Overlapping Generations

I. Introduction

We want to study how asset markets allow individuals, motivated by the need to provide income for their retirement years, to finance capital accumulation for economic growth. We consider a highly simplified model, in which people live two "periods". The periods therefore have to be thought of as about 30 years long. At any one date *t* there are N_t young people born at that date and N_{t-1} old people who were born at the previous date. The young people are born without any wealth, but are each endowed with one unit of labor, which they sell to provide themselves with the produced good. If they want to consume in their old age (second period of life), they cannot consume all of their wage earnings, but instead must when they are young put some of their earnings aside to finance their consumption when old.

II. Consumer Behavior

We assume that an individual born at t maximizes a utility function that has the special form

$$U\big(C_1(t), C_2(t+1)\big) = C_1(t)^a \, C_2(t+1)^{1-a} \quad . \tag{1}$$

Here $C_1(t)$ is consumption by this young person in his or her youth, i.e. at time t, while $C_2(t+1)$ is consumption in old age, the second period of life, i.e. at time $t+1$. The consumer faces two constraints,

$$C_1(t) + B(t) = W(t) \quad . \tag{2}$$

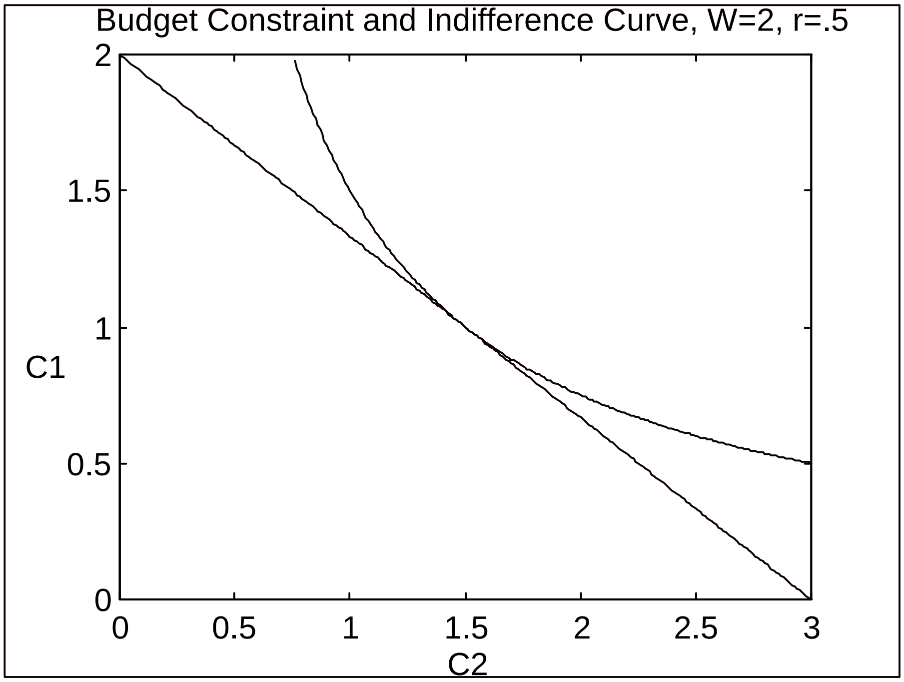
and

$$C_2(t+1) = \big(1 + r(t)\big) B(t) \quad . \tag{3}$$

Here $B(t)$ is savings and $r(t)$ is the return on savings. We use the letter "*B*" because the consumer is treating savings like a bond that pays interest rate $r(t)$, though in this version of the model there is no one "issuing" the bond. It is just output put aside for the future. $W(t)$ is both total wage income and the price of one unit of labor. Since individuals each have one unit of labor and do not gain utility from leisure, they supply their labor inelastically (i.e., they work for one unit of time, regardless of the wage). If we solve (2) and (3) to eliminate *B*, we obtain a single budget constraint in which $1/(1+r)$ is the price of C_2 in terms of C_1:

$$C_1(t) + \frac{C_2(t+1)}{1 + r(t)} = W(t) \quad . \tag{5}$$

This produces a diagram of the conventional consumption-theory type, as shown below.



Note that the interest rate of 50% in the diagram is not unrealistically high, because the time period of about 30 years makes a 50% rate correspond to an annual rate of 1.4%. As the interest rate rises, the budget constraint at a give value of *W* swings outward, rotating around the fixed point $C_2 = 0$, $C_1 = W$. A rise in *r* must increase C_2, but in general, because of offsetting income and substitution effects, may increase or decrease C_1. With our particular, Cobb-Douglas, form for the utility function, it turns out that C_1 is a fixed fraction, α, of *W*, regardless of the value of *r*.

To see this, we solve the constrained optimization problem of maximizing (1) subject to (2) and (3). The Lagrangian for this problem is

$$C_1(t)^{\alpha} C_2(t+1)^{1-\alpha} - \lambda \cdot \big(C_1(t) + B(t) - W(t)\big) - \mu \cdot \big(C_2(t+1) - (1+r(t)) \cdot B(t)\big) \ . \qquad (6)$$

Because we are looking for a competitive equilibrium solution, consumers are assumed to treat the prices – *W* and *r* – as beyond their control. Choice variables for a consumer born at *t* are $C_1(t)$, $C_2(t+1)$, and $B(t)$. The first-order conditions (FOC's) with respect to these three variables are

$$\frac{\partial}{\partial C_1(t)}: \qquad \alpha \left(\frac{C_1(t)}{C_2(t+1)} \right)^{\alpha - 1} = \lambda \qquad (7)$$

$$\frac{\partial}{\partial C_2(t+1)}: \qquad (1-\alpha) \left(\frac{C_1(t)}{C_2(t+1)} \right)^{\alpha} = \mu \qquad (8)$$

$$\frac{\partial}{\partial B(t)}: \qquad \lambda = \big(1 + r(t)\big)\mu \qquad (9)$$

It is not hard to verify that these three equations can be solved to eliminate λ and μ, delivering

$$\frac{C_2(t+1)}{C_1(t)} = \frac{(1-a)(1+r(t))}{a} \ . \tag{10}$$

Combining (10) with the budget constraint (5) written in terms of *C*'s alone gives us the conclusion we already announced above,

$$C_1(t) = aW(t) \ . \tag{11}$$

From (11) and the constraints (2) and (3), we can find the values of the other two choice variables as

$$B(t) = (1-a)W(t) \ , \quad C_2(t+1) = (1+r(t))(1-a)W(t) \ . \tag{12}$$

III. Production

We assume that at each date *t* there are $M(t)$ identical firms, each of which hires labor from young people and purchases capital held over from the previous period by old people, combining them to produce new output. The firm aims to maximize profits, which are given by

$$AK(t)^b L(t)^{1-b} + (1-d)K(t) - W(t)L(t) - Q(t)K(t) \ . \tag{13}$$

Here *K* is capital, *L* is labor, and *Q* is the market price of capital. The first term in (13) is new product, given by a Cobb-Douglas production function with constant returns to scale. The second term is the proportion of the capital stock that is not used up in production and that can therefore be sold as output along with new product. The last two terms are the cost of inputs. Maximizing (13) with respect to the firm's two choice variables $K(t)$ and $L(t)$ produces as FOC's

$$\frac{\P}{\P K(t)}: \qquad bAK(t)^{b-1} L(t)^{1-b} + (1-d) = Q(t) \tag{14}$$

$$\frac{\P}{\P L(t)}: \qquad (1-b)K^b L^{-b} = W(t) \ . \tag{15}$$

Though (14) and (15) constitute two equations in the two unknown decision variables *K* and *L*, it turns out that they cannot actually be solved for *K* and *L*. The reason is that both involve $K(t)$ and $L(t)$ only as the ratio $k(t) = K(t)/L(t)$. Since either equation can be solved for $k(t)$, together they require a certain relation between *W* and *Q*. Using (15) to get rid of *k* in (14) we arrive at

$$Q(t) = 1 - d + bA\left(\frac{W(t)}{1-b}\right)^{1-\frac{1}{b}} \ . \tag{16}$$

This is a standard situation when the technology is constant returns to scale. Factor prices determine the ratios of inputs, but the absolute levels of them are indeterminate. Also factor prices have to be such as to leave profits zero. If profits are positive, scaling production up always increases profit, without bound. If profits are negative, the maximum profit of zero is obtained by scaling production back to zero. And if profits are zero, they are zero at every scale of production. This result, that with constant returns to scale and competitive equilibrium profits are zero, is sometimes summarized as "factor payments exhaust the product."

Note that we have not said who owns these firms. Presumably the firms' motivation for maximizing profits is that there are owners receiving profits as income who instruct the firms to do this. Yet our discussion of the consumer made no mention of income from ownership of the firm. We can ignore ownership and profits in the consumer problem because, with our constant-returns-to-scale setup, profits are zero in competitive equilibrium. Ownership of the zero-profit firm has no effect on the consumer's maximization problem.

IV. Equilibrium

We now must add equations enforcing consistency between choices of firms and choices of consumers. Purchases of inputs by firms must match sales of them by consumers:

$$M(t)K(t) = N(t-1)B(t-1) \tag{17}$$

$$M(t)L(t) = N(t)\ . \tag{18}$$

The number of people per generation is assumed to grow at the rate n per period, so

$$N(t) = (1+n)N(t-1) \tag{19}$$

in every period. Taking the ratio of (18) to (17), we get

$$k(t) = \frac{B(t-1)}{1+n}\ . \tag{20}$$

Using (15) and the first equation in (12), we obtain

$$B(t) = (1-a)(1-b)Ak(t)^b\ . \tag{21}$$

$Q(t)$ is the same thing as $1+r(t-1)$; we used separate notation just to emphasize the interpretation of it as an interest rate. Recognizing this, we can use (14) and (21) to rewrite (20) entirely in terms of k:

$$k(t) = \frac{(1-a)(1-b)Ak(t-1)^b}{1+n} \tag{22}$$

The steady-state level of k, which we call $\bar{k}$, is obtained by setting $k(t) = k(t-1)$ and solving (22),

$$\bar{k} = \left(\frac{(1-a)(1-b)A}{1+n}\right)^{\frac{1}{1-b}} \tag{23}$$

Note that $1-a$ is the rate of saving out of labor income (not exactly the s of the Solow model, which is the rate of saving out of total gross product including depreciation). [A good exercise to check understanding: compare this formula for the steady state to that for the Solow model. Which parameters correspond? Do similar parameters have similar effects on steady state in each case? The two that clearly don't are d and b.]

V. Golden Rule and Dynamic Efficiency

Just as in the case of the Solow model, we can ask here what is the highest level of utility that can be sustained forever. When we ask this question, we take the viewpoint of a planner that can choose $C_1(t)$, $C_2(t)$ and $B(t)$ directly at every date. We assume the solution will make these variables the same at every date. (This is not completely obvious. If you want a *really* challenging mathematical problem, try to prove rigorously that in this model the highest feasible constant level of utility is obtained with $C_1(t)$, $C_2(t)$ and $B(t)$ all constant.) The planner's problem is

$$\max_{C_1, C_2, B} \left\{ C_1^a C_2^{1-a} \right\} \tag{24}$$

subject to

$$C_1 + \frac{C_2}{1+n} + B = A\left(\frac{B}{1+n}\right)^b + (1-d)\frac{B}{1+n} \ . \tag{25}$$

[Where does this equation come from? Why the 1+*n* divisor for C_2? Why the 1+*n* divisor for *B* in some places but not others?] Introducing l as the Lagrange multiplier for the constraint (25), we obtain FOC's

$$\frac{\P}{\P C_1}: \qquad a C_1^{a-1} C_2^{1-a} = l \tag{26}$$

$$\frac{\P}{\P C_2}: \qquad (1-a) C_1^a C_2^{-a} = \frac{l}{1+n} \tag{27}$$

$$\frac{\P}{\P B}: \qquad 1 = bA\left(\frac{B}{1+n}\right)^b \frac{1}{B} + \frac{1-d}{1+n} \ . \tag{28}$$

Equation (28) implies that, if this allocation were consistent with a competitive equilibrium, the interest rate would satisfy

$$r = n \ , \tag{29}$$

a result similar to that for the Solow model. [You should be able to prove this. Using (23), you should be able to show that the competitive equilibrium steady state is not in general a golden rule steady state, and indeed that it can be inefficient, in that it can make $r < n$.] {}