Spurious Welfare Reversals in International Business Cycle Models

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First Draft: May, 1998
This Draft: October 13, 1999

JEL Classification: F3, F4, E3;

Key Words: linearization, stochastic steady state, welfare, risk sharing.

*Comments from Marianne Baxter, Henning Bohn, Yongsung Chang, Jon Faust, Ken Judd, Jinho Kim, Bob King, Andy Levin, Chris Sims, Mike Woodford, Jonathan Wright, and seminar participants at Stanford Institute for Theoretical Economics 1999 summer workshop, the Korea macroeconomics workshop, 1999 Society for Computational Economics conference at Boston College, T2M conference at UQAM, Clark University, UCSB, UQAM, Georgetown University, and the Federal Reserve Board are appreciated. J. Kim thanks a Bankard grant from the University of Virginia.

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Abstract

Several papers on international business cycles have documented spurious welfare reversals, in that incomplete market economies can produce higher welfare than the complete market economy. This paper demonstrates how conventional linearization, as used in King, Plosser, and Rebelo (1988), can generate approximation errors that are large enough to result in such reversals. Using a two-country production economy without capital, we argue that spurious welfare reversals are not only possible but also plausible under reasonable parameter values. As a constructive alternative, this paper proposes an approximation method that modifies the conventional linearization method by a bias correction—the linear approximation around a ‘stochastic’ steady state. We show that this method can be easily implemented to accurately approximate the exact solution and therefore produce the correct welfare ordering. The accuracy of the proposed method is far better than that of the conventional linearization method and as good as that of a method involving a second-order expansion.
1. Introduction

Following Kydland and Prescott (1982) and King, Plosser, and Rebelo (1988), the literature using dynamic stochastic general equilibrium (DSGE) models to study the aggregate economy has extensively used the method of linear approximation around a deterministic steady state. A number of papers have analyzed the accuracy of the loglinear approximation method and concluded that this method works well in many respects.\(^1\)

Since Backus, Kehoe, and Kydland (1992) used a DSGE model to study international business cycles, the linearization method has been commonly used in the international business cycle literature.\(^2\) An issue in the literature, discussed by Backus, Kehoe, and Kydland (1992), Tesar (1995), and Kim (1997), is the size of welfare gains from international risk sharing. This paper investigates the validity of the linearization method in welfare calculation, especially in calculating international risk sharing gains. We show crucial results that the conventional linearization method can be so inaccurate in calculating welfare levels as to reverse welfare ordering between autarky and the complete market economy. We also propose an alternative method that approximates the exact nonlinear solution more accurately than the conventional linearization method, thereby producing a correct welfare ordering.

In a two-country model, the complete market economy should produce a higher world welfare than any incomplete market economies. This is a direct application of the first welfare theorem which states that the competitive equilibrium in the complete market economy should be Pareto optimal. Therefore, reversal of welfare ordering under the linearization method implies that approximation errors exist and are significantly large.

A few papers have documented examples of welfare reversals. Tesar (1995) reports negative risk sharing gains in several cases.\(^3\) Kim (1997) also documents some cases of welfare reversal. van Wincoop (1999) gives an example of welfare reversal with a three-state shock and suggests that the linearized decision rule may cause large approximation errors.\(^4\) However, even though this phenomenon

\(^1\)See, for example, Taylor and Uhlig (1990) and other papers in the same conference volume.
\(^2\)See, for example, Baxter and Crucini (1995) and Chari, Kehoe, and McGrattan (1998).
\(^3\)For example, in her Table 7, welfare reversals are observed in half of the model specifications.
\(^4\)The example is in his footnote 7. Our work is more general than van Wincoop’s in the sense that his analysis is based on a shock with a discrete distribution, while this paper uses a continuous distribution for the shock. He focuses on the linearized decision rule as a potential cause of approximation errors, while this paper formally expands this conjecture and shows the
of spurious welfare reversals is apparently paradoxical, no effort has been made to understand and to solve this puzzle in a formal way. We investigate this puzzle by adopting a two-country DSGE model without capital, which enables us to derive closed-form solutions.

In this paper, we select two economies—an endowment economy and a production economy in which labor is the only input—and compute risk sharing gains from autarky to the complete market economy. The complete market economy is solved by four methods—‘exact’, ‘deterministic’, ‘stochastic’, and ‘quadratic’ methods. The ‘exact’ method uses a system of exact nonlinear first-order conditions, while the ‘deterministic’ and ‘stochastic’ methods make use of linearization. The ‘deterministic’ method is the conventional method of loglinearizing the first-order conditions around a deterministic steady state. The ‘stochastic’ method incorporates some nonlinearity of the model and modifies the conventional linearization method with a bias correction which amounts to the linear approximation around a stochastic steady state. The ‘quadratic’ method is an application of the perturbation method which approximates the exact solution up to the second order in exogenous variables.

By comparing the deterministic solution with the exact solution, we show that the conventional loglinearization underestimates risk sharing gains: the deterministic solution of the complete market economy always produces lower welfare than the exact solution does. Under reasonable parameter values, these approximation errors can be large enough to reverse welfare ordering between autarky and the complete market economy. We propose the ‘stochastic’ solution method and demonstrate that this method not only computes the correct welfare ordering but also generates relatively accurate approximations of the exact nonlinear solution. The accuracy of the ‘stochastic’ method is comparable to that of the ‘quadratic’ method.

The remaining structure of the paper is as follows. Section 2 presents the two-country endowment economy under autarky and the complete market economy. We calculate approximation errors produced by the ‘deterministic’ method and demonstrate a possibility of spurious welfare reversal. The ‘stochastic’ method is explained in detail and shown to produce a correct welfare ordering. We provide plausibility of spurious welfare reversals.  
  
5Both autarky and the complete market economy have a simple static structure that enables us to reduce an infinite horizon model to a static one. We do not consider an incomplete market model with only bonds since the exact solution, due to its dynamic nature, is not expressed in an analytically tractable form. See Kim, Kim, and Levin (1999) for a detailed analysis of such incomplete market models.
an intuitive explanation for the ‘stochastic’ method in relation to the ‘quadratic’ method and show that the former is as accurate as the latter in approximating the exact nonlinear solution. Section 3 adopts a production economy with labor and shows that spurious welfare reversals are plausible under reasonable parameter values. If the elasticity of labor supply is larger than unity, the expected utility of the complete market economy derived from the ‘deterministic’ method is always lower than that of autarky. The intuition and the accuracy results regarding the ‘stochastic’ method apply also to the production economy. Section 4 serves as a conclusion.

2. Endowment Economy

We first introduce the endowment economy under autarky. Since the exact solution itself is loglinear, it is redundant to discuss approximate solutions. On the other hand, the complete market economy produces a non-loglinear exact solution, and we compute approximate solutions following the deterministic and stochastic methods. We provide intuitive explanations for the stochastic method and check its accuracy relative to the quadratic method.

2.1. Autarky

Each country, denoted by subscript $i$, consists of a representative agent who maximizes a power utility function:

$$
\max \mathbb{E} \left[ \frac{C_i^{1-\gamma} - 1}{1-\gamma} \right]
$$

subject to

$$
C_i = Y_i.
$$

The parameter, $\gamma (\geq 0)$, represents the degree of relative risk aversion. The endowment process, $Y_i$, is assumed to be lognormally distributed. Specifically, we assume that $(\log Y_i)$ has a normal distribution with mean zero and variance $\sigma_y^2 (> 0)$ and that the endowment processes are independent across countries. That is, the probability density function of an endowment is

$$
f_Y(y) = \frac{1}{y\sqrt{2\pi}\sigma_y} \exp \left[ -\frac{1}{2} \left( \frac{y}{\sigma_y} \right)^2 \right].
$$
It is trivial that consumption under autarky, denoted by the superscript ‘autarky’, is equal to the endowment process:

\[ C_i^{\text{autarky}} = Y_i, \quad (2.4) \]

and the properties of the lognormal distribution lead to the following expected utility:

\[ EU^{\text{autarky}} = \exp \left[ \frac{(1-\gamma)^2 \sigma^2_y}{2} \right] - 1 \frac{1}{1 - \gamma}. \quad (2.5) \]

Taking the inverse utility function on both sides of (2.5), we derive the certainty equivalent consumption which is defined as the amount of consumption which generate this level of utility and denoted by \( C_{CE}^{\text{autarky}} \):

\[ C_{CE}^{\text{autarky}} = \exp \left[ \frac{(1-\gamma)^2 \sigma^2_y}{2} \right]. \quad (2.6) \]

In the special case of the logarithmic utility when \( \gamma = 1 \), the expected utility is zero and the certainty equivalent consumption is unity.

### 2.2. Complete market economy

Complete market economy assumes that a complete set of Arrow-Debreu securities exists and provides complete risk sharing. Instead of introducing Arrow-Debreu securities directly in the model, we solve the complete market economy as a world command optimum problem implied by the first welfare theorem. That is, we solve the model by maximizing the average of two countries’ utilities subject to the world resource constraint:

\[
\max \mathbb{E} \left[ \frac{1}{2} \left( C_1^{1-\gamma} - 1 + C_2^{1-\gamma} - 1 \right) \right] \quad (2.7)
\]

subject to

\[ C_1 + C_2 = Y_1 + Y_2. \quad (2.8) \]

Equal weights of half can be interpreted as an assumption on symmetry between the two countries, and this simplifies the calculations to come.

The optimality condition of the complete market economy is

\[ C_1 = C_2. \quad (2.9) \]

The economy is described by the resource constraint, (2.8), and the optimality condition, (2.9). We start with the exact solution and then move to the solutions by the deterministic and stochastic methods.
2.2.1. Exact solution

By solving the model exactly, we derive the following consumption level;

\[ C_{i}^{\text{exact}} = \frac{Y_1 + Y_2}{2}, \quad (2.10) \]

where the superscript ‘exact’ denotes the exact solution of the complete market economy. The probability density of the exact solution is

\[ f_C(c) = \frac{1}{c^2 \pi \sigma_y^2} \int_0^1 \frac{1}{t(1-t)} \exp \left[ -\frac{1}{2} \left( \frac{\log c + \log t}{\sigma_y} \right)^2 - \frac{1}{2} \left( \frac{\log c + \log (1-t)}{\sigma_y} \right)^2 \right] dt, \quad (2.11) \]

which is neither normal nor lognormal. Instead of relying upon this complicated density, we calculate the results of the exact method by simulating the model with a random number generator. The results are based on 25,000 independent random drawings of two-dimensional shock processes using Matlab.

According to the first welfare theorem, the complete market economy should produce a higher expected utility than autarky;

\[ EU_{i}^{\text{exact}} > EU_{i}^{\text{autarky}}, \quad (2.12) \]

which implies the following relationship in terms of certainty equivalent consumption;

\[ C_{i}^{\text{exact}}^{CE} > C_{i}^{\text{autarky}}^{CE}. \quad (2.13) \]

In this paper, we compute welfare gains by comparing the two levels of certainty equivalent consumption.\(^6\)

2.2.2. Linearization around a deterministic steady state

If we solve the first-order conditions, (2.8) and (2.9), using the conventional loglinearization around a deterministic steady state (\( \bar{C} = \bar{Y} = 1 \)), then the consumption level becomes

\[ \log C_{i}^{\text{deterministic}} = \frac{1}{2}(\log Y_1 + \log Y_2), \quad (2.14) \]

\(^6\text{This method is slightly different from the measurement of welfare gains in Cho, Cooley, and Phaneuf (1997) and Bils and Chang (1998). They calculate welfare gains as percentage increases in consumption relative to the steady-state level, which would compensate for the utility differential. These two methods produce similar results when shock variances are small. Our method is analytically more convenient in the setup of this paper.} \)
where the superscript ‘deterministic’ denotes the solution of the conventional log-linearization.\(^7\)

The relationship between consumption levels of the exact and deterministic solutions in each state of the economy is

\[ C_i^{\text{deterministic}} = \sqrt{Y_1 Y_2} \leq \frac{Y_1 + Y_2}{2} = C_i^{\text{exact}}, \]  

(2.15)

where the equality holds when the two endowments are identical.

This state-by-state inequality is a sufficient condition for the first-order stochastic dominance of the exact solution over the deterministic solution, captured by the cumulative density function (CDF) of consumption. Figure 1 draws the two CDFs assuming that the variance of endowment is unity \((\sigma_y = 1)\).\(^8\) The CDF corresponding to the exact method lies to the right, which implies that the consumption calculated from the exact method stochastically dominates that from the deterministic method. The distance between the two CDFs in Figure 1 represents the approximation errors generated by the deterministic method.

Under this conventional linearization, the log of consumption is normally distributed with mean zero and variance \(\sigma_y^2/2\). According to the properties of the lognormal distribution, the level of expected utility is

\[ EU^{\text{deterministic}} = \exp \left[ \frac{(1-\gamma)^2 \sigma_y^2}{4} \right] \frac{1}{1 - \gamma}, \]  

(2.16)

and the certainty equivalent consumption is

\[ C_{\text{CE}}^{\text{deterministic}} = \exp \left[ \frac{(1-\gamma) \sigma_y^2}{4} \right]. \]  

(2.17)

Comparing this equation with the certainty equivalence consumption under autarky (2.6), we derive the following relationship;

\[ \frac{C_{\text{CE}}^{\text{deterministic}}}{C_{\text{CE}}^{\text{autarky}}} = \exp \left[ \frac{-(1-\gamma) \sigma_y^2}{4} \right] \leq 1 \iff \gamma \leq 1. \]  

(2.18)

\(^7\)It is true that linearization in levels would not create any approximation errors in the endowment economy. However, it still creates approximation errors in a production economy with labor, as described in the next section. Furthermore, it is more conventional to linearize in logs, rather than in levels.

\(^8\)Such a high variance is used only for the graphical purpose and will be used for drawing all other cumulative density functions. All the results are valid for a smaller and more realistic variance.
If the degree of risk aversion is less than unity, approximation errors are large enough to reverse welfare ordering—autarky produces a higher welfare than the complete market economy. Under the logarithmic utility function ($\gamma = 1$), the deterministic solution of the complete market economy generates the same level of welfare as in autarky. That is, the approximation errors completely wipe out true welfare gains.

The welfare reversal can be easily explained by approximating utility function up to the second order. Taking the second order Taylor expansion of the utility function with respect to log consumption around its deterministic steady state, we have

$$\frac{C^{1-\gamma} - 1}{1-\gamma} \approx \log C + \frac{1-\gamma}{2} (\log C)^2. \quad (2.19)$$

The expected value of log consumption is zero under both autarky and the deterministic solution. Therefore, when $\gamma < 1$, the complete market economy solved by the deterministic method generates a lower expected utility than autarky, since the variance of the former economy is smaller than that of the latter.

### 2.2.3. Linear approximation around a stochastic steady state

The stochastic method assumes linearity around a stochastic steady state which is defined as the expected value of a variable.$^9$ The stochastic method is rooted in differences between the deterministic and stochastic steady states. That is, the stochastic steady state of log consumption is larger than its deterministic steady state;

$$E \left[ \log C_i^{\text{exact}} \right] = E \left[ \log \left( \frac{Y_1 + Y_2}{2} \right) \right] > E \left[ \log \sqrt{Y_1 Y_2} \right] = E \left[ \log C_i^{\text{deterministic}} \right] = \log \bar{C} = 0. \quad (2.20)$$

Even though the mean of log consumption from the exact method is positive, the conventional linearization forces the mean to be equal to its deterministic steady state which is zero. The stochastic method improves the accuracy of linear approximation by relaxing this assumption. Instead, we impose an assumption which locates the mean close to the stochastic steady state.

However, the exact stochastic steady state of a variable can neither be calculated without solving the model nor be expressed in a compact form even if

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$^9$Bohn (1998) and his comments are instrumental in implementing this new method.
the solution exists. As shown before, the exact solution of the complete market economy has a distribution which is neither normal nor lognormal. Therefore, we approximate the stochastic steady state as follows.

First, we assume that the variable of interest has a lognormal distribution. In the endowment economy, the log of consumption is assumed to have a normal distribution,

$$\log C_i^{stochastic} \sim N(\mu_c, \sigma_c^2),$$

where the superscript ‘stochastic’ represents the linear approximation around the stochastic steady state.

Second, we plug this distribution into the model economy. In this example, after we take the expectation of the resource constraint, (2.8), the following relationship is obtained by the properties of the lognormal distribution

$$\mu_c + \frac{\sigma_c^2}{2} = \frac{\sigma_y^2}{2}.$$  

(2.22)

Note that we have one equation with two unknowns, $\mu_c$ and $\sigma_c^2$.

Lastly, but most importantly, the stochastic method assumes that the linear relationship between endogenous and exogenous variables derived by the deterministic method holds in the stochastic method as follows:

$$\log C_i^{stochastic} = \mu_c + \log C_i^{deterministic}.$$  

(2.23)

That is, the stochastic method implements a bias correction onto the deterministic method. Using the deterministic solution, we produce the following stochastic solution;

$$\log C_i^{stochastic} - \mu_c = \frac{1}{2} (\log Y_1 + \log Y_2).$$  

(2.24)

This assumption produces an additional restriction that serves as the second equation for the complete system with the two unknowns. Squaring and taking expectation of the equation (2.24), we have

$$\sigma_c^2 = \frac{\sigma_y^2}{2}.$$  

(2.25)

---

10 Suppose that $\log (X)$ has a normal distribution with mean $\zeta$ and variance $\sigma^2$, then $E[X^k] = e^{k\zeta + \frac{1}{2}k^2\sigma^2}$.

11 Note that we do not directly linearize the first-order conditions around a stochastic steady state. Such linearization would also create as much approximation errors as the deterministic method.
Therefore, the stochastic steady state of log consumption becomes

\[ E[\log C_{\text{stochastic}}^i] = \mu_c = \frac{\sigma_y^2}{4}, \tag{2.26} \]

which is positive and larger than its deterministic steady state of zero. Recapitulating, the log of consumption derived from the stochastic method is

\[ \log C_{\text{stochastic}}^i = \frac{\sigma_y^2}{4} + \frac{1}{2} (\log Y_1 + \log Y_2) \sim N \left( \frac{\sigma_y^2}{4}, \frac{\sigma_y^2}{2} \right). \tag{2.27} \]

According to the stochastic method, the level of expected utility is

\[ EU_{\text{stochastic}} = \exp \left[ \frac{(1-\gamma)(2-\gamma)\sigma_y^2}{4} - 1 \right] \frac{1}{1-\gamma}, \tag{2.28} \]

and the certainty equivalent consumption is larger than that of autarky;

\[ C_{\text{CE}} = \exp \left[ \frac{(2-\gamma)\sigma_y^2}{4} \right] \geq \exp \left[ \frac{(1-\gamma)\sigma_y^2}{2} \right] = C_{\text{CE}}^{\text{autarky}}. \tag{2.29} \]

### 2.3. Intuition and accuracy of the stochastic method

The proposed method, relative to the conventional linearization method, is based on the concept of bias correction. We present two intuitive explanations for the size of the bias correction. First, it is easy to see from the exact solution that there is no welfare gain from risk sharing when the utility function is linear in consumption, \((\gamma = 0)\). It is desirable for the proposed method to reproduce this property, which is confirmed by observing that (2.29) holds as an equality in such a case.

Further intuition comes from the second-order perturbation method.\(^\text{12}\) Had we solved for log consumption up to the second order with respect to log endowment around the deterministic steady state \((\bar{C} = \bar{Y}_i = 1)\), we would have derived the following solution:

\[ \log C_{\text{quadratic}} = \frac{1}{2} (\log Y_1 + \log Y_2) + \frac{1}{8} [(\log Y_1)^2 - 2 (\log Y_1) (\log Y_2) + (\log Y_2)^2]. \tag{2.30} \]

\(^\text{12}\)See Gaspar and Judd (1997) and Collard and Juillard (1999) for more on the perturbation method.
The superscript ‘quadratic’ indicates the solution from the second-order expansion around the deterministic steady state.

Comparing the quadratic solution (2.30) with the stochastic solution (2.27), we can easily notice that the stochastic solution replaces the second-order terms in the quadratic solution with their means,

\[
\log C_{\text{stochastic}} = \frac{1}{2} (\log Y_1 + \log Y_2) + \frac{1}{8} \mathbb{E} \left[ (\log Y_1)^2 - 2 (\log Y_1) (\log Y_2) + (\log Y_2)^2 \right].
\]

(2.31)

In the special case of the logarithmic utility function, welfare implications are invariant to the choice between the quadratic and stochastic methods.

The advantage of the stochastic method over the quadratic method is that the stochastic method generates a relatively simple solution and thus admits an analytic expression for the expected utility. An unexpected advantage is that we can rationalize the common use of the deterministic method in the DSGE literature. As far as the second moments of a variable—such as variances and covariances—are concerned, the deterministic and stochastic methods produce the same results. However, if we were to push for the quadratic method literally, then the second moments based on the deterministic method should all be recalculated.

Now we assess how accurate the stochastic method is relative to the deterministic and quadratic methods. Figure 2 reports the CDFs of consumption from the four methods. It is clear that approximation errors, represented by the distance from the exact solution, are much smaller for the stochastic and quadratic methods compared to the deterministic method. The bias correction by the stochastic method overstates the level of consumption when the consumption realization is close to unity, and understates when an outlier occurs. The quadratic method consistently overestimates the exact solution by a small amount.

To formally compare the accuracy of the three approximation methods, we compute a diagnostic statistic for each of the three approximate solutions relative to the exact solution. We use the Kolmogorov-Smirnov statistic which measures the maximum distance between the CDF from each approximation method and that from the exact method. Table 1 shows the results of the test statistics. It is clear that the deterministic method is much worse than the stochastic and quadratic methods. The Kolmogorov-Smirnov statistic favors the quadratic method, but other statistics in the Cramer-von Mises class favor the stochastic method.\(^\text{13}\) This can be understood from Figure 2, where the CDF from the sto-

\(^{13}\text{See Serena (1980) for detailed theoretical explanations on the Kolmogorov-Smirnov statistic and other statistics in the Cramer-von Mises class.}\)
chastic method crosses over that from the exact method while the CDF from the quadratic method does not.

3. Production Economy with Labor

3.1. Autarky

Under autarky, each country behaves as follows;

$$\max E [U(C_i, L_i)]$$

subject to

$$U(C_i, L_i) = \frac{C_i^{1-\gamma} - 1}{1-\gamma} + \nu \left(1 - L_i^{\frac{1}{\gamma}}\right),$$

$$C_i = A_i L_i,$$

where $A_i$ is a lognormally distributed productivity shock. We assume that $(\log A_i)$ has a normal distribution with mean zero and variance $\sigma^2$. $A_i$ is independent across countries. The linear technology is assumed without loss of generality. Additive separability between consumption and labor helps to derive an analytic expression for the expected utility. The parameter $\nu$, taking a value between 0 and 1, is related to the degree of elasticity of labor supply. The elasticity of labor supply is $\nu / (1 - \nu)$, which is increasing in $\nu$ and takes a value between 0 and $\infty$.

The solution under autarky is

$$L_{i, autarky}^\gamma = A_i^{(1-\gamma)\nu},$$

$$C_{i, autarky}^\gamma = A_i^{\frac{1}{1+\nu}}.$$

Since the first order conditions are loglinear, the solution is also loglinear. Therefore, it is easy to see that the four methods would produce the same solution. In particular, there is no bias to be corrected if we execute the stochastic method.

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14 Appendix A presents the detailed solution of a more general production economy when the production function exhibits a decreasing marginal return.

15 The endowment economy in the previous section corresponds to this production economy with $\nu = 0$, where labor supply becomes constant and $C_i = A_i$. If $\gamma = 1$, labor supply becomes constant because income and substitution effects are cancelled out.
The expected utility is

$$EU_{autarky} = \left( \frac{1 - \nu + \nu \gamma}{1 - \gamma} \right) \left( \exp \left[ \left( \frac{1 - \gamma}{1 - \nu + \nu \gamma} \right)^2 \frac{\sigma^2}{2} \right] - 1 \right).$$  \hspace{1cm} (3.6)

In the production economy, we define certainty equivalent consumption as the level of consumption producing the same level of expected utility while fixing labor supply at the steady-state level of unity. Under the logarithmic utility function, the expected utility is zero and the certainty equivalent consumption is unity.

### 3.2. Complete market economy

The complete market economy is solved as a command optimum problem as before:

$$\max E \left[ \frac{1}{2} U(C_1, L_1) + \frac{1}{2} U(C_2, L_2) \right]$$  \hspace{1cm} (3.7)

subject to

$$C_1 + C_2 = A_1 L_1 + A_2 L_2.$$  \hspace{1cm} (3.8)

The first-order conditions of this problem are

$$C_1^\gamma = A_1 L_1^{1-\nu},$$  \hspace{1cm} (3.9)

$$C_2^\gamma = A_2 L_2^{1-\nu},$$  \hspace{1cm} (3.10)

$$C_1 = C_2.$$  \hspace{1cm} (3.11)

We solve the model using the three solution methods as in the endowment economy case.

#### 3.2.1. Exact solution

Solving the four equations, (3.8)–(3.11), which describe the complete market economy, we derive the following exact solutions;

$$L_1^{\text{exact}} = \left[ \frac{2A_1^{1-\nu}}{1 + A_1^{1-\nu} A_2^{1-\nu}} \right]^{\frac{1-\nu^2}{1-\nu+\nu^2}},$$  \hspace{1cm} (3.12)

$$C_1^{\text{exact}} = \left[ \frac{A_1^{1-\nu} + A_2^{1-\nu}}{2} \right]^{\frac{1-\nu}{1-\nu+\nu^2}}.$$  \hspace{1cm} (3.13)
We use the same simulation method to derive the expected utility and $C_{CE}$ of the exact solution. Note that $C_1 = C_2$ at every state due to the additive separability of utility function.

3.2.2. Linearization around a deterministic steady state

If we loglinearized the economy around a deterministic steady state, the solution would be

$$
\log L_1^{\text{deterministic}} = \left( \frac{\nu}{1 - \nu} \right) \frac{[2(1 - \nu + \nu\gamma) - \gamma] \log A_1 - \gamma \log A_2}{2(1 - \nu + \nu\gamma)}, \quad (3.14)
$$

$$
\log C_1^{\text{deterministic}} = \frac{\log A_1 + \log A_2}{2(1 - \nu + \nu\gamma)}. \quad (3.15)
$$

Approximation errors generated by the deterministic method are represented by the distance between the CDFs from the deterministic and exact methods, as in Figures 3-1 and 3-2. These two figures are based on $\gamma = 1$ and $\nu = 0.5$, and the variance of shock is set at unity ($\sigma_y^2 = 1$). The CDF of log consumption derived from the deterministic method lies to the left of that from the exact method, which means that the deterministic method understates the consumption level. Regarding the labor input, the deterministic method overstates the amount of labor. These two observations imply that the deterministic method produces a lower utility than the exact solution does.

Figure 4 draws the contours of $C_{CE}^{\text{deterministic}}/C_{CE}^{\text{autarky}}$ to show the range of parameter values, $\nu$ and $\gamma$, where welfare reversal occurs.$^{16}$ Welfare reversal occurs in the area above the contour at unity, where the $C_{CE}^{\text{deterministic}}$ is less than the $C_{CE}^{\text{autarky}}$. The graph suggests that the approximation errors always lead to a welfare reversal when labor is more than unit elastic ($\nu > 0.5$), regardless of the value for $\gamma$. Welfare reversal also occurs when $\gamma$ is less than one irrespective of the value of $\nu$, whose special case was shown in the endowment economy.

$^{16}$The variance of output/consumption, $\sigma_y^2$, depends not only on $\sigma_y^2$ but also on the parameter values. That is, $\sigma_y^2 = (1 - \nu + \nu\gamma)^2 \sigma_y^2$ under autarky. In order to obtain realistic results, we endogenize $\sigma_y^2$ to maintain $\sigma_y^2$ constant at 0.027$^2$, a conventionally used value in the literature, with different values of $\nu$ and $\gamma$.

$^{17}$This can be easily proven by showing that the two levels of expected utility are equal when $\nu = 0.5$ and $\gamma \to \infty$. 

15
3.2.3. Linear approximation around a stochastic steady state

The stochastic method assumes;

\[
\log L_{i}^{\text{stochastic}} \sim N(\mu_l, \sigma_l^2), \quad (3.16)
\]
\[
\log C_{i}^{\text{stochastic}} \sim N(\mu_c, \sigma_c^2). \quad (3.17)
\]

Taking expectation of the resource constraint (3.8), the above assumptions imply the following relationship

\[
\mu_c + \frac{\sigma_c^2}{2} = \mu_l + \frac{\sigma_l^2 + 2\sigma_{al} + \sigma_t^2}{2}, \quad (3.18)
\]

where \(\sigma_{al}\) is the covariance between logged technology shocks and logged labor. Likewise, the optimality condition, (3.9) or (3.10), produces another relationship

\[
\nu\gamma\mu_c = -(1 - \nu)\mu_l. \quad (3.19)
\]

As in the case of the endowment economy, we use the same assumptions in solving for the values for \(\mu\)'s and \(\sigma\)'s. That is, the following relationship holds between the stochastic and deterministic solutions,

\[
\log L_{i}^{\text{stochastic}} = \mu_l + \log L_{i}^{\text{deterministic}}, \quad (3.20)
\]
\[
\log C_{i}^{\text{stochastic}} = \mu_c + \log C_{i}^{\text{deterministic}}. \quad (3.21)
\]

>From (3.14) and (3.15), we compute the second moments of the endogenous variables,

\[
\sigma_l^2 = \left(\frac{\nu}{1-\nu}\right)^2 \frac{(2\Gamma^2 - 2\Gamma \gamma + \gamma^2)}{2\Gamma^2} \sigma_a^2, \quad (3.22)
\]
\[
\sigma_{al} = \left(\frac{\nu}{1-\nu}\right) \frac{(2\Gamma - \gamma)}{2\Gamma} \sigma_a^2, \quad (3.23)
\]
\[
\sigma_c^2 = \frac{\sigma_a^2}{2\Gamma^2}, \quad (3.24)
\]

where

\[
\Gamma = 1 - \nu + \nu\gamma.
\]
Finally, plugging these second moments into (3.18) and (3.19), the stochastic steady states of log labor and log consumption become

\[
\mu_l = -\frac{\nu}{(1 - \nu)^2} \frac{\gamma \sigma_a^2}{4 \Gamma},
\]

(3.25)

\[
\mu_c = \left( \frac{1}{1 - \nu} \right) \frac{\sigma_a^2}{4 \Gamma}.
\]

(3.26)

It is trivial to show that the expected utility of the stochastic solution is always greater than that of the deterministic solution. While the means of log consumption and labor from the deterministic solution are zero, \(\mu_l\) is negative and \(\mu_c\) is positive in the stochastic solution, as in (3.25) and (3.26). Therefore, the level of expected utility from the stochastic method is greater than that from the deterministic method.

Using the results from the stochastic method, we derive the following expected utility;

\[
EU_{\text{stochastic}} = \frac{\Gamma}{1 - \gamma} \left( \exp \left[ \frac{(1 - \gamma) (2 \Gamma - \gamma)}{4 \Gamma^2 (1 - \nu)} \sigma_a^2 \right] - 1 \right).
\]

(3.27)

We can easily check whether the stochastic method corrects the welfare reversal by comparing the expected utility from the stochastic method with that of autarky. Subtracting the autarky expected utility (3.6) from the expected utility from the stochastic method (3.27) and simplifying, we have

\[
EU_{\text{stochastic}} - EU_{\text{autarky}} = \frac{\Gamma}{1 - \gamma} \exp \left( \frac{(1 - \gamma)^2 \sigma_a^2}{2 \Gamma^2 \sigma_a^2} \right) \left[ \exp \left( \frac{\gamma (1 - \gamma)}{4 \Gamma^2 (1 - \nu)} \sigma_a^2 \right) - 1 \right].
\]

(3.28)

If \(\gamma < 1\), then all terms in (3.28) are positive. Under the log utility (\(\gamma = 1\)), we can show the positivity of (3.28) by applying the L'Hopital’s rule to the first and third terms. If \(\gamma > 1\), then the first and third term become negative which makes the whole value positive. Therefore, the value of (3.28) is always positive and there is no welfare reversal under the stochastic method.

3.3. Intuition and accuracy of the stochastic method

The two intuitive explanations for the size of the bias correction given in the endowment economy hold in the production economy with labor. First, when the
utility function is linear in consumption ($\gamma = 0$), the solution derived from the exact method is

$$L_1^{\text{exact}} = A_1^{1/\nu},$$  \hspace{1cm} (3.29)

$$C_1^{\text{exact}} = \frac{A_1^{1/\nu} + A_2^{1/\nu}}{2}.$$  \hspace{1cm} (3.30)

Plugging these into the utility function, we can notice that there is no welfare gain from risk sharing. This property is preserved under the proposed method, since the level of the expected utility generated by the stochastic method, (3.27), is same as that of autarky, (3.6).

Our intuition involving the perturbation method holds for the production economy as well. The quadratic approximations of log consumption and log labor are

$$\log L_1^{\text{quadratic}} = \frac{\nu [(2\Gamma - \gamma) \log A_1 - \gamma \log A_2]}{2\Gamma (1 - \nu)} - \frac{\nu \gamma [(\log A_1)^2 - 2 (\log A_1) (\log A_2) + (\log A_2)^2]}{8 (1 - \nu)^2},$$  \hspace{1cm} (3.31)

$$\log C^{\text{quadratic}} = \frac{\log A_1 + \log A_2}{2\Gamma} + \frac{(\log A_1)^2 - 2 (\log A_1) (\log A_2) + (\log A_2)^2}{8 (1 - \nu) \Gamma}.$$  \hspace{1cm} (3.32)

The stochastic steady states, $\mu_l$ and $\mu_c$, are equal to the expected values of the second-order terms.

The fact that the stochastic method produces a correct welfare ordering is a necessary condition for an accurate approximation but not a sufficient condition. Now we compare the four solution methods for the complete market economy and assess the accuracy of the stochastic method. The graphs in Figures 3-1 and 3-2 are based on the following parameter specifications: $\gamma = 1, \nu = 0.5, \text{ and } \sigma_a = 1$.

The consumption CDFs in Figure 3-1 indicate that the deterministic method is worse than the stochastic and quadratic methods, except for outliers which drive the quadratic method to perform the worst. The CDFs of labor drawn in Figure 3-2 suggest mixed results. Therefore, we again compute the Kolmogorov-Smirnov statistics for the three approximation methods relative to the exact method.

The numbers in Table 2 show that, for both consumption and labor, the deterministic method is worse than the stochastic and quadratic methods. Comparison
of accuracy between the stochastic and quadratic methods generates mixed results: the quadratic method is the better as far as consumption is concerned, but the stochastic method produces more accurate approximation for labor.

Next, we take the log utility case \((\gamma = 1)\) and analyze welfare implications of various solution methods under different values for \(\nu\) and \(\sigma^2_a\). Under the log utility, the expected utility under autarky is

\[
EU^{\text{autarky}} = 0.
\]  

(3.33)

The deterministic solution produces

\[
EU^{\text{deterministic}} = \nu \left(1 - \exp \left[\frac{\sigma^2_a}{4 (1 - \nu)^2}\right]\right) < 0.
\]  

(3.34)

This inequality indicates spurious welfare reversals. According to the stochastic method, the expected utility is

\[
EU^{\text{stochastic}} = \frac{\sigma^2_a}{4 (1 - \nu)} > 0,
\]  

(3.35)

which confirms a correct welfare ordering. Since it is not possible to derive exact-but-tractable analytic expressions for the expected utility and certainty equivalent consumption under the exact and quadratic methods, we instead use simulations to derive welfare implications under these two methods.\(^{18}\)

Figure 5 plots the \(C_E\)’s of the five cases—autarky, exact, deterministic, stochastic, and quadratic—with respect to \(\nu\) assuming log utility and \(\sigma_a = 0.027\). The graphs show that, under the log utility, the conventional linearization always underestimates welfare and leads to a welfare reversal. Certainty equivalence consumption levels produced by both stochastic and quadratic methods are not only greater than that of autarky but also very close to the exact solution. As the elasticity of labor supply increases, the approximation errors increase. In particular, when \(\nu\) approaches unity, \(C_E^{\text{deterministic}}\) approaches zero and \(C_E^{\text{stochastic}}\) approaches infinity.\(^{19}\)

\(^{18}\)An alternative to simulations is to derive approximate analytic expressions by plugging second order approximations of the variables into a second order approximation of the utility function such as (2.19). This approach is adopted by Kim, Kim, and Levin (1999).

\(^{19}\)In the general case of decreasing returns, they do not approach to zero and infinity. See Appendix A for details.
Figure 6 plots the levels of expected utility of the five cases on the variance of shock $(\sigma_a^2)$ setting $\gamma = 1$ and $\nu = 0.5$.\textsuperscript{20} The three lines at the top corresponding to the quadratic, exact and stochastic methods are very close to each other, which demonstrates that both stochastic and quadratic methods generate an accurate approximation of the exact solution. Mathematically speaking, both stochastic and quadratic methods characterize the exact solution correctly up to the first order of $\sigma_a^2$.

As seen in Figure 6, welfare reversal always occurs except when $\sigma_a^2 = 0$. With any positive variance, however small it is, the conventional loglinearization reverses welfare ordering. The previous literature on the accuracy of loglinearization argues that the approximate solution becomes more accurate as the variance of shocks approaches zero. However, this argument doesn’t apply to the case of welfare ordering between complete and incomplete market economies. The approximation errors that do not affect the conventional metrics become a major factor determining the welfare ordering.

4. Conclusion and Further Research

We have demonstrated how spurious welfare reversals could happen in a static international business cycle model and have provided a modified method of linear approximation to compute an accurate welfare level. This paper could be extended in two ways.

First, we have not emphasized the size of welfare gains from risk sharing when we apply our proposed method. The level of welfare gains are readily computable, but this paper focuses on qualitative aspects of accurate welfare calculation.\textsuperscript{21} Furthermore, a more interesting comparison in terms of risk sharing gains is between the complete market economy and an incomplete market economy such as the bond-only economy, rather than between the complete market economy and autarky. The conventional linearization method becomes more problematic since there are multiple deterministic steady states in the incomplete market model. Kim, Kim, and Levin (1999) are currently pursuing this line of research.

\textsuperscript{20}Note that when $\gamma = 1$, we have $\sigma_a = \sigma_y$.

\textsuperscript{21}In Appendix B, we briefly document the size of risk sharing gains in several cases. We found that risk sharing gains can reach up to 5% of world consumption under reasonable parameter values with labor production economy. This number is larger than those produced by previous risk sharing papers using a production economy with both labor and capital or an endowment economy.
Second, even though the deterministic method produces incorrect results for welfare comparison in our international setting, this method works well in some other cases. In a certain class of closed-economy monetary-policy models where labor is the only input, such as Rotemberg and Woodford (1997) and Henderson and Kim (1999), the deterministic method provides accurate results for welfare implications of various policies when the shock variances are small. Sufficient conditions for the validity of the conventional method are derived by Woodford (1999). However, there is as yet no result in the literature regarding how to use linear approximations to calculate correct welfare levels in a wider and more interesting class of models. For example, in the class of models with both labor and capital as in Ireland (1997), Cho, Cooley, and Phaneuf (1997) and Biis and Chang (1998), the deterministic method can generate significantly large approximation errors that potentially produce incorrect welfare implications. It is important to develop a linear approximation method that can be used for welfare analysis. We are currently working on the validity of the stochastic method for welfare analysis in a general class of DSGE models.
References


22


Table 1. Accuracy of the three approximation methods: endowment economy

<table>
<thead>
<tr>
<th>approximation method</th>
<th>Kolmogorov-Smirnov statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>deterministic</td>
<td>0.114</td>
</tr>
<tr>
<td>stochastic</td>
<td>0.034</td>
</tr>
<tr>
<td>quadratic</td>
<td>0.021</td>
</tr>
</tbody>
</table>

Table 2. Accuracy of the three approximation methods: production economy

<table>
<thead>
<tr>
<th>approximation method</th>
<th>Kolmogorov-Smirnov statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>consumption</td>
</tr>
<tr>
<td>deterministic</td>
<td>0.17</td>
</tr>
<tr>
<td>stochastic</td>
<td>0.12</td>
</tr>
<tr>
<td>quadratic</td>
<td>0.07</td>
</tr>
</tbody>
</table>
Figure 1: Approximation errors of the deterministic method

Cumulative density vs. consumption

- **Exact**
- **Deterministic**
Figure 2: Comparison of the three approximations

cumulative density vs. consumption

- exact
- deterministic
- stochastic
- quadratic
Figure 3–1: Cumulative density function of consumption ($\gamma=1, \nu=0.5$)

- exact
- deterministic
- stochastic
- quadratic
Figure 3–2: Cumulative density function of labor ($\gamma=1, \nu=0.5$)

- **exact**
- **deterministic**
- **stochastic**
- **quadratic**
Figure 4. Contour of welfare ratio plotted on $\gamma$ and $\nu$: $C_{CE}^{\text{deterministic}}/C_{CE}^{\text{autarky}}$.
Figure 5: Plot of welfare on $v$ ($\gamma=1, \sigma=0.027$)

- certainty equivalent consumption
- quadratic
- exact
- stochastic
- autarky
- deterministic
Figure 6: Plot of expected utility on $\sigma^2$ ($\gamma=1, \nu=0.5$)

- Expected utility
- Quadratic
- Exact
- Stochastic
- Autarky
- Deterministic