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DERIVATIVES IN FINANCIAL MARKETS WITH STOCHASTIC VOLATILITY, BY JEAN-PIERRE FOUQUE, GEORGE PAPANICOLAOU AND K. RONNIE SIRCAR, CAMBRIDGE UNIVERSITY PRESS 2000. XIV + 201PP. ISBN 0-521-79163-4
Reviewed by Mark Davis, Imperial College London

In an ideal Black-Scholes world, implied volatility would be constant across all strikes and maturities. Needless to say, it isn't. The 'volatility surface' of traded option implied volatilities exhibits the familiar 'smile' or 'smirk' behaviour with, typically, out-of-the-money calls and puts quoted at lower implied vols. Further, a plot of daily implied volatility of at-the-money options with fixed time to maturity looks quite 'stochastic'. Of course, building these facts into your model is an essential part of option trading and risk management. The question is how to do it.

There are, roughly speaking, three ways:

- Make volatility a function of price level; thus no extra randomness is introduced and vol only fluctuates because price does.
- Introduce a stochastic process model – generally mean-reverting – to represent the instantaneous vol in a price model that otherwise looks like Black-Scholes.
- Replace the Black-Scholes price model by something completely different, for example a Lévy process model with jumps.

The first is the least radical departure from classic Black-Scholes. Its advantage is that the market is still complete, so that pricing and hedging à la Black-Scholes is still correct, and models can be built that fit the observed smile. On the other hand, these models may be over-parametrized for hedging, and even cursory econometric analysis tells you that price and volatility are not perfectly correlated. The second alternative – emphasized for example by Hull and White – has many attractions: it leads to quite simple formulas, and it gives a realistic representation of the evolution of implied vol over time. The market is now incomplete; it can be completed by including a traded option as an independent asset, but in many different ways. And it is not clear that realistic levels of vol of vol are enough to explain observed smiles. The third and most radical alternative is probably the right one for VaR analysis, where 'goodness-of-fit' is the main criterion, but is not so attractive for trading because so little of Black-Scholes remains. Thus each approach has its advantages, but no-one can say the problem is conclusively solved.

Fouque, Papanicolaou and Sircar ['FPS'] have come up with something genuinely new in this area, explained with admirable clarity in this extremely well-written book. In spirit it is close to the second alternative above, in that a mean-reverting process is hypothesized as an explanation for movements in volatility. The new twist is a treatment of this process by asymptotic analysis, which gives both a certain model-independence and easily-computed corrections to the standard Black-Scholes pricing and hedging formulas.

They start with the Ornstein-Uhlenbeck process, a mean-reverting Gaussian process Y_t satisfying the SDE

$$dY_t = \alpha(m - Y_t)dt + \beta dW_t, \tag{1}$$

where W_t is Brownian motion. This is an asymptotically stationary process, whose stationary distribution is $N(m, \beta^2/2\alpha)$ (the normal distribution with mean m and variance $b^2/2a$, density function $\phi(y)$.) Further, it is ergodic, in that 'time average = ensemble average'. This means that if we take a (say bounded) function f then

$$\frac{1}{T}\int_0^T f(Y_t)dt \to \int f(y)\phi(y)dy \text{ as } T \to \infty \tag{2}$$

The parameter α controls the speed of convergence to the stationary distribution. Indeed, increasing α is equivalent to speeding up the process, so if we increase α while keeping the stationary variance $\beta^2/2\alpha$ constant then the integral on the left of (2) keeping T fixed will converge to the average on the right. Suppose we now let $\sigma^2(t) = f(Y_t)$ be the stochastic volatility in our price model. Then the integral on the left of (2) is just the average realized variance over the interval [0,T]. If α is very large this is effectively constant, and we are back to Black-Scholes. The central argument of FPS is that in reality α is *large but not infinitely large*, so we can treat the stochastic volatility model as a small perturbation from Black-Scholes, which can be quantified by an asymptotic expansion.

The bottom line is a corrected pricing formula giving the option price P as $P = P_0 - P_1$ where P_0 is the Black-Scholes price based on long-run average volatility and the correction P_1 is given by

$$(T-t)\left(V_2 x^2 P_0^{(2)} + V_3 x^3 P_0^{(3)}\right).$$

Here $P_0^{(2)}, P_0^{(3)}$ are the second and third derivatives with respect to price (so $P_0^{(2)}$ is the option gamma). This is initially derived from the stochastic vol model (1), but the authors point out that the formula is universal in that a wide range of vol models lead to exactly the same formula, albeit with different V_2, V_3. These parameters can be used to calibrate the model to a given implied volatility surface by a simple procedure. There is also an associated hedging strategy based, roughly speaking, on delta hedging with the corrected price. As FPS point out this hedging strategy is not self-financing, but it is a 'minimum variance' strategy. Applications to exotic options and to interest rate contingent claims are also developed.

Have FPS cracked the stochastic volatility problem? Not really, but they have certainly provided an innovative approach that merits attention and complements the methods listed above. The approach is applicable to pricing and hedging but not to VaR analysis. Pricing is relatively benign in that the calibration process tends to wash out the differences between models except in the case of really sensitive exotics. In hedging, it is very hard to establish that any method of volatility prediction is consistently superior in terms of portfolio hedge performance using historical price and implied vol data. It remains to be seen how the FPS method stacks up in this respect.

As a prelude to their main theme, FPS have provided a compact but very readable summary of a number of topics in mathematical finance, stochastic analysis and Markov processes, putting a welcome emphasis on the effects of model error in dynamic hedging. Finally, the book is short and to the point, and the production quality is high. Buy it.