Expectations, Networks, and Conventions

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Consensus Expectations

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  - what is the average expectation of random variable? What is the average representation of the average expectation? And so on...
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  - characterize equilibrium of a network game with asymmetric information, where best responses are linear and coordination motive is (arbitrarily) high: “conventions”
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- Consensus expectations matter:
  - characterize equilibrium of a network game with asymmetric information, where best responses are linear and coordination motive is (arbitrarily) high: “conventions”
  - characterize asset prizes in a (stylized) over-the-counter market with (arbitrarily) frequent trading and market segmentation
Main Contributions

- Substantive contribution: what are consensus expectations?
  - Complete Information
    - Eigenvector centrality weighted beliefs = outcome of de Groot learning (de Groot 1974, BEN: add refs vayanos et al, golub?)

- Common Prior Assumption
  - Ex ante expectation (Samet 98)

- Common Prior on Types
  - Eigenvector centrality weighted ex ante expectation

- "Compatible Marginals on Types" (to be defined)
  - "separability": Eigenvector centrality weighted pseudo (but network free) ex ante expectation

- Common Certainty of (Mild) Optimism - everyone expects others to be (a bit) more optimistic than him
  - Highest possible interim expectation

- Least Informed Agent
  - Ex ante expectation of least informed agent

- Methodological contribution: unified treatment of network structure and asymmetric information
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Outline

- Define Linear Best Response game and Consensus Expectations simultaneously
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- Return to Asset Market application/interpretation
Network is a row-stochastic $\Gamma$; each agent chooses $a^i \in \mathbb{R}$; payoffs:

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$$a^i_{BR} = (1 - \beta) + \beta$$

**best response = weighted average of**

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Reason: write $a = (1 - \beta)y + \beta \Gamma a$ and substitute repeatedly to get $a = (1 - \beta)\sum_{k=1}^{\infty} \beta^k \Gamma^k y$, which is $\approx \Gamma^\infty y$. Each row of $\Gamma^\infty$ is the same, equal to $e$. 


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5. The external state $\theta$ is revealed and payoffs are enjoyed.
Network Game with Asymmetric Information and Linear Best Response

Fix \( y : \Theta \rightarrow \mathbb{R} \), i.e., \( y \in \mathbb{R}^{\Theta} \). Agents choose \( a^i \in [y_{\min}, y_{\max}] \); ex post payoffs:

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best response = weighted average of conditional expectation of $y$ and conditional expectation of neighbors’ average action
agents $i, j \in N = \{1, 2, \ldots, I\}$

states of the world, types (finite sets) $\theta \in \Theta$, $t^i \in T^i$

$T = T^1 \times T^2 \times \cdots \times T^I$
agents $i, j \in N = \{1, 2, \ldots, I\}$

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belief functions $\pi^i_{t^i} \in \Delta(T \times \Theta)$

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network weights $\Gamma = (\gamma^{ij})_{i,j}$, nonnegative

$I$-by-$I$, row-stochastic
agents \( i, j \in N = \{1, 2, \ldots, I\} \)

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belief functions

(subjective) expectation operators

\[ \pi_{ti}^i \in \Delta(T \times \Theta) \]

\[ E_{ti}^{(i)} \]
e.g. for \( y \in \mathbb{R}^\Theta \), \( E^{(i)}y \in \mathbb{R}^{T^i} \)
(Interim) Environment

agents

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\[ i, j \in N = \{1, 2, \ldots, I\} \]

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\[ \pi^i_t \in \Delta(T \times \Theta) \]

\[ E_{t^i}^{(i)} \]

\[ \text{e.g. for } y \in \mathbb{R}^\Theta, \quad E^{(i)} y \in \mathbb{R}^{T^i} \]

\[ \Gamma = (\gamma^{ij})_{i,j}, \text{ nonnegative} \]

\[ I\text{-by-}I, \text{ row-stochastic} \]
Agents have priors $\mu^i \in \Delta(T \times \Theta)$.

These priors induce belief functions:
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$$\pi^i_{\hat{t}_i}(E) = \mathbb{P}_{\mu^i}(E \mid t^i = \hat{t}^i) = \frac{\mu^i(\{t^i = \hat{t}_i\} \cap E)}{\mu^i(\{t^i = \hat{t}_i\})}.$$
A Markov Process

\[ \begin{array}{cccc}
\pi^1 & w & 1 - w & \pi^3 \\
1 - \pi^1 & 1 - w & w & 1 - \pi^3 \\
\end{array} \]

black: transition probabilities
A Markov Process

\[
\frac{w\pi^1 + (1-w)\pi^3}{2}
\]

black: transition probabilities
blue: ergodic distribution

\[
\frac{w(1 - \pi^1) + (1-w)(1 - \pi^3)}{2}
\]
Four-Player Network Game

\[
\frac{w \pi^1 + (1 - w) \pi^3}{2}
\]

\[
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Limit action = dot product of blue weights and ideal actions.
Four-Player Network Game

Limit action = dot product of blue weights and ideal actions.

\[
\frac{w}{2}y^1 + \frac{w\pi^1 + (1-w)\pi^3}{2}y^2 + \frac{w(1-\pi^1) + (1-w)(1-\pi^3)}{2}y^3 + \frac{1-w}{2}y^4
\]
Three-Player Game of Incomplete Information

green: subjective probabilities
red: network weights
Three-Player Game of Incomplete Information

black: network $\times$ belief
Three-Player Game of Incomplete Information

\[ w\pi^1 + (1 - w)\pi^3 \]

\[ w(1 - \pi^1) + (1 - w)(1 - \pi^3) \]

black: network × belief
blue: player-type weights
Network Game with Linear Best Response in new (more explicit) notation

Fix $y \in \mathbb{R}^\Theta$. Agents choose $a^i \in [y_{\text{min}}, y_{\text{max}}]$; ex post payoffs:

$$u^i = -(1 - \beta)(a^i - y(\theta))^2 - \beta \sum_j \gamma^{ij}(a^i - a^j)^2.$$ 

$$a_{\text{BR}}^i = (1 - \beta)E^{(i)}y + \beta E^{(i)} \sum_j \gamma^{ij}a^j$$

The best response is defined as the weighted average of the conditional expectation of $y$ and the conditional expectation of the neighbors' average action:

$$a_{t_i}^i = \sum_k \pi_{t_i}^i (\theta_k) \theta_k + \sum_{j \in N} \gamma^{ij} \sum_{t_j \in T_j} \pi_{t_i}^i (t^j) a_{t_j}^j$$
Iterated Average Expectations (cf. Samet 98)

- Fix a random variable $y \in \mathbb{R}^\Theta$ (measurable w.r.t. state of the world).
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- Define $x_t^i(1) = E_{t^i} y$;

- This is an expectation of an average (taken across population). It is second-order: an expectation over first-order expectations.
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- Fix a random variable \( y \in \mathbb{R}^{\Theta} \) (measurable w.r.t. state of the world).
- Define \( x_{t_i}(1) = E_{t_i}^{(i)} y; \)
  - first-order expectations of \( y; \)
- Define
  \[
  x_{t_i}(2) = E_{t_i}^{(i)} \sum_j \gamma_{ij} x_{t_i}(1). 
  \]
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- Define
  $$x^i_t(2) = E^{(i)}_{t} \sum_j \gamma^{ij} x^j(1).$$
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- Fix a random variable $y \in \mathbb{R}^\Theta$ (measurable w.r.t. state of the world).

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  - first-order expectations of $y$; a random variable measurable with respect to $i$’s information.

- Define
  $$x^{i}_{ti}(2) = E^{(i)}_{ti} \sum_{j} \gamma^{ij} x^{j}(1).$$
  - This is an expectation of an average (taken across population).
  - It is second-order: an expectation over first-order expectations.
Iterated Average Expectations (cf. Samet 98)

- Fix a random variable $y \in \mathbb{R}^\Theta$ (measurable w.r.t. state of the world).

- Define $x^i_t(1) = E^{(i)}_{t_i} y$;
  - first-order expectations of $y$; a random variable measurable with respect to $i$’s information.

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Iterated Average Expectations (cf. Samet 98)

- Fix a random variable \( y \in \mathbb{R}^\Theta \) (measurable w.r.t. state of the world).

- Define \( x_{ti}^i(1) = E_{ti}^{(i)} y \);
  - first-order expectations of \( y \); a random variable measurable with respect to \( i \)'s information.

- Define

  \[
  x_{ti}^i(n + 1) = E_{ti}^{(i)} \sum_j \gamma_{ij} x^j(n).
  \]

  - This is an expectation of an average (taken across population).
  - It is \((n + 1)^{\text{st}}\)-order: an expectation over \( n^{\text{th}}\)-order expectations.
Professional investment may be likened to those newspaper competitions [in which] each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view . . . We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees.

— Keynes, *The General Theory* . . . (1936)
Iterated Average Expectations: Example

\[ y^{31} = 1 \quad y^{12} = 1 \quad y^{23} = 1 \]

\[ E^{(1)}y \]
Iterated Average Expectations: Example

\[ E^{(3)} E^{(1)} y \]
Iterated Average Expectations: Example

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Iterated Average Expectations: Example

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Network Game with Linear Best Responses

Fix $y \in \mathbb{R}^\Theta$. Agents choose $a^i \in [y_{\text{min}}, y_{\text{max}}]$; ex post payoffs:

$$u^i = -(1 - \beta)(a^i - y(\theta))^2 - \beta \sum_j \gamma^{ij}(a^i - a^j)^2.$$ 

$$a^i_{\text{BR}} = (1 - \beta) E^{(i)} y + \beta E^{(i)} \sum_j \gamma^{ij} a^j$$

In any rationalizable strategy profile, actions played are

$$a^i_{t,i} = \sum_{n=0}^{\infty} (1 - \beta) \beta^n x^i_{t,i}(n + 1)$$

as $\beta \uparrow 1, a^i_{t,i} \to \lim_{n \to \infty} x^i_{t,i}(n)$ when limit exists.
Defining Consensus Expectation

For any \( y \in \mathbb{R}^{\Theta} \)

- If limit

\[
\lim_{\beta \uparrow 1} (1 - \beta) \sum_{n=0}^{\infty} \beta^n x^i_{ti} (n + 1)
\]

- Will equal (if well-defined)

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Convergence to a Consensus Expectation: Theorem

Suppose beliefs and interactions are jointly connected (definition to follow). Then for any $y \in \mathbb{R}^\Theta$

- The Limit

\[
c = \lim_{\beta \uparrow 1} a_i^t
\]

\[
c = \lim_{\beta \uparrow 1} (1 - \beta) \sum_{n=0}^{\infty} \beta^n x_i^t (n + 1)
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exists for every agent $i$ and every type $t^i \in T^i$. 
Convergence to a Consensus Expectation: Theorem

Suppose beliefs and interactions are jointly connected (definition to follow). Then for any \( y \in \mathbb{R}^\Theta \)

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- This consensus expectation is a weighted average of various types’ first-order beliefs:

  $$c = \sum_i \sum_{t^i \in T^i} p(t^i) E_t^{(i)} y,$$

  where $\sum_i \sum_{t^i \in T^i} p(t^i) = 1$ and $p$ depends only on $\pi_t^i(t^i)$’s and $\Gamma$. 
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Convergence Result: Proof Idea

- Define $S = \bigcup_{i \in N} T^i$, union of type spaces.
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  - So we can write the iteration conveniently via an interaction structure matrix, $B$:

$$x(n + 1) = Bx(n).$$

Then analysis comes down to powers of this $B$ matrix – and these are well-understood.
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  - So we can write the iteration conveniently via an interaction structure matrix, $B$:
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  Then analysis comes down to powers of this $B$ matrix – and these are well-understood.

- Alternate proof: reduction of incomplete-information to just another network game.
The Interaction Structure: The Matrix $B$

\[ x^i_{t_i}(n+1) = \sum_j \gamma^{ij} E_{t_i}^{(i)} x^j(n) \]
The Interaction Structure: The Matrix $B$

\[ x^i_{t_i}(n + 1) = \sum_j \gamma^{ij} E^{(i)}_{t_i} x^j(n) = \sum_{j \in N} \gamma^{ij} \sum_{t_j \in T^j} \pi^i_{t_i}(t^j) x^j_{t_j}(n) \]
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\[ x(n + 1) = B x(n). \]
The Interaction Structure: The Matrix $B$

$$x_{t_i}^i(n+1) = \sum_j \gamma^{ij} E_{t_i}^{(i)} x^j(n) = \sum_{j \in N} \gamma^{ij} \sum_{t_j \in T} \pi^{i}(t^j) x_{t_j}^j(n)$$

$$x(n+1) = B x(n).$$
The Interaction Structure: The Matrix $B$

\[ x^i_t(n+1) = \sum_j \gamma^{ij} E^{(i)}_t x^j(n) = \sum_{j \in N} \sum_{i \in T_j} \pi^i_{t^j}(t^j) x^j_{t^j}(n) \]

\[ x(n+1) = Bx(n). \]

⇒ \[ x(\infty) = B^\infty x(1) \]
The Interaction Structure: The Matrix $B$

$x_{t_i}^i(n+1) = \sum_j \gamma_{ij}^{ij} E_{t_i}^{(i)} x^j(n) = \sum_{j \in N} \gamma_{ij}^{ij} \sum_{t_j \in T_j} \pi_{t_i}^i(t^j) x_{t_i}^j(n)$

$x(n+1) = B x(n)$.

$\Rightarrow x(\infty) = B^\infty x(1) = p' x(1) 1$, where $p'$ is stationary distribution of $B$
The Interaction Structure: The Matrix $B$

$x^i_{ti}(n + 1) = \sum_j \gamma^{ij} E^{(i)}_{ti} x^j(n) = \sum_{j \in N} \sum_{t^j_i \in T^j} \pi^i_{t^j_i}(t^j_i) x^j_{ti}(n)$

$x(n + 1) = Bx(n)$.

$\Rightarrow x(\infty) = B^\infty x(1) = p'x(1)1$, so $c = \sum_i \sum_{t^i} p^i(t^i)x(1)$
Beliefs and Interactions are Jointly Connected

We say **beliefs and interactions are jointly connected** if for any nonempty proper subset of types $R \subseteq T^1 \cup T^2 \cup \cdots \cup T^I$ there is some $t^i \in R$ and $t^j \notin R$ so that $\gamma^{ij} \pi^i_{t^i}(t^j) > 0$. 

![Diagram](attachment:image.png)
The Interaction Structure

Case 1: Complete Information

- Define $e$ to be the eigenvector centrality of the network $\Gamma$: unique vector summing to 1 so that

$$e^i = \sum_j \gamma^{ji} e^j \quad \forall i$$

- If there is no incomplete information, type spaces are singletons and $B = \Gamma$. So $p = e$.

- Now consensus expectation is eigenvector weighted complete information expectation

$$c = \sum_i e^i E^i y.$$  

- Ballester, Calvó-Armengol, and Zenou (06) on network games.
Case 2: Common Prior Assumption

- There exists $\mu^* \in \Delta(T \times \Theta)$ such that $\mu^1 = ... = \mu^n = \mu^*$
- Now consensus expectation is (common prior) ex ante expectation
  \[ c = E^* y. \]
- implied by Samet 98
- argument to follow
Case 3: Common Prior on Types

- There exists $\mu^* \in \Delta(T \times \Theta)$ such that

$$\mu^1(\hat{T}) = \ldots = \mu^n(\hat{T}) = \mu^*(\hat{T})$$

for all $\hat{T} \subseteq T$

- Now

$$c = \sum_i e^i E^i y.$$

- argument to follow
Type Weights Sum to Agent Centralities

**Proposition**

total weight on $i$’s types $=$ $i$’s network centrality

$$\sum_{t^i \in T^i} p(t^i) = e^i$$
Type Weights Sum to Agent Centralities

Proposition

total weight on $i$’s types $= i$’s network centrality

$$
\sum_{t^i \in T^i} p(t^i) = e^i
$$

Therefore, can write

$$p(t^i) = e^i r(t^i) \quad \text{where} \quad \sum_{t^i \in T^i} r^i(t^i) = 1.
$$

$r^i(t^i)$, the type weight on $t^i$, can be thought of as a pseudoprior on type $t^i$ of $i$. 
Type Weights Sum to Agent Centralities

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- \(r^i(t^i)\), the **type weight** on \(t^i\), can be thought of as a **pseudoprior** on type \(t^i\) of \(i\).

\[
c = \sum_{i \in N} e^i \sum_{t^i} r^i_{\pi, \Gamma}(t^i) E_{t^i} y
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- \( r^i(t^i) \), the **type weight** on \( t^i \), can be thought of as a **pseudoprior** on type \( t^i \) of \( i \).

\[
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\]

- In general, \( r^i_{i, \Gamma}(t^i) \) depends on information structure \( \pi \) and the network \( \Gamma \).
Separating Effects of Network and Beliefs

**Definition**

Beliefs \( \pi = (\pi^i)_{i \in N} \) have **compatible marginals** if there is a profile \( (\tilde{r}^i \in \Delta(T^i))_{i \in N} \) such that for any \( i \), any \( t^i \in T^i \), and any \( j \in N \)

\[
\tilde{r}^i(t^i) = \sum_{t^j \in T^j} \tilde{r}^j(t^j) \pi^j_{ij}(t^i).
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Weaker than assuming beliefs arise from a common prior over $T$. 
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Weaker than assuming beliefs arise from a common prior over \( T \).

Proposition

The following are equivalent:

1. Beliefs \( \pi \) have compatible marginals.
2. For all irreducible \( \Gamma \), type weights \( (r_{\pi,\Gamma^i})_{i \in N} \) are the same, \( (\tilde{r}_\pi^i)_{i \in N} \).
Separating Effects of Network and Beliefs

Definition

Beliefs $\pi = (\pi^i)_{i \in N}$ have **compatible marginals** if there is a profile $(\tilde{r}^i \in \Delta(T^i))_{i \in N}$ such that for any $i$, any $t^i \in T^i$, and any $j \in N$

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The following are equivalent:

1. Beliefs $\pi$ have compatible marginals.
2. For all irreducible $\Gamma$, type weights $(\bar{r}_{\pi,\Gamma^i})_{i \in N}$ are the same, $(\bar{r}_{\pi^i})_{i \in N}$.

If either condition holds, then $\tilde{r}^i = \bar{r}^i$ for each $i$. 
Case 4: Compatible Marginals

- So compatible marginals implies:

\[ c = \sum_{i \in N} e^i \sum_{t^i \in T^i} \bar{r}_{\pi^i}(t^i) E^i_{t^i} y \]
So compatible marginals implies:

\[ c = \sum_{i \in N} e^i \sum_{t^i \in T^i} \bar{r}_{\pi^i}(t^i) E^i_{t^i} y = \sum_{i \in N} e^i \cdot "i’s prior expectation of y." \]
Losing Separability and Optimism: A Counterexample

Without type-consistency, no separation \( c = \sum_{i \in N} e^i \sum_{t_i \in T^i} \bar{r}_\pi (t^i) E^i_{t_i y} \).

- An agent observing signal \( g \) assigns probability \( p \) to state \( G \).
- An agent observing signal \( b \) assigns probability \( 1 - p \) to state \( G \).
- An agent observing signal \( g \) assigns ...
Losing Separability and Optimism: A Counterexample

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\[ c = \sum_{i \in \mathbb{N}} e^i \sum_{t^i \in T^i} \bar{r}_\pi^i (t^i) E_{t^i}^i. \]

- An agent observing signal $g$ assigns probability $p$ to state $G$.
- An agent observing signal $b$ assigns probability $1 - p$ to state $G$.
- An agent observing signal $g$ assigns ...
  - ...probability $1 - \varepsilon$ to agent $i + 1$ observing signal $g$;
  - ...probability $\frac{1}{2}$ to agent $i - 1$ observing signal $g$.
- An agent observing signal $b$ assigns ...
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\[
\gamma^{31} = 1 \quad \gamma^{12} = 1 \quad \gamma^{23} = 1
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- An agent observing signal \( g \) assigns...
  - ...probability \( 1 - \varepsilon \) to agent \( i + 1 \) observing signal \( g \);
  - ...probability \( \frac{1}{2} \) to agent \( i - 1 \) observing signal \( g \).
- An agent observing signal \( b \) assigns...
  - ...probability \( \frac{1}{2} \) to agent \( i + 1 \) observing signal \( g \);
  - ...probability \( \varepsilon \) to agent \( i - 1 \) observing signal \( g \).

Under the above (clockwise) network the consensus expectation is

\[
\frac{p + 2\varepsilon (1 - p)}{1 + 2\varepsilon} \approx p
\]
Losing Separability and Optimism: A Counterexample

Without type-consistency, no separation
\[ c = \sum_{i \in N} e_i \sum_{t_i \in T^i} \bar{r}_{\pi^i}(t^i) E_{t^i}^i. \]

- An agent observing signal \( g \) assigns probability \( p \) to state \( G \).
- An agent observing signal \( b \) assigns probability \( 1 - p \) to state \( G \).
- An agent observing signal \( g \) assigns...  
  - ... probability \( 1 - \varepsilon \) to agent \( i + 1 \) observing signal \( g \);
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- An agent observing signal \( b \) assigns...  
  - ... probability \( \frac{1}{2} \) to agent \( i + 1 \) observing signal \( g \);
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\[ \gamma_{13} = 1 \quad \gamma_{21} = 1 \quad \gamma_{32} = 1 \]

Under the above (counterclockwise) network the consensus expectation is
\[ \frac{1 - p + 2\varepsilon p}{1 + 2\varepsilon} \approx 1 - p \]
Cyclic Optimism

- Each agent has ex ante probability $\frac{1}{2}$ of state $G$
Cyclic Optimism

- Each agent has ex ante probability $\frac{1}{2}$ of state $G$
- Consensus probability is higher (or lower) than any agent’s ex ante probability
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- “Second order optimism (or pessimism)”
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- Each agent has ex ante probability $\frac{1}{2}$ of state $G$
- Consensus probability is higher (or lower) than any agent’s ex ante probability
- “Second order optimism (or pessimism)”
Case 5: Optimism

- Each agent is sure that each other agent is at least as optimistic as him
- Type weight must put mass only on the more optimistic types
- Consensus expectation is the highest possible interim expectation
- Can approximate
Case 6: One Least Informed Player

- Suppose that one agent knows (or believes that he knows) nothing while other agents all know (or believe that they know) something.
- Consensus expectation is the *ex ante* expectation of the least informed agent.
- Can approximate.
Whose Beliefs Matter?

- Work with explicit ex ante stage.

- Suppose agents $i$ receive signals of what the state is, equal to the true state with probability $1 - \varepsilon^i$, and erroneous with probability $\varepsilon^i$. 
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- If all $\epsilon^i \downarrow 0$, but one of them ($j$) much slower than the others, then only $j$’s priors over states will matter.
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    Must assume
    $$\lim \frac{\varepsilon^i}{(\varepsilon^j)^m} \to 0,$$
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    $$\lim \frac{\epsilon^i}{(\epsilon^j)^m} \rightarrow 0,$$
    where $m$ depends on structure of uncertainty near certainty.
  - Study limits of ergodic distribution of $B$ as some edges are going to 0.
Consider $B(\zeta)$ where $\varepsilon^i$ is a function of $\zeta$. 
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Consensus depends on ergodic distribution of $B(\zeta)$. 
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Whose Beliefs Matter? Via Limits of Interaction Structures

- Consider $B(\zeta)$ where $\varepsilon_i$ is a function of $\zeta$.

- Consensus depends on ergodic distribution of $B(\zeta)$.

- $B(0)$ is disconnected.

- Skeleton of “leading edges” will determine stationary distribution in the low-$\zeta$ limit.

- Argument is via Markov chain tree lemma.
Other Related Literature


- de Martí and Zenou (2014) “Network Games with Incomplete Information.”

- Bergemann, Heumann, and Morris (JET 2015), “Information and Volatility.”


Connections with DeGroot Updating (DeGroot 1974)

\[ x^i(t + 1) = \sum_j W_{ij}x^j(t) \]

- DeGroot and his literature (Delphi method, Lehrer and Wagner 1981) were seeking a normatively justified way to pool different estimates or forecasts.

Looks crazy to us in view of Aumann's (1976), "Agreeing to Disagree," Geanakoplos Polemarchakis (1982), "We Can't Disagree Forever." But there is a way to reconcile their approach with standard treatment of beliefs: make it about aggregating priors rather than posteriors. Somewhat related to Harsanyi's thoughts about coming up with a common prior from expert views.
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- In different ways they defended this dynamic process as a normatively reasonable scheme to allocate weights to initial views.
- But there is a way to reconcile their approach with standard treatment of beliefs: make it about aggregating priors rather than posteriors. Somewhat related to Harsanyi’s thoughts about coming up with a common prior from expert views.
An Over-the-Counter Market

- $I$ populations of agents: continuum of each population, with each individual in population $i$ having the same type or signal $t^i$. 

- With probability $1 - \beta$, state is realized and the agent consumes the realization of the asset.

- If not, then with (subjective) probability $\gamma_{ij}$ agent $i$ must sell the asset in a market where he faces population $j$.

- Competitive market, so the price is equal to population $j$’s subjective valuation.

- As $\beta \uparrow 1$, valuation of agent $i$ tends to $\lim_{\beta \uparrow 1} (1 - \beta) \sum_{n=0}^{\infty} \beta^n x^i (n + 1)$. 

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$\lim_{\beta \to 1} \left(1 - \beta\right) \sum_{n=0}^{\infty} \beta^n x^i (n+1)$.
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- As $\beta \uparrow 1$, valuation of agent $i$ tends to
  \[
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  \]
Conclusion

- Consensus expectations exist and have economically interesting properties, interpretations and applications.

- By studying “interaction structure” $B$ that treats network and beliefs symmetrically (à la Morris 2000, “Contagion”), can generalize both classical beauty contest results and complete-information network results.

- Compatible Beliefs is a nice middle ground between common priors and “anything goes.”
  - Reduces “whose priors matter” question to network centrality.
  - Has implications also for Samet iterated expectations (separate paper).

- Can use “physical” intuitions about Markov chains to understand, e.g., whose priors matter.
Separability of Network Structure and Type Weights: Proof

1. \( p \in \Delta(S) \) defined by \( pB = p \).
2. \( p(t^i) = e^i r^i(t^i) \) by the proposition.
3. Plug (2) into (1) to reduce characterization of \( p \) to finding weights \( r^i(t^i) \in \Delta(T^i) \) such that

\[
e^i r^i(t^i) = \sum_{j \in N} \gamma_{ji} e^j \sum_{t^j \in T^j} r^j(t^j) \pi_{t^j}(t^i).
\]

\((*)\)
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$$e^i r^i(t^i) = \sum_{j \in N} \gamma^i_j e^j \sum_{t^j \in T^j} r^j(t^j) \pi^j_{t^j}(t^i). \quad (*)$$

4. Assume $\pi$ have consistent marginals. Set $r^i = \tilde{r}^i \in \Delta(T^i)$ to be the marginals in the definition of type-consistency. Then (*) boils down to $e = \Gamma e$, which holds by definition.
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$$e^i r^i(t^i) = \sum_{j \in N} \gamma^i e^j \sum_{t^j \in T^j} r^j(t^j) \pi^j_{ti}(t^i).$$

4. Assume $\pi$ have consistent marginals. Set $r^i = \bar{r}^i \in \Delta(T^i)$ to be the marginals in the definition of type-consistency. Then (*) boils down to $e = \Gamma e$, which holds by definition.

5. Conversely, suppose that $r^i(t) = \bar{r}^i(t)$, independent of $\Gamma$. Then use $e^i = \sum_{j \in N} \gamma^i e^j$ to write

$$\sum_{j \in N} \gamma^i e^j \bar{r}^i(t^i) = \sum_{j \in N} \gamma^i e^j \sum_{t^j \in T^j} \bar{r}^j(t^j) \pi^j_{ti}(t^i).$$

6. Because we can vary $\gamma^i e^j$ freely, this implies type-consistency.
Convergence Result: Proof

\[ x^i_k(n + 1) = \sum_j \gamma^{ij} \mathbb{E}^i [x^j(n) | t^i_k] \]

Introduce a vector stacking expectations by agent and type:

\[
\begin{bmatrix}
  x^1_1 \\
  \vdots \\
  x^1_K \\
  \vdots \\
  x^I_1 \\
  \vdots \\
  x^I_K \\
\end{bmatrix}
\]

\[ Bx(n) \]

Exactly the DeGroot theory without DeGroot goofiness.
Convergence Result: Proof

\[ x_k^i(n + 1) = \sum_j \gamma^{ij} \mathbb{E}^i [x^j(n) \mid t_k^i] \]

\[ = \sum_j \sum_\ell \gamma^{ij} \pi^i(t_\ell^j \mid t_k^i) x_\ell^j(n). \]
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Introduce a vector stacking expectations by agent and type:

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x(n) = \begin{bmatrix}
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    \vdots \\
    x_K^1 \\
    \hline
    x_1^K \\
    \vdots \\
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  \vdots \\
  x_K^1 \\
  \hline
  \vdots \\
  \hline
  x_1^l \\
  \vdots \\
  x_K^l
\end{bmatrix}
\]

\[ x(n + 1) = Bx(n) \]
Convergence Result: Proof

\[ x_k^i(n + 1) = \sum_j \gamma^{ij} \mathbb{E}^i [x^j(n) | t_k^i] \]

\[ = \sum_j \sum_\ell \gamma^{ij} \pi^i (t_\ell^j | t_k^i) x_\ell^j(n). \]

Introduce a vector stacking expectations by agent and type:

\[ x(n) = \begin{bmatrix} x_1^1 \\ \vdots \\ x_K^1 \\ \vdots \\ x_1^K \\ \vdots \\ x_1^L \\ \vdots \\ x_K^L \end{bmatrix} \]

\[ x(n + 1) = Bx(n) \]

where

\[ B_{(i,k),(j,\ell)} = \gamma^{ij} \pi^i (t_\ell^j | t_k^i) \]
Convergence Result: Proof

\[ x_k^i(n + 1) = \sum_j \gamma^{ij} \mathbb{E}^i [x^j(n) | t_k^i] \]

\[ = \sum_j \sum_\ell \gamma^{ij} \pi^i(t_\ell^j | t_k^i) x_\ell^j(n). \]

Introduce a vector stacking expectations by agent and type:

\[ x(n) = \begin{bmatrix} x_1^1 \\ \vdots \\ x_K^1 \\ \vdots \\ x_1^{K'} \\ \vdots \\ x_K^{K'} \end{bmatrix} \]

\[ x(n + 1) = B x(n) \]

where \( B_{(i,k),(j,\ell)} = \gamma^{ij} \pi^i(t_\ell^j | t_k^i) \)

Exactly the DeGroot theory without DeGroot goofiness.
Proof Using $B$ Matrix

Consider the simple case where $B^n$ converges.

$$B^n \rightarrow \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & \ddots \\ & & & & p \end{bmatrix}$$
Proof Using $B$ Matrix

Consider the simple case where $B^n$ converges.

$B^n \rightarrow \begin{bmatrix} p & \cdots & p \\ p & \cdots & p \\ \vdots \\ p & \cdots & p \end{bmatrix}$

$x_k^i(n + 1) \rightarrow pf = \sum_i \sum_k p_k^i \mathbb{E}[Y | t_k^i]$
Proof Using $B$ Matrix

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$p$ is a vector of player-type weights
Proof Using $B$ Matrix

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$$B^n \rightarrow \begin{bmatrix} p & p & \cdots & p \\ p & p & \cdots & p \\ \vdots \\ p & p & \cdots & p \\ \end{bmatrix}$$

$$x_k^i(n+1) \rightarrow pf = \sum_i \sum_k p_k^i \mathbb{E}[Y | t_k^i]$$

$p$ is a vector of player-type weights

$$pB = p,$$ so $p$ is a left-hand eigenvector of $B$ w/ eigenvalue 1