν. We also have $\int_{\mathbb{R}^m} f(\nu) d\nu = 1$.

For each $\lambda \in \mathbb{R}^m$ whose Euclidean norm $\|\lambda\|$ is normalized to 1, we define $\delta(\lambda) > 0$ by

$$\int_{0 \leq \lambda \cdot x \leq \delta(\lambda)} f(x) dx = p.$$

Note that $\delta(\lambda)$ is continuous in λ.

For each positive real number r, let $B_r = \{\nu \in \mathbb{R}^m \mid \|\nu\| \leq r\}$ and $C_r = \{\nu \in \mathbb{R}^m \mid r < \|\nu\| \leq r + \delta(\nu/\|\nu\|)\}$. Since B_r and C_r approximately "flat" boundaries for large r, there exists r_1 such that for any $r \geq r_1$,

$$\int_{B_r} f(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}, \qquad \int_{B_r \cup C_r} f(\xi - \nu) d\xi \geq \frac{1}{2} + p - \frac{\varepsilon}{3}$$

if $\nu \in B_r$, and

$$\int_{B_r} f(\xi - \nu) d\xi \geq \frac{1}{2} - p - \frac{\varepsilon}{3}, \qquad \int_{B_r \cup C_r} f(\xi - \nu) d\xi \geq \frac{1}{2} - \frac{\varepsilon}{3}$$

if $\nu \in C_r$.

For each integer k, let $\hat{B}_k = \{x \in \mathbb{Z}^m \mid \|x\| \leq k\}$ and $\hat{C}_{k,n} = \{x \in \mathbb{Z}^m \mid k < \|x\| \leq k + n\delta(x/\|x\|)\}$. Because $\{(\mathbb{Z}^m, P_n)\}$ is well-behaved, we can apply the dominated convergence theorem to show that there exists n_1 such that for any $n \geq n_1$,

$$\left| \sum_{y \in \hat{B}_k} P_n(y - x|0) - \int_{B_{k/n}} f(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3},$$

$$\left| \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y - x|0) - \int_{B_{k/n} \cup C_{k/n}} f(\xi - x/n) d\xi \right| \leq \frac{\varepsilon}{3}$$

for any $x \in \mathbb{Z}^m$ and k. Therefore, there exists $n_2 \geq n_1$ such that for any $n \geq n_2$ and $k \geq r_1 n$,

$$\sum_{y \in \hat{B}_k} P_n(y|x) \geq \frac{1}{2} - \varepsilon, \qquad \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y|x) \geq \frac{1}{2} + p - \varepsilon$$

for any $x \in \hat{B}_{k+1}$, and

$$\sum_{y \in \hat{B}_k} P_n(y|x) \geq \frac{1}{2} - p - \varepsilon, \qquad \sum_{y \in \hat{B}_k \cup \hat{C}_{k,n}} P_n(y|x) \geq \frac{1}{2} - \varepsilon$$

for any $x \in \hat{C}_{k+1,n}$.

Now we show that 0 is contagious in $(\mathbb{Z}^m, P_n)$ if $n \geq n_2$. The proof is similar to that of Lemma 1(a). Pick an integer $K \geq r_1 n$, and consider any best response sequence (σ^t) such that $\sigma^0(x) = 0$ for all $x \in \hat{B}_K$ and $\sigma^0(x) = 1$ for all $x \in \hat{C}_{K,n}$. Then we can show by induction on k that for any $k \geq K$, there exists T_k such that for any $T \geq T_k$, we have $\sigma^t(x) = 0$ for all $x \in \hat{B}_k$ and $\sigma^0(x) \leq 1$ for all $x \in \hat{C}_{k,n}$.

This argument also shows that 0 is uninvadable in $(\mathbb{Z}^m, P_n)$ because for any initial configuration that satisfies $\sigma^0_{P_n}(\{1,2\}) < \infty$, there exists a translation Y of $\hat{B}_K \cup \hat{C}_{K,n}$ such that $\sigma^0(x) = 0$ for all $x \in Y$. ▮

A.6 The Case Where Pareto-Dominance and Risk-Dominance Coincide

For completeness, we report the contagion and uninvadability result also for the case where action 0 is both Pareto-dominant and pairwise risk-dominant. The game u,

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} a & a & b \\ a-e & a-e & d-e \\ c & d & d \end{pmatrix} \end{array}, \tag{A.15a}$$

now satisfies

$$c \leq d < a, \quad d - b < a - c, \quad e > 0. \tag{A.15b}$$

Theorem A.6. *Let u be the bilingual game given by* (A.15)*. 0 is always contagious and uninvadable.*

Proof. In light of Lemma 1(a-i) and Lemma 3, it suffices to show that condition (4.1) holds for some p and that 0 is a strict MP-maximizer. If $e \leq (d-b)/2$, we have $(c-b)e < (a-d)(d-b)/2$. Therefore, these follow from the argument in case (α) in the proof of Lemma 2(1) and Claims 1–3 in the proof of Lemma A.3. If $e > (d-b)/2$, they follow from the symmetric arguments for 0 in place of 2 as in case (β) in the proof of Lemma 2(1) and Lemma A.5. ▮

The contagion part of this theorem has been shown by Goyal and Janssen (1997, Theorem 3) in their circular network setting with a continuum of players.

Immorlica et al. (2007) consider the current case with a payoff parameter restriction $a = 1 - q$, $b = c = 0$, and $d = q$, so the game is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1-q & 1-q & 0 \\ 1-q-e & 1-q-e & q-e \\ 0 & q & q \end{pmatrix} \end{array}, \qquad 0 < q < \frac{1}{2}. \tag{A.16}$$

(Note that by reversing the order of the actions, we know from Theorem A.6 that action 2 is uninvadable if $q > 1/2$.) They focus on the class $\mathcal{G}_\Delta$ of Δ-regular networks; for a natural number Δ, a Δ-regular network is a constant-weight local interaction system where each player has Δ neighbors. They consider the "epidemic region" $\Omega(G) \subset (0, 1/2) \times \mathbb{R}_{++}$, the set of points (q, e) for which action 0 spreads contagiously in a network G, and show that for any Δ, there exists a point $(q, e) \notin \Omega_\Delta = \bigcup_{G \in \mathcal{G}_\Delta} \Omega(G)$, and in particular, Ω_Δ is not convex. On the other hand, since the linear network constructed in Lemma 1(a-i) (with a choice of a rational number p) can be replicated by a Δ-regular network, our Theorem A.6 implies that $\Omega^* = \bigcup_\Delta \Omega_\Delta$ covers the whole space $(0, 1/2) \times \mathbb{R}_{++}$ and is convex.

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