Direct Numerical Simulations of Turbulent Ekman Layers with Increasing Static Stability: Modifications to the Bulk Structure and Second-Order Statistics

STIMIT K. SHAH and ELIE BOU-ZEID

Department of Civil and Environmental Engineering, Princeton University, Princeton, NJ 08540, USA

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Direct Numerical Simulations (DNS) of Ekman layers are conducted to study the effect of increasing static stability on turbulence dynamics in wall-bounded flows at three moderate Reynolds numbers, and the implications for turbulence models. The flow field is examined by looking at mean profiles of wind speed, potential temperature, momentum flux, as well as streamwise velocity and temperature spectra. The maximum stabilizing buoyancy flux that a flow can sustain while remaining fully turbulent is found to depend on the Reynolds number. The flows with the highest Reynolds numbers display well-developed inertial range and logarithmic layer, and are found to bear many similarities to much higher Reynolds number flows like the ones encountered in the atmospheric boundary layer; in particular, the mean profiles follow the Monin-Obukhov similarity theory. However, several flow features are found to maintain significant dependence on the Reynolds number. Subsequently, the budgets of turbulence kinetic energy (TKE), vertical velocity variance, momentum and buoyancy fluxes, and temperature variance are analyzed. The DNS results indicate that the effect of stability on turbulence is first directly manifested in the vertical velocity variance budget, and results in damping of vertical motions. This then leads to a reduction in the downward transport of horizontal momentum components towards the surface, and consequently to a decrease in the shear production term in the turbulence kinetic energy budget: changes in the vertical profile of TKE shear production are significant with increasing Richardson number and have a larger impact than direct buoyancy destruction. The reduction in vertical velocity variance also results in significant drops in the production terms in the momentum flux, buoyancy flux, and temperature variance budgets. Various assumptions and parameters related to low-order turbulence closures are finally investigated. The results suggest that the vertical velocity variance is a better parameter, than the full TKE, to base eddy-diffusivity and viscosity models on.

Key words: Turbulent Ekman layers, Stable Boundary Layers, TKE budget, Flux budget, Temperature variance budget

1. Introduction

Understanding and parameterizing turbulent fluxes in statically stable boundary layers, where buoyant forces destroy turbulence kinetic energy, remains a challenging yet
very important problem in geophysical fluid dynamics (Ivey et al. 2008; Fernando & Weil 2010). A particular flow of interest is the one in the atmospheric boundary layer (ABL), which roughly spans the lowest 0.2 to 2 km of the atmosphere and which is ubiquitously affected by stability (Stull 1988). Some of the crucial experiments in understanding ABL flow and turbulence date back to as early as the 1960s-70s (Faller 1963; Tatro & Mollo-Christensen 1967; Caldwell & VanAtta 1970; Kaimal et al. 1972; Caldwell et al. 1972). The stable ABL was recognized early on as a uniquely challenging and important fluid dynamics problem. Arya (1968), Brost & Wyngaard (1978), Nieuwstadt (1984), and Hunt et al. (1985) suggested simple models to get mean profiles of turbulent quantities in the stable ABL. Arya (1975), for example, carried out experiments on a flat plate with stabilizing temperature gradients and illustrated that the mean temperature and velocity profiles are affected by stability, both in the inner and outer layers. André et al. (1978) developed a higher-order closure model for the diurnal evolution (where stability varies with time) of the ABL, which included equations for up to third order statistics. Zilitinkevich et al. (1992) also proposed a model for an unsteady ABL based on similarity theory. Mahrt (1999) assessed the complications in modeling of stable layers and formulated surface flux parameterizations for numerical models. One such complication is that under very stable conditions, turbulence in the atmosphere has been observed to collapse. In a recent study, Flores & Riley (2011) developed a criteria for identifying such a collapse under strongly stable conditions using DNS of open channel flows. van de Wiel et al. (2011) also discuss the condition in the atmospheric boundary layer where winds become weak resulting into very low levels of turbulence in the atmosphere. They analyze these quasi-laminar flows using DNS and a simplified model to understand collapse of turbulence.

Numerical simulations, with their ability to provide 3D, time-varying information on turbulent structures and dynamics, have recently emerged as a powerful tool to further our understanding of the stable ABL problem. Some of the first numerical simulations of neutral and unstable (convective) boundary layers were carried out by Deardorff (1970, 1972) using large eddy simulations. Deardorff investigated profiles of turbulence statistics, the effect of buoyancy on mean profiles, and the effect of latitude upon turbulence and its structure. Later on Mason & Thomson (1987), Moeng & Sullivan (1994), Cai & Steyn (1996), Kosović & Curry (2000), Jiménez & Cuxart (2005), Bou-Zeid et al. (2004, 2007), Stoll & Porté-Agel (2008), and Zhou & Chow (2011), to name a few, also did similar studies of neutral and non-neutral ABL flows relying mainly on the LES technique with wall-modeling (Pope 2000). Remmler & Hickel (2012) proposed a turbulence model based on Adaptive Local Deconvolution Method (ALDM) for simulations at large Reynolds numbers, which are otherwise not possible using DNS, and tested it for homogeneous stably stratified flow. However, for flows under highly stable conditions, some uncertainties regarding the performance of LES remain for two reasons. Firstly, due to stability, there is a reduction in the size of the eddies (Mason & Derbyshire 1990; Andren 1995). This reduces the fraction of the turbulence kinetic energy that the simulation can resolve and the resolution with which the most energetic eddies are captured, and hence leads to an increased importance of the subgrid scales and models (Huang & Bou-Zeid 2013). Secondly, under strongly stable conditions, turbulence can become spatially and temporally intermittent as well as highly anisotropic (Ohya et al. 1997; Mahrt 1999); this invalidates some of the basic assumptions often used in developing subgrid scale models such as the availability of statistically-homogeneous dimensions and the applicability of Kolmogorov’s theories to the subgrid range of eddies (Bou-Zeid et al. 2010).

While recent a priori (Bou-Zeid et al. 2010) and a posteriori (Kumar et al. 2006; Huang & Bou-Zeid 2013) studies tested and validated some facets of the LES approach for weak
and moderate stabilities, their scope does not include very high stabilities where the turbulence can become intermittent or completely damped. Furthermore, Taylor & Sarkar (2008) recently investigated bottom Ekman layers under an external stratification using DNS, wall-resolved LES and wall-modeled LES; their main aim being to evaluate wall-resolved and wall-modeled LES, along with the investigation of the thermal properties of the Ekman layer. They observed that LES can capture mean velocity profiles well, but had biases in reproducing some higher-order quantities (fluxes near the top inversion of the stable boundary layer). Higher-order statistics in LES are more sensitive to SGS contributions and models than the means, and the reduced quality in simulating these statistics is related to the increasingly important role of the SGS scales and budgets with increasing stabilities. Hence, to probe in detail how stability modifies turbulent structure and higher-order budgets, in this paper, we rely on direct numerical simulations despite the limit this technique imposes on the highest achievable Reynolds number (Spalart et al. (2008, 2009) could achieve a maximum Reynolds number $Re_f$ of 2000, while in the atmosphere the observed $Re_f \sim 10^5$).

We perform a series of DNS at moderate Reynolds numbers with the aim of better understanding flow physics and improving current turbulence models under strongly stable conditions. The setup of the numerical experiments in this paper is very similar to the DNS studies on statically stable Ekman layers carried out by Coleman et al. (1992) at low Reynolds numbers. Coleman et al. (1990) and Coleman et al. (1992) examined the structural differences in neutral and stable Ekman layers by analyzing, among others, the vertical profiles of various parameters and the eddy size. They also evaluated some of the parameterizations recommended by Brost & Wyngaard (1978), Nieuwstadt (1984), and Hunt et al. (1985) for stable atmospheric boundary layers. The main aim of Coleman et al. was to extrapolate, using theoretical arguments, certain DNS results to high Reynolds numbers applicable to atmospheric boundary layers. Their analysis suggests that the models proposed by Hunt and Brost & Wyngaard are valid if the effects of varying Reynolds number are accounted for. Their analysis also shows that simple gradient closures for temperature variance and heat flux are not very sensitive to Reynolds number variation. However, later on, Coleman (1999) simulated neutral Ekman layer at Reynolds number of 1000 based on the geostrophic wind and the Coriolis parameter. They found that the flow has a distinct logarithmic region and that the profiles of shear stress are indeed similar to the high Reynolds number flow simulated using LES by Andrén & Moeng (1993), but they did not find a complete Reynolds number similarity. This suggests that further investigations of turbulent Ekman layers, where both the Reynolds (Re) and Richardson (Ri) numbers are varied over wide ranges, are needed to further understand the effect of stability and to develop simple but accurate closures, that account for stability, for weather and climate models. Marlatt et al. (2011) evaluated two such closure models for neutrally stratified Ekman layers at moderate Reynolds numbers, which approximate eddy diffusivities as a function of height using cubic polynomial. In this paper, we perform DNS with varying Reynolds and Richardson number to understand the physical mechanisms through which stability affects the budgets of first- and second-order quantities, with the long-term aim of developing physical models of turbulence in the stable wall-bounded flows for any Re-Ri combination.

One of the few studies with analysis of second-order budgets was performed by Garg et al. (2000), who investigated the budgets of TKE, fluxes and density variance in a channel flow using wall-resolved LES with a dynamic model and showed that stability weakens the coupling between the inner and outer layers. Their Reynolds numbers based on the friction velocity $u_*$ were limited to 180 and 360, while in this paper, and due to the increase in computational power since their paper came out, we are able reach over values
of 1180. In addition, Coriolis force was not included in their simulations and the upper part of the channel remained turbulent due to mechanical production at the upper wall, which would affect the bottom stable part of the flow in a channel but not in a boundary layer flow. Hattori et al. (2007), in their DNS of stable and unstable developing boundary layers without Ekman turning, examined the budgets of TKE, fluxes, and temperature variance. The main aim of their study was to probe the near wall structures and turbulent quantities under the influence of stability and as such their simulations were performed at a constant Reynolds numbers, and only one stable simulation (one Richardson number) was carried out. Marlatt et al. (2010) on the other hand analyzed turbulence energy budgets in Ekman layer simulations, but did not study the effect of varying Richardson and Reynolds numbers. Hence, to the best of our knowledge Garg et al. (2000), Hattori et al. (2007) and Marlatt et al. (2010) are the only studies that considered the budgets of turbulence kinetic energy, fluxes, and temperature variance, but many flow conditions remain unprobed and many questions remain open.

In this paper we discuss the effect of Reynolds number and stability, quantified using the Richardson number, on turbulent Ekman boundary layers. On a neutrally stratified flow, stable temperature profile is imposed keeping the surface temperature constant. The equation of the imposed profile is given in section §2. The main questions we would like to answer are: 1) How does buoyancy in stable layers affect the profiles of mean second-order turbulent quantities like spectra, the various terms of the TKE budget equation, fluxes, and temperature variance? 2) Are there mechanisms, other than direct buoyancy destruction of TKE, through which stability impacts the flow and can these mechanisms be quantified? 3) Is Monin-Obukhov similarity theory valid under stable conditions at relatively lower Reynolds numbers than those seen in the atmospheric boundary layer?

In the next section §2, the equations describing the simulated flow here are given. In §3 we discuss the numerical methods and simulations carried out. Validation of the numerical methods and the code has been discussed in the §4. §5 presents the results and analyses and §6 contains discussion and conclusions.

2. Governing equations and boundary conditions for DNS

The incompressible Navier-Stokes equations and the thermal energy budget equation, which govern the flow of a statically stable fluid, are numerically integrated in time. The momentum budget equations are solved in rotational form to ensure conservation of kinetic energy (Orszag & Pao 1974). The flow is driven by a steady pressure gradient, equivalent to a geostrophic forcing, and it experiences steady rotation (Coriolis force). We use the Boussinesq approximation to account for buoyancy effects in the fluid. The governing equations are as follows:

\[
\frac{\partial u_i}{\partial t} + u_j \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = -\frac{1}{\rho_\infty} \frac{\partial p*}{\partial x_i} + 2 \epsilon_{ijk} \Omega_j (G_k - u_k) + \delta_{i3} g \Phi + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j},
\]

\[
\frac{\partial \Phi}{\partial t} + u_i \frac{\partial \Phi}{\partial x_i} = \alpha \frac{\partial^2 \Phi}{\partial x_j \partial x_j},
\]

\[
p* = p + (1/2) \rho_\infty u_j u_j,
\]

\[
\frac{\partial P}{\partial x_i} = -2 \rho_\infty \epsilon_{ijk} \Omega_j G_k + \delta_{i3} \rho_\infty g,
\]

\[
\Phi = (\theta - \theta_\infty)/\theta_\infty,
\]
where $u_i$ is the velocity vector; $\theta$ is the local potential temperature and $\theta_\infty$ is the reference potential temperature; $g = 9.81 \text{m}^2\text{s}^{-2}$ is the gravitational acceleration; $\epsilon_{ijk}$ is the alternating unit tensor; $\delta_{ij}$ is the Kronecker delta; $\Omega_j$ is the angular velocity vector of the Earth and is in the opposite direction as gravity (the model here assumes ‘f-plane’ approximation; see Zikanov et al. (2003) for details of the effect of non-zero horizontal component of the Earth’s rotation vector); $G_k$ is geostrophic wind vector; $p$ is the pressure deviation from the mean pressure ($P$) field imposed via the geostrophic forcing and hydrostatic balance, and $\rho_\infty$ is constant reference density. $p^*$ is the modified pressure term, and is computed by solving a Poisson equation (resulting from taking the divergence of the momentum equation and using the continuity equation to simplify it) at each time step. $\nu$ and $\alpha$ are the kinematic viscosity and thermal diffusivity, respectively. Note that the actual potential temperature conservation equation has the term $\alpha \frac{\partial^2 \theta}{\partial x_j \partial x_i}$. But at small scales where thermal diffusivity is important, the ratio $\frac{\theta}{T}$ varies negligibly in space and hence the diffusion term simplifies to $\alpha \frac{\partial^2 \theta}{\partial x_j \partial x_i}$ (Wyngaard 2010). This term we write as $\alpha \frac{\partial^2 \Phi}{\partial x_j \partial x_i}$.

Horizontal boundary conditions are periodic. Vertical boundary conditions consist of a no-slip condition ($u = 0$ and $v = 0$) with a smooth wall at the bottom surface, and a stress-free with no-penetration/impermeability boundary condition along with no heat flux at top of the domain:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial z} = 0 \quad \text{at the top.} \quad (2.7)$$

A constant temperature, which is lower than the temperature $\theta_\infty$ above the boundary layer, is imposed at the wall; this results in a downward stabilizing heat flux across the whole boundary layer that varies with time. Consequently, a stationary state is not attained in stable cases; however, the change in downward heat flux is sufficiently slow such that a quasi-steady state is reached where the turbulence is very close to equilibrium with the mean fields at any time. This setup is very similar to the one used by Coleman et al. (1992). Taylor et al. (2005) simulated an open channel flow using LES with heat flux at top of the domain:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{and} \quad \frac{\partial \theta}{\partial z} = 0 \quad \text{at the top.} \quad (2.7)$$

The initial temperature profile (Coleman et al. 1992) is given by the following equation:

$$\theta(z) = \frac{a_s}{2} \left( -\frac{\pi}{\ln(0.01)} \right)^{1/2} \frac{\partial}{\partial z} \left[ \frac{z/a_s}{(-\ln(0.01))^{1/2}} \right] \theta_0, \quad (2.8)$$

where $\text{erf}$ is error function, $\Gamma_{0,0}$ is initial surface temperature lapse rate (here lapse rate is defined as $\Gamma = d(\theta)/dz$), $\theta_0$ is surface temperature that remains fixed throughout any given simulation, and $a_s$ is the height where the vertical temperature gradient $d(\theta)/dz$ is 1% of $\Gamma_{0,0}$. Throughout this paper, single 0 subscript is used to indicate surface values and double (0, 0) subscript is used to indicate surface values at initial time, $t = 0$. Further details about the initialization of the temperature field can be found in Coleman et al. (1992). Note that since we have imposed constant temperature at the surface, the mean fluid temperature will evolve in a manner determined by the time-dependent heat flux.

Some of the important non-dimensionalization variables we will use include: the Coriolis parameter $f = 2 \Omega \sin \phi$, where $\Omega$ is angular velocity of the Earth and $\phi$ is the latitude (here we assume a latitude of 90° giving a value of $f = 1.145 \times 10^{-4} \text{rad} \text{s}^{-1}$); the laminar Ekman layer depth $D = (2\nu/f)^{1/2}$; the neutral case turbulent Ekman layer length scale $\delta_t = u_s/f$ (a good approximation of the actual boundary layer depth in our simulations would be $C_h u_s/f$; where $C_h = 0.7$ is a constant. The range of values of $C_h$ used in
literature vary from 0.5-0.7 (Caldwell et al. 1972; Coleman et al. 1990; Zilitinkevich & Mironov 1996). Friction velocity $u_*$ is defined as follows:

$$u_* = \left[ \left( \frac{\partial u}{\partial z} \right) _0^2 + \left( \frac{\partial v}{\partial z} \right) _0^2 \right]^{1/2},$$

(2.9)

where the subscript 0 indicates that the gradients are taken at the surface. Also note that the the kinematic viscosity of air $\nu = 1.14 \times 10^{-5} m^2 s^{-1}$.

The important set of non-dimensional parameters based on these variables are: the Reynolds number $Re_f = GD/\nu = G/(\nu f/2)^{1/2}$, the bulk Reynolds number $Re_b = G\delta /\nu$; the surface (initial) Richardson number $Ri_{0,0} = g\Gamma_0D^2/\theta_\infty G^2$; the bulk Richardson number $Ri_b = g \Delta \theta \delta_\infty/\theta_\infty G^2$ where $\Delta \theta = \theta_\infty - \theta_b$; the Prandtl number $Pr = \nu/\kappa$; the non-dimensional thermal length scales $L_V = a_*/D$ and $L_T = a_*/\delta_b$; and the non-dimensional time $t_* = tf$. Note that $U_g$ is the $x$-component of $\mathbf{G}$, but since the imposed mean geostrophic wind is in $x$-direction, $U_g$ and $G$ (the magnitude of $\mathbf{G}$) are the same.

3. Physical and numerical setup of the simulations

The code used here was initially developed as an LES code by Albertson and Parlange (1999; 1999) and then later modified by Porté-Agel et al. (2000) and Bou-Zeid et al. (2005). The domain is periodic in the spanwise and streamwise directions, and this is associated with the use of pseudo-spectral numerical schemes in those directions ($x$ & $y$). A second-order centered-difference scheme on uniform grid with staggering is used in the vertical (z). Time integration uses the second-order Adams-Bashforth scheme. Nonlinear terms are dealiased using the 3/2 rule (Orszag 1971). Message passing interface (MPI) is used to carry out the simulations on multiple cores. Simulations at Reynolds numbers ($Re_f$) of 400, 600 and 900 were carried out on 128, 288 and 416 cores, respectively, on computing clusters at the National Center for Atmospheric Research (through project P36861020) and at Princeton University. In total 15 cases were simulated.

3.1. Physical parameters

Simulations details are given in the table 1. Unlike Coleman et al. (1990) and Coleman et al. (1992), here we vary all the parameters: the Reynolds number ($Re_f$) from 400 to 900 and the surface Richardson number ($Ri_{0,0}$) from 0.001 to 0.010. Parameters that have been kept constant are the latitude $\phi = 90^\circ$ that was selected to give the largest Coriolis effect, the Prandtl number ($Pr = 0.7$ for air), and the ratio of the thermal length scale to the approximate turbulent length scale $L_T = a_*/\delta_b = 0.75$. This values that has been used for $a_*$ here corresponds to late night or early morning temperature profile over land, where the effect of lower surface temperature has reached to larger heights. Some analysis with temperature profiles corresponding to early evening can be found in Coleman et al. (1992). For the three Reynolds numbers ($Re_f = 400, 600, 900$), the geostrophic wind speeds are 0.0115, 0.0172 and 0.0259 m/s, respectively; the domain sizes used are $L_x \times L_y \times L_z = 26D \times 26D \times 24D, 36D \times 36D \times 25D$ and $50D \times 50D \times 33D$, respectively; and the heights of the computational domains in terms of the boundary layer height are approximately $2\delta_b, 1.5\delta_b$ and $1.35\delta_b$, respectively. Coleman et al. (1992) found that $Ri_{0,0} \sim 0.001$ is the near maximum surface Richardson number that can be imposed on a neutral flow with $Re_f = 400$. Our simulation results agree with findings of Coleman et al. (1992); for higher surface Richardson number (0.005 and 0.010) at $Re_f = 400$, the flow laminarizes. However, we observed that for $Ri_{0,0} = 0.002$, turbulence intensity dies down to very small values, and then increases again after build-up of shear. With a higher Reynolds number ($Re_f = 600$), the surface Richardson number $Ri_{0,0}$ where
this oscillatory behavior in turbulence kinetic energy is seen increases to around 0.005, while for \( Ri_{0,0} = 0.010 \) the flow laminarizes almost completely. At \( Re_f = 900 \), the flow remains fully and steadily turbulent up to \( Ri_{0,0} = 0.005 \); hence, an increase in the critical Richardson number with increasing Reynolds number (Schlichting 1979) is confirmed.

The extent of the domain is governed by the ability to capture the largest dynamically-important turbulent scales. Jiménez (1998), Abe & Kawamura (2002), and others have indicated the existence of very large-scale structures that could require very large simulation domains. However, Coleman et al. (1990) found no evidence of such large-scale structures in their studies. Shingai & Kawamura (2004) studied neutral Ekman layers at higher Reynolds numbers and on larger domains than Coleman et al. (1990) to verify the existence of these large-scale structures. They observe that energy spectra and two-point correlations do indicate the existence of large-scale structures away from the wall; however, turbulence statistics near the surface are not affected by the size of the computational domain and averaging time, which suggests that these very large-scale structures are not very active dynamically near the wall. Thus, despite the continued research into the topology and behavior of such very large-scale structures, for simulations focusing on the dynamics in the lower part of the boundary layer, domain sizes similar to the ones used here are sufficient.

3.2. Numerical parameters

Computations for all cases at a particular Reynolds number were performed using the same number of grid points for all stabilities at a given Reynolds number; the corresponding grid resolution, domain sizes, and other simulation parameters are given in the table (1) below. As Coleman et al. (1992) argue, vorticity fluctuations are reduced in the stable cases and so is the rate of dissipation of kinetic energy throughout the domain. Even if stability dampens some of the largest length scales present, the Kolmogorov scale (\( \eta \)) will be larger in the stable cases than in the neutral one with the same \( Re_f \) (see appendix A for detailed profiles of the Kolmogorov and other pertinent length scales). This can also be verified from figure 1. Here we plot the normalized streamwise velocity spectra against the normalized wavenumber at a different heights \( z/\delta_t \), for \( Re_f = 900 \). The energy content at all the levels drops with stability. For the highest level analyzed, which is at the top of the boundary layer, the inset in figure 1b clearly shows that the smallest scales in the stable case are over-resolved. Similarly, the smallest scale over which fluctuations in scalar field are maintained is given by the Batchelor scale \( \eta_s \), which for \( Pr < 1 \) is larger than \( \eta \) (Tennekes & Lumley 1972). Hence, the number of points used for a neutral case are also sufficient to capture the Kolmogorov scale eddies in the stable cases at the same Reynolds number. A further requirement in DNS is the proper resolution of the viscous sublayer. We ensure that by placing around 10 points in the vertical direction within the first \( z^+ \) of 10 in the neutral simulations, the expected depth of that viscous sublayer, to yield a non-dimensional vertical grid spacing in wall units (\( \Delta z^+ = \Delta z \ u_* / \nu \approx 1 \)). Again this is more than sufficient for the stable cases at the same \( Re \) in which we maintain the same dimensional \( \Delta z \): the friction velocity at the surface is reduced for stronger stability simulations, leading to a reduction in the grid spacing in wall units \( \Delta z^+ \), as reported in table 1, and hence to a better resolution of the viscous sublayer.

Simulations are run for at least 60 physical hours with no static stability (i.e. neutral case) as a warm-up to allow the flow field to fully develop before any statistics are collected for analysis. Initial conditions imposed for these warm-up runs were laminar profiles with random perturbations of about 5% of free stream values. The flow fields at the end of these warm-up runs (for different Reynolds numbers) are subsequently used to initialize the runs used to generate the analyses data; the analyses
Figure 1. Streamwise velocity spectra for (a) neutral case ($Ri_{0,0} = 0$) (b) stable case with $Ri_{0,0} = 0.002$ at Reynolds number ($Re_f$) of 900. The spectral energy is non-dimensionalized using the plane-averaged dissipation at that height ($\epsilon$) and molecular viscosity ($\nu$), while the wavenumber is non-dimensionalized by the Kolmogorov scale ($\eta = (\nu^3/\epsilon)^{1/4}$) at that height. The $-5/3$ slope corresponding to the inertial subrange is also depicted. The inset in (b) shows a zoomed-in plot of the spectra at the higher levels and wave numbers. Here the height is normalized by neutral case turbulent length scale so that we can compare spectra at the same physical height in the stable and neutral cases.

Simulations are run for one inertial period $= 2\pi/f$ (here $f = 2\Omega \sin(\pi/2) = 2\Omega = 2 \times 2\pi$ radians/24 hours; thus one inertial period = 12 hours), for both neutral and stable cases. However, for cases with stability, the temperature profile is imposed on these velocity fields and the buoyancy term is added to the momentum equations. While stable cases are run for one inertial period, the data used for analysis of these cases is from the last half of this period. This allows the stable simulations to reach a quasi-equilibrium state. Perturbations do not need to be added to the initial temperature field (Coleman et al. 1992) for the stable cases; they develop automatically due to velocity perturbations.

The time step in inner coordinates ($\Delta t^+ = \Delta t u^+_2/\nu$) for the neutral cases is around 0.015 ($\Delta t^+ = 0.011, 0.0104$ and 0.017 for $Re_f = 400, 600$ and 900, respectively). The corresponding dimensional time steps thus ranged from 0.1 to 0.225 s. For stable cases, since the friction velocity is smaller than the neutral value, the time steps in inner coordinates are also somewhat smaller.

3.3. Statistics

Throughout this paper, the notation used is as follows: $u_j = \langle u_j \rangle + u'_j = U_j + u'_j$, $p = \langle p \rangle + p'$, $\rho = \langle \rho \rangle + \rho'$ and $\theta = \langle \theta \rangle + \theta'$, where angle brackets $\langle \rangle$ indicate Reynolds averaging and primes $'$ indicate the fluctuating component. Sub-scripts denoting directions $x$, $y$ and $z$ are used interchangeably with 1, 2 and 3, respectively. Pressure is computed from the modified pressure solved for by the DNS code by subtracting the trace term as per equation 2.4. Averaging has been carried out over $xy$-planes and in time $t$, at each $z$, for about half an inertial period, or $\pi/f$ seconds, unless otherwise noted. The start and end points of the averaging period are later indicated in the surface Richardson number plots (for example 8a) by cross marks.
4. Validation

4.1. Neutral mean profiles

We first present basic plots to validate our methodology and the DNS code. Figure 2 shows the mean velocity profiles for the neutral simulations at various Reynolds numbers in comparison with the experimental studies of Caldwell & VanAtta (1970) and the numerical simulations of Shingai & Kawamura (2004) and Morris et al. (2011). Our simulations reproduce the log-law (given by $M = \frac{1}{\kappa} \log (z^+) + B$, where $M$ is the mean streamwise velocity magnitude, $\kappa = 0.41$ is the the von Kármán constant, and $B = 5.5$ is a constant) very well and match both experiments and other simulations. The simulated log-region can also be observed in figure (3), where a plateau of constant $\kappa$ depicting the slope in the log-region is seen (Marlatt et al. 2011). The value of the von Kármán constant $\kappa$ can be computed using the surface-layer similarity relationship (Stull 1988):

<table>
<thead>
<tr>
<th>$Re_f$</th>
<th>$Ri_{0,0}$</th>
<th>$\frac{Re}{U_0}$</th>
<th>$\delta_t = \frac{\nu}{\delta T}$ (m)</th>
<th>Grid</th>
<th>$\Delta x^+, \Delta y^+, \Delta z^+$</th>
<th>$Re_{\delta_1}$</th>
<th>$Re_{\tau} = \frac{\delta_{l, \tau}}{\nu}$</th>
<th>$\delta_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>0</td>
<td>0.0650</td>
<td>5.147</td>
<td>128$^2 \times 512$</td>
<td>5.28$^2 \times 1.21$</td>
<td>5200</td>
<td>338</td>
<td>0</td>
</tr>
<tr>
<td>0.001</td>
<td>0.0636</td>
<td>5.036</td>
<td>5.17$^2 \times 1.18$</td>
<td>5087</td>
<td>330.7</td>
<td>0.052</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.002</td>
<td>0.0610</td>
<td>4.830</td>
<td>4.96$^2 \times 1.13$</td>
<td>4880</td>
<td>317.2</td>
<td>0.104</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>0.0601</td>
<td>4.759</td>
<td>4.88$^2 \times 1.11$</td>
<td>4808</td>
<td>312.5</td>
<td>0.261</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.010</td>
<td>0.0595</td>
<td>4.711</td>
<td>4.83$^2 \times 1.10$</td>
<td>4759</td>
<td>309.4</td>
<td>0.523</td>
<td></td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>0</td>
<td>0.0595</td>
<td>7.067</td>
<td>256$^2 \times 864$</td>
<td>5.02$^2 \times 1.03$</td>
<td>10,710</td>
<td>637</td>
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<td>609.8</td>
<td>1.934</td>
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Table 1. Simulation parameters: $^\dagger$ for highest stability cases where intermittency occurs, the values of the friction velocity $u_*$ (and hence the corresponding $\delta_t$, wall units, and $Re_{\tau}$) are not very meaningful due to the large amplitude oscillatory behavior with periods comparable to the simulation time (larger than averaging time); see figures: 8c and 9c.
Figure 2. Mean velocity profiles for neutral cases at different Reynolds numbers, in comparison with other numerical studies (Shingai & Kawamura (2004) and Morris et al. (2011)) and experimental observations (Caldwell et al. (1972)) at comparable Reynolds number ($Re_L = 1159$).

Figure 3. Kármán measure for the neutral case with $Re_t = 637$ ($Re_f = 600$) and 1190 ($Re_f = 900$) in comparison with channel flow at $Re_t = 2003$ (Hoyas & Jiménez 2006; Spalart et al. 2009). Typical range of values of von Kármán constant $\kappa$ reported in the literature (0.38-0.42) is indicated by the two black dotted lines.

\[
\kappa z = \left[ \left( \frac{\partial \langle u \rangle}{\partial z} \right)^2 + \left( \frac{\partial \langle v \rangle}{\partial z} \right)^2 \right]^{1/2}, \tag{4.1}
\]

The exact value is not exactly known, but experiments in laboratories, in the atmosphere, and direct numerical simulations indicate values from 0.38 to 0.42 (Miyashita et al. 2006; Spalart et al. 2008, 2009; Marlatt et al. 2011). The above equation can be further written in inner coordinates as:

\[
\frac{1}{\kappa} = z^+ \left[ \left( \frac{\partial (u^+)}{\partial z^+} \right)^2 + \left( \frac{\partial (v^+)}{\partial z^+} \right)^2 \right]^{1/2}, \tag{4.2}
\]

where $u^+ = u/u_*$ and $z^+ = z u_*/\nu$.

The values we obtain for the simulations with $Re_f = 600$ and $Re_f = 900$ are in the same range as reported by others for more than 100 wall units (the log-layer in the $Re_f = 400$ simulation is too shallow to compute a value).

4.2. Velocity and temperature spectra

In figure 4 a-d, we plot one dimensional (streamwise) normalized premultiplied spectra of each of the three velocity components and temperature against normalized wavenumber at different levels of stability (at $Re_f = 900$ and $Ri_0,0 = 0.001, 0.002, 0.005$) defined by the parameter $z/L$, where

\[
L = - \frac{u_*^3 \theta_0}{\kappa g H_{k,s}}, \tag{4.3}
\]

is Obukhov length. Here $\theta_0$ is the surface temperature and $H_{k,s} = -k d(\theta)/dz|_0$ is the surface kinematic heat flux. The spectra are normalized using Monin-Obukhov parameters namely: $u_*$, $\theta_*$ and $z$, where $\theta_* = -H_{k,s}/u_*$ is a surface temperature scale (Kaimal...
et al. 1972; Wyngaard & Coté 1972; Chung & Matheou 2012). In the inertial range, dimensional arguments (see for example Kaimal et al. 1972) lead to the following models for these spectra (we write only the equations for \( u \) and \( \theta \) spectra, the equations for the spectra of \( v \) and \( w \) are similar to \( u \) but with an additional 4/3 factor on the right hand side):

\[
\frac{k_x E_u(k_x)}{u^2_*} = \frac{\beta_1}{k^{2/3}} \phi^{2/3}_e (k_x z)^{-2/3},
\]

\[
\frac{k_x E_\theta(k_x)}{\theta^2_*} = \frac{\beta_2}{k^{2/3}} \phi^{-1/3}_e \phi_{\chi\theta} (k_x z)^{-2/3},
\]

where \( \beta_1 \approx 0.5 \) and \( \beta_2 \approx 0.8 \) (Kaimal et al. 1972), and \( \phi_e = \epsilon k z / u^3_* \) and \( \phi_{\chi\theta} = \chi_{\theta\theta} k z / (u_* \theta^2_*) \) are the non-dimensionalized dissipation rates for turbulence kinetic energy and \( \langle \theta' \theta' \rangle / 2 \) respectively (\( \chi_{\theta\theta} = \epsilon \theta^2 / 2 \), where \( \epsilon \theta^2 \) is the molecular dissipation of temperature variance). The advantages of this non-dimensionalization is that, regardless of the level of static stability imposed on the flow, the spectra collapse in the inertial sub-range (if it exists i.e. the Reynolds number is high enough) on to a \(-2/3\) power law. One can then examine the energy content of the larger scales as a function of \( z/L \). Note that the inertial subrange in the \( v \) and \( w \)-spectra has a higher energy content than that in \( u \)-spectra by a factor of 4/3.
In neutral and near-neutral cases \((z/L \approx 0)\), there is a small (given the Reynolds number, the extent of the subrange is limited as one would expect) inertial range indicated by the segments of the spectra with a slope of \(-2/3\). Higher Reynolds number simulations will be useful to further confirm the inertial subrange. At the lower end of the wave numbers, the spectral lines for different values of \(z/L\) bifurcate as has been observed by Kaimal \textit{et al.} (1972). The overall behavior at all the scales agrees quite well with the generally observed spectra in atmospheric boundary layers. Notice also from the plots in figure 1 that the energy content of the large scales at different heights drops with increasing stability, indicating that the buoyancy damping effect is mostly acting on the larger scales (in agreement with LES results reported in Huang & Bou-Zeid 2013). This reduces the span of scales that fall in the inertial subrange (see further discussion on this in appendix A). The amount of total energy at a given height decreases with stability for large heights (e.g. \(z/\delta_t = 0.652 \text{ and } 1.01\); here \(\delta_t\) is neutral case boundary layer height), again underlining the important effect of buoyancy on the total turbulence kinetic energy content away from the wall.

5. Results

5.1. Mean profiles with stability

Log-linear plots similar to the one in figure 2 for \(Re_f\) of 600 and 900, but with positive Richardson numbers, are compared to the neutral case in figures (5a) and (5c). A widely-used correction to the log-law in stable cases was given by Monin & Obukhov (1954). The Monin Obukhov similarity theory (MOST), which theoretically holds for infinite Reynolds number only, suggests that the non-dimensional velocity and temperature gradients \((\phi_m\text{ and } \phi_h\text{ respectively})\) in the log-region for stable cases are functions of the height normalized by the Obukhov length \(L\); these functions have been experimentally shown to be linear in \(z/L\) (for \(z/L\) less than order of one) in stable conditions (Brutsaert (1982), Stull (1988), see also theoretical justification in Li \textit{et al.} (2012)):

\[
\phi_m\left(\frac{z}{L}\right) = \kappa \frac{z}{u_*} \left[ \left(\frac{\partial \langle u \rangle}{\partial z}\right)^2 + \left(\frac{\partial \langle v \rangle}{\partial z}\right)^2 \right] = 1 + \beta_m \frac{z}{L}, \tag{5.1}
\]

\[
\phi_h\left(\frac{z}{L}\right) = \kappa \frac{z}{\theta_*} \frac{\partial \langle \theta \rangle}{\partial z} = \alpha_h + \beta_h \frac{z}{L}, \tag{5.2}
\]

where \(\beta_m\), \(\alpha_h\) and \(\beta_h\) are empirical constants. Physically, \(z/L\) is an approximation of the ratio of buoyant destruction or production of TKE to TKE production by shear, while \(L\) indicates the height at which buoyancy destruction/production is of the same order as TKE production by shear. The magnitude of \(L\) is infinite under neutral conditions, where \(H_{k,s} = 0\), and the non-dimensional velocity gradient \(\phi_m\) reverts to its law-of-the-wall value of 1.

The behavior of the mean velocity gradients in the log-layer for statically stable cases follows MOST despite the finite and relatively moderate \(Re\), as can be seen from figures 5b and 5d. The value of \(\beta_m\) at the Reynolds numbers simulated here (about 9), however, is higher than typical values observed in atmospheric flows with higher Reynolds numbers which are about 5 for \(z/L < 1\) (Webb 1970; Dyer 1974; Kondo \textit{et al.} 1978; Stull 1988). This suggest that these MOST coefficients are Reynolds numbers dependent, i.e. viscosity has an effect on \(\beta_m\). This effect was in fact analyzed by Chung & Matheou (2012). By considering a stable regime where shear stress is mainly carried by eddies of sizes ranging
between the Kolmogorov scale ($\eta$) and $L$, they get an expression of the following form:

$$1 \approx \frac{3}{4} \frac{\gamma_1}{(2\pi\kappa)^{4/3}} \beta_m (\beta_m - 1)^{1/3} \left(1 - \frac{\eta}{L}\right)^{4/3},$$

(5.3)

where $\gamma_1 \approx 0.15$ is the constant in the u-w cospectra model for the inertial range (Lumley 1967; Wyngaard & Coté 1972; Chung & Matheou 2012). The expression above illustrates that the value of $\beta_m$ has to increase as the term $1 - (\eta/L)^{4/3}$ decreases. But since $L/\eta$ is a ratio of the largest scale $L$ to the smallest scale $\eta$, it is a Reynolds number. As such, as this Reynolds number decreases, $1 - (\eta/L)^{4/3}$ also decreases and $\beta_m$ increases. This decrease in $\beta_m$ leads to a reduction in critical flux Richardson number ($Ri_f = z/L\phi_m$), Chung & Matheou (2012) at lower Reynolds numbers, which is indeed what we observed in our simulations as noted above. Similar arguments can be applied to conclude that $\beta_h$ also increases when the Reynolds number ($L/\eta$) is reduced.
Figures 6a to 6c depict the separate variations in the mean profiles of $u$ and $v$ as
stability increases for $Re_f = 400, 600$ and $900$, respectively, while Figure 6d shows the variation of these velocity components with time during one inertial period $2\pi/f$ (= 12 hours) for the case $Re_f = 600$ at $Ri_{0,0} = 0.002$. Increasing stability changes the Ekman rotation angle at a given height. It also reduces flow velocities very close to the wall (see inset in Figure 6c), while increasing them away from the wall (see Figure 5). This is consistent with the reduced mixing and momentum transport from the upper levels of the boundary layer, where the flow becomes faster, to the near-wall region, where the flow becomes slower, as stability increases. The stable profiles are hence more “laminar-like”.

The neutral case turbulent Ekman layer length scale can be defined as $\delta_t = u_*/f$. As illustrated in table 1, the friction velocity $u_*$ and correspondingly the Ekman layer depth increase with increasing $Re$, and decrease with increasing $Ri$. As the Obukhov length and vertical mixing decrease for the more stable simulations, a shallower boundary layer is produced (a common feature of stable atmospheric boundary layers: Kaimal & Finnigan (1994)). The results obtained here are consistent with this trend. A good indicator of the boundary layer depth is the height at which there is a maximum in the velocity profile; this depth decreases as shown in Figure 6. However, the magnitudes of the maximum velocities are higher for stronger stabilities. The bands of “super-geostrophic” velocities
Figure 8. \( Re_f = 400 \). (a) Surface Richardson number \( (R_i_{0,t}) \) versus non-dimensional time \( (t_f) \) for different stabilities. Cross marks in the figure indicate the start and end times of averaging period used for calculating means and the averages of budgets. (b) Non-dimensional volume integrated turbulence kinetic energy per unit area \( (E) \). (c) Friction velocity \( (u_*) \) and its variation with time and stability. (d) Variation of the angle \( (\beta^\circ) \) between the geostrophic wind direction and the surface shear stress direction with time and stability.

around the peaks observed in Figures 5 and 6, where the flow speed in the boundary layer can be higher than the geostrophic value far from the wall, are also a common feature seen in stable atmospheric boundary layers (Stull 1988).

Comparison of our results with plots from Taylor & Sarkar (2008) is also shown in Figure 7. Taylor & Sarkar (2008) carried of LES to study stratification effects in the bottom Ekman layer in the ocean (Zikanov et al. (2003) also used LES to study wind-driven surface Ekman layers in the ocean but without stratification). The Reynolds numbers they simulated hence are much larger (two orders of magnitude larger) than the ones in our DNS. Despite the differences in Reynolds numbers, the profiles for neutral case match almost perfectly away from the viscous sublayer. With stratification the trends obtained in moderate Reynolds number flow are also very similar to the high Reynolds number flows simulated by Taylor & Sarkar (2008). This shows that much can be deduced by carrying out DNS of relatively smaller \( Re \). Note that the physics of the flows simulated by Taylor & Sarkar (2008) and us are not exactly same.

5.2. Evolution of turbulence and stability

Figures 8a, 9a and 10a show the time evolution of the surface Richardson number \( (R_i_{0,t}) = g\Gamma_{0,t} D^2/\theta_0 G^2 \) at various stabilities, indicated in the legend by bulk Richardson number \( R_i_b = g\delta_t \Delta \theta/\theta_0 G^2 \), where \( \Delta \theta = \theta_\infty - \theta_0 \). \( R_i_b \) remains almost constant throughout
Figure 9. $Re_f = 600$. (a) Surface Richardson number ($Ri_{0,t}$) versus non-dimensional time ($tf$) for different stabilities. Cross marks in the figure indicate the start and end times of averaging period used for calculating means and the averages of budgets. (b) Non-dimensional volume integrated turbulence kinetic energy per unit area ($E/DG$). (c) Friction velocity ($u^*$) and its variation with time and stability. (d) Variation of the angle ($\beta^\circ$) between the geostrophic wind direction and the surface shear stress direction with time and stability.

Note that the cross-marks on the plots show the period over which the statistics were averaged, which spans 12 hours. Coleman et al. (1992) argue that the first $tf = 1$ is enough for the development of temperature fluctuations in the field, which would be in “equilibrium” with the velocity field. However, in this analysis, the time before the first cross mark at $tf = \pi$ is not used for averaging since during this period the changes in turbulence statistics due to stability seem to remain significant in our simulations. The time period after $tf = \pi$ has a more or less constant surface Richardson number, confirming that the flow is in quasi-equilibrium, and justifying the averaging we perform over this half inertial time period. For the stronger stability simulations at each value of $Re$, intermittency is observed whereby turbulence is damped and then regenerated as discussed previously. This behavior continues when the simulations are run longer and is an important physical phenomenon in stable boundary layers that we will analyze separately in the future. The statistics accumulated over the half inertial period for these intermittent cases are not very informative; other analysis approaches are needed. From the plots, it can be seen that the gradient at the surface initially increases since the initial temperature profiles are not in “equilibrium” with the turbulent velocity fields; therefore, the gradients at the surface need time to adjust. Subsequently, $Ri$ falls back and stabilizes.

In figures 8b, 9b and 10b, we plot the variation of volume-integrated turbulence kinetic
energy per unit area \( (E) \), non-dimensionalized by the laminar Ekman layer depth \( (D) \) and the square of the geostrophic wind velocity \( (G^2) \), versus time, for various stabilities. The volume-integrated turbulence kinetic energy (per unit area) averaged over a horizontal plane is defined as:

\[
E = \frac{1}{2} \int_{z_{\text{top}}}^{0} \left\langle u'_i u'_j \right\rangle_{x,y} \, dz.
\]  

(5.4)

Here \( \left\langle \right\rangle_{x,y} \) indicates averaging over \( xy \)-planes, and \( z_{\text{top}} \) indicates the top of the domain. The turbulence kinetic energy drops down with increasing stability (the mechanisms behind this drop are discussed later in the paper), with most of the drop occurring when the flow transitions from neutral to weakly stable. For Reynolds numbers of 600 and 900, the total TKE in the domain for the least stable cases is about half of that for the neutral simulation. For the highest Richardson numbers imposed at every value of the Reynolds number, the TKE in the domain drops almost to zero, but with strong intermittent bursts. The bulk Richardson number at which this intermittency sets in varies with \( Re_f \) (recall the discussion of the variation of the critical Richardson number with \( Re \) above). We also observe that the stronger the imposed stability, the faster the rate at which TKE drops at initial times. Note that all the simulations have the same initial state (fully turbulent neutral Ekman layer), and hence this comparison is meaningful.
Figures (8c), (9c) and (10c) show the evolution of the friction velocity (non-dimensionalized by the geostrophic wind velocity \( G \)) versus time. Static stability reduces the friction velocity (surface momentum flux). The initial rate at which the friction velocity drops from the neutral-case initial value also increases with stability. At large stabilities, there are bursts in the friction velocity, associated with the turbulent bursts, that increase \( u_* \) to values comparable to those seen in the neutral cases.

Figures 8d, 9d and 10d shows the variation of the angle formed between the geostrophic wind velocity vector and the shear stress vector at the surface versus time, for different stabilities. Note that the angle for the turbulent neutral case is denoted by the by black lines, while the value of the angle at the surface for the laminar limit is 45°. This laminar limit can be obtained by solving the equations of motion with simplifying assumptions like steady state, homogeneity in the horizontal directions, neutral stratification and eddy diffusivity being zero for laminar flows. With increasing stability, as the TKE drops and the flow becomes less turbulent, the angle at the surface increases and approaches the laminar limit of 45° for the laminarizing intermittent cases.

In figure 11 we examine the decay rate of r.m.s. of vertical velocity (\( \sigma_w/u_* \)) and turbulence kinetic energy (\( q/u_*^2 \)) at a height of \( z^+ \approx 15 \). Both r.m.s. of vertical velocity and turbulence kinetic energy when normalized by friction velocity \( u_* \) at various stabilities fall on one curve. Flores & Riley (2011) argue that at initial times, buoyancy flux is the only term damping variance of vertical velocity. With order of magnitude analysis of the vertical velocity variance budget, at short times, they were able conclude that the time for turbulence collapse (\( t_{c1} \)) due to this buoyancy flux scales as \( L/u_* \). Figure 11a shows that the time needed for collapse of r.m.s. of vertical velocity is indeed on the order of \( L/u_* \), consistent with order of magnitude analysis of Flores & Riley (2011). Here we extend the analysis and apply to the full TKE budget equation.

To that end we assume that the terms responsible for drop in TKE in statically stable cases are drop in turbulence shear production and buoyant destruction. Changes in the other terms can be neglected as they are relatively small. Note that viscous dissipation also drops significantly (although with some lag with turbulence shear production; not shown here), but that is due lesser turbulence kinetic energy at that instant. Hence, the viscous dissipation term is not considered here while analyzing the time scale for collapse.
Figure 12. Average angle between shear stress at the surface and wind velocity with height (in inner coordinates), $\langle \gamma \rangle = \tan^{-1}(v/u) - \beta$, in the surface layer (a) $Re_f = 400$, (b) $Re_f = 600$ and (c) $Re_f = 900$ at different stabilities.

Scaling individual terms in the above equation with the assumptions that velocity variance in each direction and turbulent fluxes near the wall (in buffer region) scale with $u_*^2$, we get

$$\frac{3}{2} \frac{u_*^2}{t_c^2} \approx -2u_*^2 \frac{u_s}{\kappa z} + 2u_*^2 \frac{u_s}{\kappa z} (1 + \beta_m \frac{z}{L}) + \frac{u_*^3}{\kappa L} \rightarrow t_c \approx \frac{L}{u_*} \frac{3\kappa}{4\beta_m + 2}$$

Note that we have considered the drop in production to be between neutral case statistically steady state and quasi-steady stable case. These two terms have been scaled using MOST for the order of magnitude analysis (for example $\langle u'v' \rangle \approx u_*^2$ and $\partial U/\partial z \approx u_*/\kappa z$). The above result suggests that TKE in all the cases should again show the same decay rate at initial times. Figure 11b confirms corroborates the analysis.

In figure (12), we also plot the average angle, $\langle \gamma \rangle$, between the surface shear stress and the wind direction with height. In many stable boundary layer theories, it is assumed that near the surface, the turning due to Coriolis force is negligibly small, and hence this angle is zero throughout the surface layer. From the plots, it is clear that even if the effect of Coriolis forcing is small on turbulence (as compared to other terms in the budgets, analyzed later), the angle between surface shear stress and wind direction is not zero in the surface layer. However, it seems that $\langle \gamma \rangle$ does decrease with increasing Reynolds number, and so even higher Reynolds number simulations will be helpful in verifying if $\langle \gamma \rangle$ is indeed negligible in the ABL.

5.3. Profiles of the gradient Richardson number, TKE, and stress

The vertical profiles of the gradient Richardson number are depicted in figure 13. It is defined as:

$$Ri_{gr} = \frac{g}{\theta_\infty} \frac{d(\theta)/dz}{(d(u)/dz)^2 + (d(v)/dz)^2}.$$ (5.5)

As the Richardson number increases, buoyancy destruction of TKE will result in complete laminarization of the flow. The exact value at which this happens, called as the critical
Richardson number ($Ri_c$), is determined by the state of the flow. It has been shown by Miles (1961) and Howard (1961), using linear stability analysis of two-dimensional steady stable shear flows, that a critical gradient Richardson number for such flows can be expected at a value of about 0.25, beyond which turbulence dies down completely due to destruction by buoyancy. Other studies (see Canuto 2002) indicate that the critical Richardson number maybe as large as 1 or beyond, and some argue that a critical limit does not exist at all (Monin & Yaglom 1975; Yamamoto 1975; Lettau 1979; Galperin et al. 2007), depending on the state of the flow. Chung & Matheou (2012) argue in their analysis of stationary homogeneous stratified turbulence that the Richardson number is not sufficient to determine the state of the flow; a measure of the scale separation (given by the Reynolds number) is also needed. This point of view is confirmed by our analysis as discussed above. Away from the critical limit, the gradient Richardson number is a good estimate of the flux Richardson number and hence large values indicate stronger buoyant destruction and/or lower mechanical production of TKE. From the profiles in figure 13, one can see that the buoyancy starts influencing the flow significantly (large $Ri_{gr}$ exceeding some arbitrary threshold, e.g. 0.25) only at larger heights, but these critical heights at which buoyancy becomes important decrease drastically with increasing $Ri_b$.

A very similar trend is observed in the plot with $Re_f = 900$ (not shown here), indicating a decrease in the height at which “critical” Richardson number is reached. Note that the normalization of height (y-axis here) is carried out using turbulent length scale of the neutral simulations. Other scales can be used for normalization (Mironov & Fedorovich (2010) for example found two depth scales for stable boundary layers that are consistent with TKE and momentum budgets), but to emphasize the effect of static stability on turbulence at a given dimensional distance from the wall, we use the fixed neutral case turbulent Ekman layer length scale for normalization.

Figures 14 presents the vertical variation of $2q^2 = \langle u'_i u'_i \rangle$ (twice the turbulence kinetic energy) at various stabilities and at different Reynolds numbers. Increasing $Re$ brings the peak in $2q^2$ closer to the wall as expected, while increasing $Ri$ generally reduces the peak values of $2q^2$, and for the highest stabilities it seems to move the location of these peaks away from the wall. The decrease in $2q^2$ however in non-monotonic. In figure 14a, the total amount of turbulence kinetic energy at $Ri_b = 0.261$ seems to be more than lower stability cases ($Ri_b = 0.052$ and $Ri_b = 0.104$). This seems to be counter-intuitive, but a closer look at the time variation of total turbulence kinetic energy at each stability (figure 8b) shows that the turbulence kinetic energy at $Ri_b = 0.261$ in the averaging period used

![Figure 13. Gradient Richardson number as a function of height, for varying stabilities at a Reynolds number $Re_f = (a) 400$ and (b) 600](image-url)
Figure 14. Profiles of the time-averaged non-dimensional sum of the variances of the three velocity components (i.e. $2q^2$, where $q^2$ is turbulence kinetic energy) at Reynolds number ($Re_f$) of (a) 400 (b) 600 and (c) 900 at various stabilities, compared with the corresponding neutral cases (black solid line in each figure). The bulk Richardson numbers $Ri_b$ given in the legend correspond, from lowest to highest, to the following initial surface Richardson numbers $Ri_{0,0} = 0.001, 0.002, 0.005, 0.10$.

Here is higher due to oscillatory behavior. This suggests that significant turbulence and mixing can be present in stronger stability boundary layers due to this intermittency.

Figure 15 depicts the vertical profiles of magnitude of kinematic stress (or momentum flux) as a function of height, for various Reynolds numbers and stabilities. The magnitude of kinematic turbulent stress $|\tau/\rho|$ is defined as follows:

$$
|\tau/\rho| = \left(\langle u'w' \rangle^2 + \langle v'w' \rangle^2\right)^{1/2}.
$$

(5.6)

More than for the TKE, the effect of stability on the stress/flux profiles is very pronounced. The fluxes are monotonically damped with increasing stability, and the intermittent cases that resulted in higher TKE than lower stability cases do not result in higher stresses. This damping of fluxes by buoyancy will be a critical factor in reducing TKE production as will be illustrated in the next subsection.

5.4. TKE and momentum flux budgets

We have seen from vertical profiles of TKE (figure 14) that buoyancy affects turbulence at all heights, but more so away from the wall where the shear in the velocity field is weaker. It has also been observed in the evolution of TKE with time that, as static stability is increased, the total TKE in the domain decreases (figures 8b, 9b and 10b). To investigate the mechanisms causing this reduction in TKE, we study the TKE budgets for stable cases in comparison with neutral ones at the same Reynolds number. Figure 16 shows TKE budgets for different Reynolds numbers and with varying stabilities. Figure 16a shows the trends of all the terms in the TKE budget equation for the neutral simulations at a Reynolds number $Re_f$ of 600 and 900.

Notice that the peaks of the shear production $P_{ii}$, turbulent transport $T_{ii}$ and viscous diffusion $D_{ii}$ have almost the same value after normalization by the relevant velocity scale near the wall at both Reynolds numbers. However, the height at which those peaks occur is reduced with increasing Reynolds number, as expected. Figures 16b to 16d compare these terms in neutral and stable cases at $Re_f = 400, 600$ and 900, respectively. When stability is imposed, though the overall shapes remain more or less the same, the magni-
Figure 15. Time-averaged non-dimensional Reynolds stress ($|\tau/\rho|$) profiles at $Re_f$ of (a) 400 (b) 600 and (c) 900 at various stabilities compared with neutral cases. The bulk Richardson numbers $Rib$ given in the legend correspond, from lowest to highest, to the following initial surface Richardson numbers $Rib_{0} = 0.001, 0.002, 0.005, 0.010$.

The buoyancy destruction term in the TKE budgets is not shown in the TKE budget plots, but is plotted separately in Figure 17. The term is very small in comparison to other terms in the TKE budget, but it is not zero. In fact one can clearly observe the increase in its magnitude with increasing stability. This implies that the main direct impact of stability on the flow is not through TKE destruction, but rather through another mechanism: from the plots in Figure 16 one can see that a drastic drop in the shear production of TKE is induced by stability. Hence, it is the inhibition of shear production that seems to be the most drastic impact of stability on the TKE budget. Since in a stationary turbulence state with small direct buoyant destruction, domain-averaged TKE dissipation has to be equal to TKE production, with buoyancy reducing production, dissipation is also significantly reduced. Viscous diffusion terms near the wall and turbulent transport, which have smaller contributions in the TKE budget, are also decreased. The TKE budget plots for high stability cases that display oscillation in the domain integrated TKE are not shown here since the averaging carried out for the budget terms from $t_f = \pi$ to $t_f = 2\pi$ (i.e. 6 - 12 hours, which included only about one cycle of oscillations for these flows) would not be very representative of the mean state of these flows.

To elucidate the mechanism and causes of the drop in shear TKE production further, we investigate the quantities involved in this term, namely the gradients and fluxes. The normalized fluxes $\langle u'w' \rangle$ and $\langle v'w' \rangle$ and the gradients $\partial \langle u \rangle / \partial z, \partial \langle v \rangle / \partial z$ which enter into the two dominant terms of the TKE shear production:

$$\text{TKE Production} \approx -\langle u'w' \rangle \frac{\partial \langle u \rangle}{\partial z} - \langle v'w' \rangle \frac{\partial \langle v \rangle}{\partial z},$$

(5.7)
Figure 16. Turbulence kinetic energy ($q^2 = \frac{1}{2}\langle u'_i u'_i \rangle$) budget: (a) neutral case comparison for $Re_f = 600$ (dotted-dashed lines) and $Re_f = 900$ (solid lines); and comparison of neutral case with stable cases at Reynolds numbers $Re_f$ of (b) 400 (c) 600 and (d) 900. The different colors denote the different terms of the budget: $rac{\partial q}{\partial t} + U_k \frac{\partial q}{\partial x_k} = \frac{1}{2} P_{ii} + \frac{1}{2} T_{ii} + \frac{1}{2} \Pi_{ii} + \frac{1}{2} R_{ii} + \frac{1}{2} C_{ii} + \frac{1}{2} D_{ii} + \frac{1}{2} \epsilon_{ii}$, where turbulent transport $T_{ii} = -\partial \langle u'_i w'_i \rangle / \partial z$, shear production $P_{ii} = -\langle u'_i w'_i \rangle \partial U_i / \partial z$, viscous diffusion $D_{ii} = \nu \partial^2 \langle u'_i u'_i \rangle / \partial z^2$, and viscous dissipation rate $\epsilon_{ii} = -2\nu (\partial u'_i / \partial x_k) (\partial u'_i / \partial x_k)$. All the other terms which amount to very small values are not plotted here, but the expressions for them can be found in the appendix B at the end of the article. Expressions for individual terms have been written in a simplified by eliminating derivatives in directions of homogeneity. In (a) we plot the residual $R$ that should be zero for validation; all terms are normalized by $u^4_{i,neutral} / \nu$. Gray line at the center of each subplot indicates the zero-gain/loss line.

are plotted in figure 18 for $Re_f = 900$. The effect of stability on velocity gradients near the wall is significant, while away from the wall, this effect is reduced. In general, gradients are reduced or increased by a factor of about 2, with higher gradients for the more stable conditions at heights where the TKE production peaks. The effects of stability on vertical momentum fluxes (i.e. magnitude of Reynolds stresses) however is much more significant. The fluxes are reduced due to stability throughout the boundary layer compared to the neutral case, almost by one order of magnitude for the most stable simulation. This indicates that the reduction due to stability in the vertical transport of momentum from the higher elevations in the boundary layer into the near-wall region is the main mechanism for the reduction in shear production of TKE in these stable wall-bounded flow. A side note related to figure 18 is that the normalization used also makes the gradients plotted in part (b) equivalent to viscous stresses, normalized by $G^2$, while part (a) depicts the turbulent stresses also normalized by $G^2$. Hence from this figure one can compare the viscous and turbulent stresses and observe the increase in the relative
Figure 17. Profiles of buoyant destruction $B_{ii} = g\langle w'\theta' \rangle/\theta_\infty$ normalized by viscosity and neutral case friction velocity, same non-dimensionalization used in the TKE budget, at (a) $Re_f = 600$ and (b) $Re_f = 900$

Figure 18. (a) Magnitude of turbulent stress $\tau = (\langle u'w' \rangle^2 + \langle v'w' \rangle^2)^{1/2}$ and (b) magnitude of viscous stress $\nu |\langle du/dz \rangle|$ (where $|\langle du/dz \rangle| = ((du/dz)^2 + (dv/dz)^2)^{1/2}$) at $Re_f = 900$.

importance of the turbulent part as the distance to the wall increases, and as stability decreases.

To further understand the reduction of the momentum fluxes by buoyancy, we study the momentum flux budgets (for $\langle u'w' \rangle$ and $\langle v'w' \rangle$; $\langle v'w' \rangle$ budget is not shown in here, but the behavior is similar to $\langle u'w' \rangle$ budget). Figure 19a shows all the terms of the flux budget for the neutral case with $Re_f = 900$. Details of the equations for the flux budgets and all the terms in it can be found in the appendix B. For clarity, only the significant terms have been plotted in Figure 19b-d, where the effects of stability are illustrated. As seen in the TKE budget plots, the peaks in vertical profiles shift towards the wall with increasing Reynolds numbers. The magnitudes of the peak values also increase with Reynolds number. The figure illustrates that the pressure correlation and return to isotropy terms, which have opposite signs are both reduced in magnitude as stability increases; the changes in these terms for stronger stabilities partially offset each other. The flux production (which like the TKE shear production depends on the fluxes themselves) and the turbulent transport terms are also significantly reduced by buoyancy. Given that the highest gradients are in the vertical direction, the dominant
components of these terms will be \(-\partial\langle u'w'w'\rangle/\partial z\) in the transport and \(-\langle w'w'\rangle\partial U/\partial z\) in the production, both of which are strongly dependent on \(w'\). Therefore, damping of turbulent transport of momentum, which, as we showed, leads to a reduction in TKE, seems to be a result of the damping of \(\langle w'w'\rangle\), which is a component of the TKE. This circular dependence thus needs further elucidation. Note that the buoyancy sink term \(B_{13}\) increases with stability for all the Reynolds numbers. This re-enforces the mechanism of dampening of vertical motions resulting in loss of production and transport of fluxes, which in turn reduces production of TKE.

To clarify the sequence of effects stability induces in a turbulent boundary layer, we now plot the budget of the vertical velocity variance (figure 20). The plot in figure 20a compares the budget of this variance in the neutral case to the case with stability at \(Ri_{0,0} = 0.002\) for Reynolds number \(Re_f = 600\), while figure 20b gives the same comparison at \(Re_f = 900\) and \(Ri_{0,0} = 0.005\). It is observed from the plot that the change
in the buoyancy destruction here is on the same order as the change in the other terms. This is thus the only budget where the direct effect of buoyancy, via a destruction term, is significant. This implies that buoyancy first affects motions in the vertical direction, damping \( w' \), which then causes a reduction of momentum fluxes (\( \langle u'w' \rangle \) and \( \langle v'w' \rangle \)), which then leads to lower TKE production, particularly of the \( \langle u'^2 \rangle \) and \( \langle v'^2 \rangle \) components of the TKE (the shear production of \( \langle u'^2 \rangle \) is zero since it involves gradients of the mean vertical velocity). The reduction in \( \langle u'^2 \rangle \) and \( \langle v'^2 \rangle \) then reduces the return-to-isotropy or pressure-strain redistribution term as depicted in Figure 20, further reducing \( \langle u'^2 \rangle \). Figure 21 shows the profiles of the horizontal velocity variance (a) and the vertical velocity variance (b), illustrating that the reduction in \( \langle w'^2 \rangle \) relative to the neutral case, which is associated with the damping of the vertical motions, is larger than the relative reduction in \( \langle u'^2 \rangle \) and \( \langle v'^2 \rangle \), as can be anticipated. A particularly important implication of this finding is that even a small Richardson number associated with a small buoyant TKE destruction compared to shear production can have a significant impact on turbulence in wall-bounded flows: this small TKE destruction will primarily reduce the vertical variance, which then drastically reduces production compared to a neutral flow. This might explain the very rapid drop in TKE levels shortly after the evening transition in the ABL (Nadeau et al. 2011).

5.5. Buoyancy flux and temperature variance budgets

Heat flux budgets for all the stable cases are shown in Figure 22. The shear production term of heat flux given by \( P^2 = -\langle \theta'w' \rangle \partial W/\partial x_j \) is identically zero for the flows simulated here. The main source of covariance or flux (note that production here is negative since the heat flux is negative/downward, in a statically unstable boundary layer it would be
Figure 21. Reduction in the sum of the horizontal velocities variances compared (a) compared to the reduction in the vertical velocity variance (b) with increasing static stability at \(Re_f = 900\): black solid line corresponds to neutral case and dashed lines corresponds to stable cases. Blue dashed line: \(Rib = 0.193\) and red dash-dotted line \(Rib = 0.387\).

Positive) is the temperature gradient production term. The other two significant terms are the pressure terms, while other terms (namely the Coriolis term \(C\), dissipation \(\epsilon_{23}\), viscous diffusion \(D_{23}\) and turbulent transport \(T_{23}\)) are quite small for neutral simulations. For stable simulations, the reduction in the temperature gradient production term drives the buoyancy effect on this budget. However, the buoyancy destruction term becomes important at stronger stabilities, especially for the \(Re_f = 900\) simulation. Note here that for increasing stabilities, the vertical temperature gradient is larger in magnitude, and yet, the gradient production term is smaller. This results from the significant drop in the magnitudes of the variance of the \(w\)-component of velocity with stability, which was illustrated in the previous subsection. Expressions for all the terms involved in the heat flux budget can be found in the appendix B.

We also plot the budget of perturbation temperature variance \((\langle \theta' \theta' \rangle)\) in Figure 23. Similar to TKE budget plots, from these plots one can infer that the main terms being modified by stability are the variance production terms. Variance dissipation falls with decreasing production. Turbulent transport and molecular diffusion away from the wall are small, while near the wall they are very weakly affected by buoyancy.

5.6. Turbulent diffusivities under stable conditions

Most stable flows of practical engineering or geophysical relevance are at much higher Reynolds numbers than the one we attain here, and many can only be simulated using the Reynolds-averaged Navier-Stokes (RANS) approach (e.g. stable ABL in weather and climate simulations). Thus, an important follow-up on the investigation of turbulence dynamics and budgets that we performed in the previous sections is to relate the findings to the development of better turbulence closure models for stable conditions and the comparison to existing models for very high \(Re\) flows.

Figure 24 shows the vertical profiles of eddy viscosity \((\nu_T)\) and eddy diffusivity \((\alpha_T)\) for Reynolds numbers of 400, 600 and 900, non-dimensionalized by molecular viscosity.
and diffusivity. Eddy viscosity (Coleman et al. 1992) and diffusivity are computed from:

$$\nu_T = -\frac{\langle u'w' \rangle d\langle u \rangle / dz - \langle v'w' \rangle d\langle v \rangle / dz}{(d\langle u \rangle / dz)^2 + (d\langle v \rangle / dz)^2}, \tag{5.8}$$

$$\alpha_T = -\frac{\langle u'\theta' \rangle}{d\langle \theta \rangle / dz}. \tag{5.9}$$

For the neutral case, the turbulent viscosity is lower near the wall, but dominates away from the wall as expected. For cases with stability, the turbulent components of viscosity and diffusivity are reduced significantly away from the wall. For the most stable cases, the turbulent contribution to the fluxes is significantly smaller than the viscous one.

In the figure 25, we plot eddy viscosities evaluated separately for x and y directions ($\nu_{Tx} = -(u'w')/(d\langle u \rangle / dz)$ and $\nu_{Ty} = -(v'w')/(d\langle v \rangle / dz)$) to show that the assumption of eddy viscosity being isotropic (also known as the Boussinesq analogy) is not really valid.
Figure 23. Temperature variance \(\langle \theta' \theta' \rangle\) budget at (a) \(Re_f = 900\) and \(Ri_{0,0} = 0.001\) and comparison of all the significant terms from the stable cases at Reynolds numbers \(Re_f\) of (b) 400 (c) 600 and (d) 900 and at various Richardson numbers. The different colors denote the different terms of the budget: \(\frac{\partial \langle \theta' \theta' \rangle}{\partial t} + \frac{\partial \langle \theta' \theta' \rangle}{\partial x_j} U_j = P_\theta + T_\theta + D_\theta + \chi_\theta\), where production: \(P_\theta = -2\langle w' \theta' \rangle \partial \langle \theta \rangle / \partial z\), Turbulent transport: \(T_\theta = -\partial \langle w' \theta' \theta' \rangle / \partial z\), Molecular diffusion: \(D_\theta = \nu \partial^2 \langle \theta' \theta' \rangle / \partial z^2\), Molecular dissipation: \(\epsilon_\theta = -2\nu \langle (\partial \theta' / \partial x_j)(\partial \theta' / \partial x_j) \rangle\). All terms are normalized by \(u^2_\tau \theta_\infty - \theta_0)^2 / (\nu / Pr)\). Gray line at the center of each subplot indicates the zero-gain/loss line. Expressions for individual terms have been written in a simplified by eliminating derivatives in directions of homogeneity. Full budget equations can be found in the appendix B at the end of the article.

Note that the results are plotted only for values of \(z/L_z\) less that 0.2 and where \(d\langle v \rangle / dz\) is not close to zero. From the plots, we conclude that eddy viscosity in the y-direction is consistently smaller than that in the x-direction.

The ratio of \(\nu_T\) over \(\alpha_T\) gives turbulent the Prandtl number \(Pr_T = \nu_T / \alpha_T\), which is plotted in figure 26 for all the Reynolds numbers simulated here, at Richardson numbers of \(Ri_{0,0} = 0.001\) and 0.002 with height (subplots (a), (c) and (e)) and gradient Richardson number \(Ri_{gr}\) (subplots (b), (d) and (f)). The turbulent Prandtl number increases, but very slightly, with increasing stability \(Ri_{0,0}\), approaching unity in the cases we have simulated. This indicates that buoyancy affects turbulent transfer of heat and momentum in the same way in statically-stable flows. This has also been observed by Coleman et al. (1992). However, for higher Reynolds number, several studies suggest that the turbulent Prandtl increases more significantly with increasing stability (Bou-Zeid et al. 2010; Venayagamoorthy & Stretch 2010; Huang & Bou-Zeid 2013). Despite this divergence on the effect of stability on the similarity of momentum and scalar turbulent transport, a robust conclusion from all those studies is that the Reynolds analogy holds much better
Figure 24. Profiles of eddy viscosity (plots (a), (c), and (e)) and eddy diffusivity (plots (b), (d), and (f)) non-dimensionalized by their molecular counterparts under different stabilities for \(Re_f = 400\) ((a) and (b)), \(600\) ((c) and (d)), and \(900\) ((e) and (f)).

Figure 25. Profiles of eddy viscosity for \(x\) and \(y\) directions calculate separately (\(\nu_{Tx} = -\langle u'w' \rangle / (d\langle u \rangle / dz)\) and \(\nu_{Ty} = -\langle v'w' \rangle / (d\langle v \rangle / dz)\)) non-dimensionalized by their molecular counterparts under different stabilities for (a) 600 and (b) 900

in stable wall-bounded flows than in unstable (convective) ones (Li et al. 2012; Katul et al. 2013).

Figures 27 a, c, and e depict the ratio of the magnitude of Reynolds stress to the square of the vertical velocity variance (which is defined as \(b_1\) in the caption of the figure), for various stabilities and Reynolds numbers, as a function of height; figures 27 b, d, and
Figure 26. Turbulent Prandtl number ($Pr_T$) variation with height and gradient Richardson number ($Ri_{gr}$) at Reynolds numbers (a) & (b) 400, (c) & (d) 600 and (e) & (f) 900. 

f depict the ratio of the Reynolds stress to twice the TKE (defined as $a_1$ in the caption of the figure), again for various stabilities and Reynolds numbers, as a function of height. These quantities represent measures of turbulent transport efficiencies in the stable boundary layer. One can observe that these fluxes, when non-dimensionalized by the vertical velocity variance, almost perfectly collapse to one curve with minor spreads near the wall and at the top of the boundary layer. However, when non-dimensionalized with TKE, the lines do not collapse and this measure of efficiency drops down significantly with increasing stability. These results confirm that the variance of the vertical velocity is the primary component of the TKE that accounts for momentum fluxes, which is not surprising at all. The ratio $b_1$ is not very sensitive to stability since both the numerator and denominator decrease proportionally. On the other hand, horizontal variances are not reduced significantly by stability (thus the preponderance of pancake-like structures in the stable ABL: Beckers et al. (2001); Riley & deBruynKops (2003); Davidson (2004)) and hence the TKE decreases with stability at a lower rate than fluxes. This suggests that first-order or 1.5-order closure models should solve the budget equation of the vertical velocity variance or use that variance as the basis for formulating vertical eddy diffusivities in stable wall-bounded flows. Currently, most existing formulations rely on the full TKE as the basis for eddy-viscosity and diffusivity models.

Another common hypothesis invoked by eddy-viscosity models is that the fluxes and gradients are aligned. From the results obtained with the simulations carried out here (Figure 28), one can note that: (1) the fluxes and gradients are misaligned for all stabilities, (2) the misalignment however is significantly reduced as the Reynolds number is
Figure 27. Ratio of the Reynolds stress to vertical velocity variance, $b_1$, at $Re_f$ of (a) 400 and (c) 600 and (e) 900, respectively, and ratio of Reynolds stress magnitude to twice the turbulence kinetic energy, $a_1$, at $Re_f$ of (a) 400 and (c) 600 and (e) 900, respectively, as a function of height and bulk Richardson number. $b_1 = \frac{(\langle u'w'\rangle^2 + \langle v'w'\rangle^2)^{1/2}}{\sigma_w^2}$ and $a_1 = \frac{(\langle u'u'\rangle^2 + \langle v'v'\rangle^2)^{1/2}}{2q^2}$ where $2q^2 = \langle u'u_iu'_i \rangle$ and $\sigma_w^2 = \langle w'w' \rangle$.

Increased from $Re_f = 600$ to $Re_f = 900$, and (3) stability slightly increases the misalignment. A similar analysis based on LES (thus at much higher Reynolds number) indicated that the angle is much smaller and insensitive to stability (Huang et al. 2013). The sharp reduction in the angle in our analysis as the Reynolds number increases couple with these other results indicates that this alignment assumption is highly sensitive to the Reynolds number and less sensitive to the Richardson number. For the high $Re_f$ geophysical flows, it seems to be a reasonable hypothesis.

6. Conclusions

Direct Numerical Simulations (DNS) of neutral and statically stable turbulent Ekman layers were performed at three different moderate Reynolds numbers. For stable cases, a constant surface temperature is imposed at the wall, which results in a time-dependent heat flux across the boundary layer; however, analyses are based on periods during which a quasi-equilibrium state has been reached.

The simulations are validated by plotting mean profiles of velocity, turbulence kinetic energy, turbulent kinematic stress and spectra of temperature and of each of the three components of velocity. Non-dimensionalization of spectra using Monin-Obukhov parameters results in collapse of the inertial subrange spectra (pre-multiplied by $k_x$, where $k_x$
is the wavenumber) on to the $-2/3$ power-law. Spectral characteristics at lower wave numbers are very similar to those seen in the atmosphere under varying stabilities. The mean profiles are also found to follow Monin-Obukhov similarity theory (MOST) for the two highest Reynolds numbers simulated here (the lowest one displayed no log-layer even under neutral conditions). However, the value of the empirical constant $\beta_m$ in the non-dimensional gradients that was obtained at $Re_f = 600$ and $Re_f = 900$ is approximately equal to 9, suggesting that at low to moderate Reynolds numbers viscosity effects increase $\beta_m$, as has been argued by Chung & Matheou (2012). Mean profiles of velocity show that stable cases are more “laminar-like”. A maximum at a certain height away from the wall is seen (super-geostrophic velocities) and the maximum velocities are higher for stronger stabilities. Friction velocity and hence the turbulent Ekman boundary layer height ($\delta_t = u_*/f$) increase with increasing Reynolds number, and decrease with increasing Richardson number.

The total TKE levels in the domain for the weakest stabilities simulated are about half the levels for the corresponding neutral simulation at the same Reynolds number. The stronger the imposed stability, the faster is the rate at which TKE drops at initial times as the flow adjusts from the initial neutral state to the statically-stable state. Similar trends are seen in surface momentum flux (friction velocity) plots. With increasing stability, as the TKE drops the flow becomes less turbulent, the angle at the surface increases and approaches the laminar limit of $45^\circ$. For the highest stabilities simulated at each Reynolds number, the TKE drops almost to zero, and friction velocity decreases to near laminar values. However, strong intermittent bursts of turbulence are observed for these highly stable simulations. This intermittency is also seen in plots of surface Richardson number and the angle between surface shear stress with the geostrophic wind velocity. In these cases, turbulence is almost completely damped and then regenerated due to build up of shear away from the wall. The bulk Richardson number at which this type of intermittency sets in increases with increasing Reynolds number.

A detailed analysis of several budgets of second-order quantities was then conducted. The results suggest that the first direct effect of buoyancy is on motions in the vertical direction. Buoyancy damps $w'$ and reduces its variance, which then causes a reduction of the downward momentum fluxes $\langle u'w' \rangle$ and $\langle v'w' \rangle$. This in turn leads to lower TKE production, particularly of the $\langle u'w' \rangle$ and $\langle v'v' \rangle$ components of the TKE. This reduction

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure28}
\caption{Vertical profile of difference in angles of stress ($\tan^{-1}(v'w'/u'w')$) and gradient ($\tan^{-1}(\partial \langle v \rangle / \partial z / \partial \langle u \rangle / \partial z$) at different stabilities (neutral, $Ri_0 = 0.001$ and 0.002) and Reynolds number ($Re_f$) of (a) 600 and (b) 900. Note that the vertical direction uses the respective inner coordinate $z^+ = zu_*/\nu$.}
\end{figure}
in TKE production with increasing Richardson number, which is an indirect effect of buoyancy since the potential temperature does not enter into the expression of TKE production by shear, is significant and has a much larger impact on the TKE budget than direct buoyancy destruction. The reduction in $\langle w'w' \rangle$ also results in drops in the production terms in the momentum flux, buoyancy flux, and temperature variance budgets, leading to a drop in these terms. For the temperature variance and buoyancy flux budgets, the drops occur despite an increase in the vertical temperature gradient that enter into the production terms of both budgets and would have increased them if the vertical velocity perturbations were not damped even more significantly.

The last section of the paper focused on the implications of these findings for low-order turbulence closure, with a particular focus on closure for the ABL. Turbulent viscosity is lower near the wall, but dominates away from the wall for neutral case; this result is Reynolds number dependent. Under the effect of stability, the turbulent components of viscosity and diffusivity are reduced significantly away from the wall. For the most stable cases, the turbulent contribution to the fluxes is significantly smaller than the viscous one. The ratio of turbulent viscosity to turbulent diffusivity, the turbulent Prandtl number, increases very little with stability. This indicates that buoyancy affects turbulent transfer
of heat and momentum similarly for statically-stable flows; however, we note that the impacts of static stability on turbulent heat and momentum transport were found to be somewhat more dissimilar in studies in the much higher Reynolds number ABL.

When momentum fluxes are non-dimensionalized by vertical velocity variance, the profiles at different stabilities, at a given Reynolds number, collapse almost perfectly. However, when non-dimensionalized with TKE, they do not collapse and this measure of transport efficiency drops down significantly with increasing stability: horizontal variances are not reduced significantly by stability and hence the TKE decreases with stability at a lower rate than fluxes. This suggest that first-order or 1.5-order closure models for geophysical or engineering statically stable flows would be more universal if based on the vertical velocity variance, rather then the full TKE.

In general, the results confirm that both the Richardson and Reynolds number are important in statically stable wall-bounded flow. Some results are transferable to higher Reynolds number flows such as the ABL: MOST is found to predict the stability effects on spectra and mean profiles quite well despite the sensitivity of the coefficients to the Reynolds number. In addition, the general features of the effect of buoyancy on second-order budgets are also expected to hold for much higher Reynolds numbers. On the other hand, some results such as the angle between the horizontal velocity and horizontal shear stress vectors or the impact of stability on the turbulent Prandtl number seem to be quite dependent on the Reynolds number and the findings from this study might not hold for more intensely turbulent wall-bounded flows.
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Appendix A. Stability parameters and Kolmogorov, Ozmidov and Buoyancy length scales

We plot the full profiles of the Kolmogorov scale $\eta = (\nu^3/\epsilon)^{\frac{1}{4}}$ in this appendix (Figure 29) to confirm the increase in its value with increasing stability. The value of this smallest turbulent scale in the flow field increases with stability as expected due to the reduction in TKE dissipation, which follows from the reduction in TKE production. Dissipation of kinetic energy occurs at the smallest scales, and the higher the energy production and cascade, the smaller are the scales required to dissipate it. The grid resolution requirement for DNS simulations depends on the value of this smallest significant Kolmogorov scale ($\eta$). As its value is larger in stable cases, the grid points used in neutral case are sufficient for stable cases at the same Reynolds number.

The Brunt-Väisälä frequency, $N_{bv}^2 = g/\theta_{\infty}(d\langle\theta\rangle/dz)$, is another important parameter in statically stable flow. It is the frequency of oscillations a particle will undergo if displaced from an equilibrium position in a static stable environment. The stronger the stability, the higher are the gradients, and the higher is $N_{bv}^2$. Figure 30a-b shows the vertical profiles of Brunt-Väisälä frequencies for stable cases at Reynolds numbers of 400 and 600. With increasing stability, the frequency of oscillations increases for both Reynolds number cases.

The Ozmidov scale ($L_O = (\epsilon/N_{bv}^3)^{\frac{1}{2}}$) is another relevant length scale in stable flow. It is a measure of the smallest turbulent scale that is affected by stability; it decreases with increasing $Ri$ (Dalaudier & Sidi 1990). On the other hand, the Buoyancy scale ($L_B = \sigma_w/N_{bw}$) represents the degree of damping of vertical motions caused by stability (Stull 1988), which as we show in the paper is a very significant effect of stability. Figure 31 depicts the ratio of the Ozmidov scale to the Kolmogorov scale and of the buoyancy scale to the Kolmogorov scale. Drastic reduction in both ratios is observed under the influence of stability. This is due to the fact that the Kolmogorov scale increases, while the Ozmidov and buoyancy scales decrease as stability is enhanced. This trend reduces the span of the inertial subrange which is now squeezed at the high wavenumber end by the dissipation range and at the low wavenumber end by the buoyancy range. For very high stabilities, an inertial range might cease to exist.

Appendix B. Turbulence kinetic energy, momentum and buoyancy fluxes and temperature variance budget equations:

Flux budget equation: $\langle u'_i u'_j \rangle$

$$\frac{\partial \langle u'_i u'_j \rangle}{\partial t} + U_k \frac{\partial \langle u'_i u'_j \rangle}{\partial x_k} = P_{ij} + T_{ij} + C_{ij} + D_{ij} + \Pi_{ij} + \phi_{ij} + \epsilon_{ij} + B_{ij} \quad (B1)$$
where

\[ P_{ij} = - \left( \langle u'_i u'_k \rangle \frac{\partial U_i}{\partial x_k} + \langle u'_i u'_k \rangle \frac{\partial U_j}{\partial x_k} \right) \]  production rate

\[ T_{ij} = - \frac{\partial \langle u'_i u'_k \rangle}{\partial x_k} \]  turbulent transport

\[ C_{ij} = f_c [\epsilon_{ik3} \langle u'_j u'_k \rangle + \epsilon_{jk3} \langle u'_i u'_k \rangle] \]  Coriolis effects

\[ D_{ij} = \nu \frac{\partial^2 \langle u'_i u'_j \rangle}{\partial x_x \partial x_k} \]  viscous diffusion rate

\[ \Pi_{ij} = -\frac{1}{\langle \rho \rangle} \left[ \frac{\partial \langle p' u'_j \rangle}{\partial x_j} + \frac{\partial \langle p' u'_i \rangle}{\partial x_i} \right] \]  pressure transport rate

\[ \phi_{ij} = \frac{2}{\langle \theta \rangle} \langle p' S'_i \rangle \]  pressure strain redistribution

\[ \epsilon_{ij} = -2\nu \left\langle \frac{\partial u'_k}{\partial x_k} \frac{\partial u'_j}{\partial x_j} \right\rangle \]  viscous dissipation rate

\[ B_{ij} = \frac{g}{\langle \theta \rangle} \left[ \delta_{i3} \langle u'_j \theta' \rangle + \delta_{j3} \langle u'_i \theta' \rangle \right] \]  buoyancy sink term

Here that the strain-rate tensor is defined as \( S_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \). With horizontal homogeneity and no subsidence, the above equation simplifies to

\[
\frac{\partial \langle u'_i u'_j \rangle}{\partial t} = -\left( \langle u'_i w' \rangle \frac{\partial U_i}{\partial z} + \langle u'_j w' \rangle \frac{\partial U_j}{\partial z} \right) - \frac{\partial \langle u'_i u'_j w' \rangle}{\partial z} - \frac{1}{\langle \rho \rangle} \left[ \delta_{j3} \frac{\partial \langle p' u'_i \rangle}{\partial z} + \delta_{i3} \frac{\partial \langle p' u'_j \rangle}{\partial z} \right] + \frac{2}{\langle \rho \rangle} \langle p' S'_i \rangle
+ \frac{g}{\langle \theta \rangle} \left[ \delta_{i3} \langle u'_j \theta' \rangle + \delta_{j3} \langle u'_i \theta' \rangle \right] + f_c [\epsilon_{ik3} \langle u'_j u'_k \rangle + \epsilon_{jk3} \langle u'_i u'_k \rangle] + \nu \frac{\partial^2 \langle u'_i u'_j \rangle}{\partial z \partial z} - 2\nu \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \right\rangle
\]

TKE budget: \( \frac{1}{2} \langle u'_i u'_i \rangle \)

\[
\frac{\partial q}{\partial t} + U_k \frac{\partial q}{\partial x_k} = \frac{1}{2} P_{ii} + \frac{1}{2} T_{ii} + \frac{1}{2} \Pi_{ii} + \frac{1}{2} B_{ii} + \frac{1}{2} C_{ii} + \frac{1}{2} D_{ii} + \frac{1}{2} \epsilon_{ii} \quad (B2)
\]

where \( q = \frac{1}{2} \langle u'_i u'_i \rangle \). Again with horizontal homogeneity and no subsidence, the above equation simplifies to

\[
\frac{\partial q}{\partial t} = -\langle u'_i w' \rangle \frac{\partial U_i}{\partial z} - \frac{1}{2} \frac{\partial (w' w')}{\partial z} - \frac{1}{\langle \rho \rangle} \langle w' \theta' \rangle + f_c \epsilon_{ij3} \langle u'_i u'_j \rangle + \nu \frac{\partial^2 q}{\partial z \partial z} - 2\nu \left\langle \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_j} \right\rangle
\]

Buoyancy flux budget: \( \langle w' \theta' \rangle \)

\[
\frac{\partial \langle w' \theta' \rangle}{\partial t} + U_j \frac{\partial \langle w' \theta' \rangle}{\partial x_j} = P^1 + P^2 + T + \Pi_1 + \Pi_2 + D^1 + D^2 + \chi + B \quad (B3)
\]
where

\[ P_1 = -\langle w'u_j' \rangle \frac{\partial \langle \theta \rangle}{\partial x_j} \quad \text{production by mean-temperature gradient} \]

\[ P_2 = -\langle \theta'u_j' \rangle \frac{\partial w'}{\partial x_j} \quad \text{production by mean-velocity gradient} \]

\[ T = -\frac{\partial (u_j'w'\theta')}{\partial x_j} \quad \text{turbulent transport} \]

\[ \Pi_1 = -\frac{1}{\langle \rho \rangle} \frac{\partial \langle p'\theta' \rangle}{\partial z} \quad \text{transport by pressure correlation} \]

\[ \Pi_2 = \frac{1}{\langle \rho \rangle} \left\langle p' \frac{\partial \theta'}{\partial z} \right\rangle \quad \text{redistribution by return-to-isotropy} \]

\[ D_1 = \nu \frac{\partial}{\partial x_j} \left\langle \theta' \frac{\partial w'}{\partial x_j} \right\rangle \quad \text{diffusion by velocity-gradients} \]

\[ D_2 = \nu_0 \frac{\partial}{\partial x_j} \left\langle w' \frac{\partial \theta'}{\partial x_j} \right\rangle \quad \text{diffusion by temperature gradients} \]

\[ \chi = -\left( \nu + \nu_0 \right) \left( \frac{\partial w'}{\partial x_j} \frac{\partial \theta'}{\partial x_j} \right) \quad \text{molecular dissipation} \]

\[ B = \frac{\langle \theta' \theta' \rangle}{\langle \theta \rangle} g \quad \text{buoyant destruction} \]

Temperature variance budget: \( \langle \theta' \theta' \rangle \)

\[
\frac{\partial \langle \theta' \theta' \rangle}{\partial t} + U_j \frac{\partial \langle \theta' \theta' \rangle}{\partial x_j} = P_\theta + T_\theta + D_\theta + \chi_\theta \quad (B4)
\]

where

\[ P_\theta = -2\langle u_j' \theta' \rangle \frac{\partial \langle \theta \rangle}{\partial x_j} \quad \text{production rate} \]

\[ T_\theta = -\frac{\partial \langle u_j' \theta' \theta' \rangle}{\partial x_j} \quad \text{turbulent transport} \]

\[ D_\theta = \nu_0 \frac{\partial^2 \langle \theta' \theta' \rangle}{\partial x_j \partial x_j} \quad \text{molecular diffusion} \]

\[ \epsilon_\theta = -2\nu_0 \left\langle \frac{\partial \theta'}{\partial x_j} \frac{\partial \theta'}{\partial x_j} \right\rangle \quad \text{molecular dissipation} \]

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DNS of Stable Ekman Layers


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