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WHEN DOES INFORMATION LEAD TO TRADE?
Trading with heterogeneous prior beliefs
and asymmetric information

by

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Abstract

Consider an exchange economy with uncertainty where, initially, there is no information and the allocation is efficient. Suppose agents then observe signals correlated with the initial payoff relevant uncertainty. Does the allocation remain efficient after the arrival of the signals? A necessary and sufficient condition on agents' beliefs for interim efficiency is consistent concordance: agents' posterior beliefs could have been derived from prior beliefs which assigned equal probability to signals, conditional on payoff relevant uncertain events. Conditions for interim incentive compatible efficiency entail a further weakening of the common prior assumption: in addition to the weakening described above, agents may put less weight on their own signals than other agents do. An allocation is interim public efficient (there does not exist a feasible re-allocation depending only on publicly known signals which interim dominates the allocation) if and only if beliefs are public consistent concordant - agents' posterior beliefs could have been derived from prior beliefs which assigned equal probability to public signals, conditional on payoff relevant uncertain events.

These efficiency conditions can be used to characterize when there is trade and when the initial allocation is a rational expectations equilibrium, after the arrival of the signals. There is no common knowledge acceptance of trades depending only on payoff relevant uncertainty if and only if the allocation is interim public efficient. Weaker notions of acceptance (which still preclude trade under the common prior assumption) give no trade results equivalent to interim incentive compatible efficiency. The initial allocation is a rational expectations equilibrium after the arrival of the signals if the initial allocation is interim efficient. This is not necessary however: the only necessary efficiency condition is that it is ex post public efficient.

The paper provides a framework for looking at the interaction between heterogeneous prior beliefs and asymmetric information, relating "type space" and partition representations of uncertainty, and dealing with private and public information in a unified way. The paper gives necessary and sufficient conditions for "no trade" results such as Milgrom and Stokey (1982). The results are heterogeneous prior analogues of "no trade" results in Geanakoplos (1989) concerning agents with non-standard information processing.

"It is frequently said of two bettors that one is a thief and the other is an imbecile; this is true in certain cases, where one of the two bettors is better informed than the other, and knows it; but it can also happen that two men in good faith in complex situations where they possess exactly the same elements of information will arrive at different conclusions on the probabilities of an event and that betting together, each one figures... that he is the thief and the other the imbecile."

"It is difficult to imagine that the volume of trade in security markets has very much to do with the modest amount of trading required to accomplish the continual and gradual portfolio balancing inherent in our current intertemporal models. It seems clear that the only way to explain the volume of trade is with a model that is at one and the same time appealingly rational and yet permits divergent and changing opinions in a fashion that is other than ad hoc."

Section 1: Introduction

If all differences in posterior beliefs are explained by differences in information (and agents have common prior beliefs), and if agents' rationality is common knowledge, then differences in information alone cannot explain trade. Agents' willingness to trade would reveal enough of their private information to each other to preclude trade. On the other hand, if all differences in posterior beliefs derive from heterogeneous prior beliefs (and there is no asymmetric information), then agents should be prepared to bet with each other whenever they have different posterior beliefs.

In this paper, I consider the intermediate case where differences of posterior beliefs are explained by both differences in prior beliefs and asymmetric information. I can thus dispense with both the extreme results above. I can explain how sometimes agents may rationally not trade with each other, despite different prior beliefs and apparently different posterior beliefs. But other times, trade may occur despite agents making rational inferences from other agents' willingness to trade. With one important exception to be discussed below [Varian (1989)], the interaction between heterogeneous priors and asymmetric information has not been studied in this context.

Before giving an outline of the results in this paper, it will be useful to put this viewpoint in context by a review of the literature on the relation between information and trade, together with alternative ways of "getting around" the "no trade" result other than dropping the common prior assumption. I then summarize the main results in the paper, and explain how the framework is this paper unifies a number of earlier results in the literature, as well as providing new results.

Conventional accounts of day to day movements of the stock market take it for granted that the arrival of new information explains both price changes and the volume of trade. This is assumed to be
true both for private (asymmetrically distributed) information and for publicly released information - where trading volume presumably reflects a lack of consensus in the interpretation of the information. Both elements have been criticized in the economics literature: Harsanyi (1967/68) made a since popular argument that differences in posterior beliefs must be explained by some prior information, so therefore the common prior assumption must hold. This argument implies that "different interpretation" of public information is simply asymmetric private information in disguise. Meanwhile Hirshleifer (1971) made the argument that private information may inhibit, rather than encourage trade, in models where agents correctly take into account other agents' informational advantage. Aumann (1976) showed that if agents' posterior beliefs are derived from a common prior, common knowledge of those posterior beliefs implies common prior beliefs, a result that implies that the arrival of new information should result in changes in prices, but no trade. 

Since the arrival of new information surely does lead to trade, various tactics have been employed to "get around" the no trade results in explaining financial markets. One explanation is that, if markets are incomplete, the arrival of new information will lead to trades that could not be made, contingent on that information, ex ante (indeed, such re-trading can "complete" the market in some circumstances). But as the Ross quotation above notes, it is hard to believe that such re-adjustment, in the absence of differences in priors, can account for the volume of trade observed.

The most common modelling approach around the "no trade" results is to allow for some population of "noise" traders, whose behavior may be irrational, or at least determined by factors exogenous to the model. Allowing small numbers of such noise traders accounts for certain qualitative features of financial markets and provides a framework for empirical analysis of trading volume. One possible interpretation of such noise traders is that they may be expected utility maximizers who do not use the "true" prior. Thus one possible interpretation of this paper is that I am providing a theoretical framework for explicitly modelling the behavior of such noise traders, so that results can be given as a function of those "wrong" beliefs. But because I cannot think of a good way, empirically or theoretically (or philosophically, for that matter), to distinguish between those with the "true" prior and those with the "wrong" prior, all agents are treated symmetrically.  

4. Beaver (1968), discussing the relation between accounting releases and trading volume, notes "the economist's notion that volume reflects a lack of consensus regarding the price." 
5. Sebenius and Geanakoplos (1983) gave the no trade analogue of Aumann's no agreeing to disagree result. See Geanakoplos (1992) on the relation between the two kinds of results. 
8. In fact, there are typically two important asymmetries between noise traders and arbitrageurs in such models: first, the noise traders have "wrong beliefs" or unexplained objectives and, secondly, arbitrageurs anticipate what noise traders will do and not vice versa. Even if you are going to assume that some people have the wrong beliefs, it is still not clear why you should also assume that they are any less able to anticipate the behavior of those with different beliefs. 
9. As noted by Milgrom and Stokey (1982).
entails a further weakening called noisy beliefs. Suppose agent started off with consistent concordant beliefs, but some agent chose to ignore his signal. Could this difference in beliefs be exploited by a social planner? No, because the agent would be required to bet against his signal being correlated with payoff relevant events (as others believe it to be). But this would inevitably give him an incentive to lie. Yet another kind of difference in beliefs does not lead to inefficiency. Finally, for public efficiency only agents' beliefs about public and payoff relevant events, not private information, matter.

These nine efficiency concepts and the corresponding conditions on beliefs provide benchmarks for the no trade and rational expectations equilibrium results which follow. No common knowledge acceptance of trades depending on payoff relevant events is equivalent to interim public efficiency. There is a discussion in section 5 of weaker notions of acceptance of trade, while it is true that examples can be given where arbitrarily high orders of mutual knowledge of acceptance of trade are insufficient to preclude trade even under the common prior assumption, other weaker notions of acceptance are sufficient to preclude trade. 'Nash acceptance' of payoff relevant trades is shown to be precluded in this environment if the initial allocation is interim incentive compatible efficient, which is equivalent, in this environment, to Holmström and Myerson's (1983) "no trade" concept of durability.

Revolution of information in rational expectations equilibrium [REE] may or may not occur and may or may not be incentive compatible. The initial allocation will be a REE after the arrival of information if the initial allocation is interim efficient. It will not be a REE if the initial allocation is not ex post efficient. But examples show it is not possible to characterize REE more tightly in terms of the efficiency results of section 4.

An example in section 2, with two payoff relevant events, one uninformed agent and one agent who observes one of two signals, illustrates the main results of the paper.

The concluding section 7 contains some remarks on the applicability of this theoretical framework.

In the remainder of the introduction, I discuss how this paper builds on the considerable literature of "no trade" results.

An important early literature considered conditions under which changes in beliefs would lead to trade in competitive equilibrium (Rubinstein (1975), Vervicchia (1980, 1981)). Hakansson, Kunkel and Ohlson (1982) identify a condition they call "essential homogeneity of beliefs" which is necessary and sufficient to prevent trade in a competitive framework. Milgrom and Stokey (1982) showed that if all information is public, this same condition is necessary and sufficient to preclude common knowledge acceptance of trade. This condition is the same as public consistent concordance in this paper, and my result reduces to this earlier work if all information is public.

The uncertain environment in this paper is essentially that of Milgrom and Stokey (1982). Milgrom and Stokey considered private information also, and showed that with concordant beliefs (implying no incentive efficiency, as described above) common knowledge acceptance of trade and trade in a competitive environment is impossible. This paper, inter alia, provides necessary conditions on beliefs for the results in Milgrom and Stokey where only sufficient conditions were given.

Geanakoplos (1989) considers when trade occurs between agents who have common prior beliefs, but who "misinterpret" information. Each agent's information is represented by a "possibility correspondence" [PC] mapping $P_i: \Omega \to 2^/\Omega$, which does not necessarily satisfy the characteristic property of partitions, that $P_i(\omega) = \{\omega' \in \Omega \mid P_i(\omega') = P_i(\omega)\}$. Geanakoplos shows that common knowledge trade is precluded under weaker assumptions than (Nash) equilibrium trade. The result is of interest both because of the natural interpretations of the conditions for trade and because it clarifies the relationship between solution concepts. Brandeisburger, Dekel and Geanakoplos' (1989) showed that, for any decision problem facing a set of agents with common priors and possibility correspondences, there is a decision theoretically equivalent problem with heterogeneous priors and standard partitions. This paper gives the heterogeneous prior analogues of Geanakoplos' results.

Varian (1989) is the only other paper I am aware which explicitly examines the interaction between heterogeneous prior beliefs and asymmetric information in this context. He considers a model with a single normally distributed risk asset as in Grossman (1976). This framework has the advantage that it allows quantitative results about how the volume of trade is related to the heterogeneity of the interpretation of information. Because a competitive model is considered, and all information is revealed in equilibrium, this case becomes equivalent to the public information case.

Myerson, in a wide body of work, has characterized conditions for no trade in a mechanism design context. Because the effect of the uncertain environment introduced in section 3 is to reduce a many good exchange economy to a linear problem, the noisy condition on beliefs is a particular application of conditions identified more generally in Myerson's work.

10. The relation is explored in Morris (1991) chapter IV, where it is shown that by imposing restrictions on the priors of agents in this paper (most importantly, agents all believe that the distribution of their of their signals are independent of each other, conditional on verifiable events), I can show the Geanakoplos' result. Shin (1989) gives a dual interpretation of Geanakoplos' result which is related to the results in this paper.
Section 2: Example

In this section, an example with risk-neutral agents, one-sided private information and no public information is given to illustrate the main results in sections 3 to 6. At the end of the section, the role of these special assumptions in the example are discussed.

Suppose the Open Market Committee will meet tomorrow to decide whether the discount rate will go up [U] or down [D]. Art and Beth are having dinner and considering making a bet. But they both know that Art had lunch with a member of the committee, and he made a prediction as to whether the meeting would decide to go up or down (write "u" and "d" for his prediction; he didn't tell Art anything else). Thus uncertainty can be represented by a four-state space:

$$\Omega = \{ uU, uD, dU, dD \}$$

and Art has observed a partition of the state space:

$$P_A = \{ \{ uU, uD \}, \{ dU, dD \} \}$$

Art and Beth have different prior beliefs over that same space: write $q_a$ and $q_b$ for the ex ante probabilities they assign to prediction u; $a_u$ and $b_u$ for the probabilities they assign to the interest rate going up [U], conditional on a prediction that it will go up; and $a_d$ and $b_d$ for the probabilities they assign to the interest rate going up conditional on a prediction that it will go down. Thus their priors are:

$$\pi_A = \{ q_a a_u, q_a (1-a_u), (1-q_a) a_u, (1-q_a)(1-a_u) \}$$

$$\pi_B = \{ q_b b_u, q_b (1-b_u), (1-q_b) b_u, (1-q_b)(1-b_u) \}$$

Assume, without loss of generality, $a_u \geq a_d$, so that u is the signal that Art thinks correlated with U. Beth has not observed the prediction, so we will also be interested in Beth's ex ante assessment of the probability of U, $b^* = q_b b_u + (1-q_b) b_d$. Art's ex ante assessment of the probability of U is $a^* = q_a a_u + (1-q_a) b_d$.

The core of the paper (sections 4 to 6) discusses efficiency, no trade results, and competitive equilibrium in the trading environment to be introduced in section 3. It is useful to follow that structure in discussing this example. So I examine, first, what are the efficiency properties of the situation where they don't bet? Could a social planner propose a re-allocation which made both Art and Beth better off? Secondly, under what circumstances do they bet on whether U or D will occur, without the benefit of a third party to propose the re-allocation. Finally, when is a "no betting" situation supported as a competitive equilibrium? Of course, these three questions are intimately related, but the relation is a little subtle in the presence of asymmetric information.

2.1 Efficiency

An allocation is efficient if a social planner could not propose a feasible re-allocation that would make each agent better off. An allocation is interim efficient if there does not exist a re-allocation which makes each agent better off, whatever information he has observed. Under risk-neutrality, efficiency will depend only on agents' beliefs and not their endowments. The no betting situation is interim efficient if and only if agents' beliefs are consistent in the sense of Harsanyi (1967/68); agents' posterior beliefs could have been derived from a common prior, even if (in fact) they weren't. Thus in the example, beliefs are consistent if they agree on the probabilities of interest rate changes, conditional on the prediction, regardless of either agent's ex ante beliefs about the prediction; i.e. no trade is interim efficient if $a_u = b_u$ and $a_d = b_d$, regardless of the values of $q_a$ and $q_b$.

However, when agents have observed private information, the concept of interim efficiency requires us to permit re-allocations contingent on private information (in our example, contingent on what the prediction was). But suppose it is not possible for an outsider to verify what the prediction was. The following illustrates the incentive problem which arises.

Suppose agents have beliefs

$$\pi_A = \{ 1/3, 1/6, 1/6, 1/3 \}$$

$$\pi_B = \{ 3/8, 1/8, 1/8, 3/8 \}$$

In this case, beliefs are not consistent, because although they agree, ex ante, about the probability that a given signal (u or d) or a given interest rate outcome (U or D) will occur, Beth (who has not observed the signal) thinks that the signal is more closely correlated with the outcome than Art. Whichever signal (u or d) occurs, Beth thinks there is a 3/4 chance that it is accurate, while Art thinks there is only a 2/3 chance. Thus there exists a re-allocation which makes each agent better off, conditional on his information: for example, let Art pay Beth $2 if the prediction is correct (i.e. event \{ uU, dD \} occurs) and Beth pays Art $5 if the prediction is incorrect (i.e. event \{ uD, dU \} occurs).

But now if it was not verifiable what prediction had been observed, and re-allocations were made on the assumption that Art truthfully reported his signal, Art would have an incentive to lie about what signal (u or d) he had observed. If he tells the truth, his expected gain from the re-allocation described
above is 1/3 [ (1/3)5 - (2/3)2 ]. But he maximizes his expected gain by always lying (so that he gets paid when the prediction is true). Then his payoff is 8/3 [ (2/3)5 - (1/3)2 ].

Thus an allocation is said to be interim incentive compatible efficient (IICE) if there does not exist a re-allocation which makes all agents better off, whatever information they have observed, and which gives agents an incentive to truthfully report their signals. For risk-neutral agents, the no-betting situation is IICE if agents’ beliefs are noisy consistent. This means that not only do agents’ ex ante probabilities of observing their signals not matter, but also agents can ignore some of the information they have observed. In the example, beliefs are noisy consistent, and no betting is IICE, if and only if Alf - the informed agent - thinks that the signal is no more valuable than Beth - the uninformed agent - does, i.e. if

$$b_i \geq a_i \geq b^* \geq a_i \geq b_i^*.$$  

An alternative (and stronger) notion of efficiency is interim public efficiency: an allocation is interim public efficient if there does not exist a re-allocation which does not depend on agents’ private information (even in an incentive compatible way) which makes agents better off, whatever their private information. Re-allocation can depend only on non-information events (i.e. U and D) and public events, which are always known to all agents whenever they occur. In section 3, public events will be defined in terms of agents’ information partitions, so that there is an endogenous definition of what is private and public. In the example, there is no public information (Beth knows nothing). With risk-neutral agents, the initial situation is interim public efficient if agents’ prior beliefs are public consistent: each agent’s prior could be replaced by another prior with the same posterior beliefs, such that those replacement priors have the same beliefs about non-private information events. In the example, beliefs are public consistent as long as Beth’s probability of U (unconditional on Alf’s signal) is between Alf’s possible conditional probabilities, i.e.:

$$a_i \geq b^* \geq a_i.$$  

Let us finally note how these results change if ex ante rather than interim efficiency concepts are used. Now a requirement is imposed for the first time on Art’s ex ante beliefs. Conditions for efficiency, incentive compatible efficiency and public efficiency remain the same, except that it is now also required that agents’ ex ante probabilities of U and D are the same, i.e. a* = b*. Notice that it is because consistency and common ex ante probabilities of U and D imply common priors that the no betting situation is ex ante efficient only if agents have common priors.

### 2.2 No Trade

Efficiency notions underlie no trade results, but the relation is not straightforward. Let us return, then, to Alf and Beth having dinner and deciding whether to bet about whether U or D will occur, but now without the benefit of the social planner to implement the appropriate Pareto-improvements. Noise betting would occur if the agents agreed to bet about whether U or D would occur simply on the basis of their differences in posterior beliefs. Then betting would occur if signal U is observed and a_1 \neq b^* or if signal D is observed and a_2 \neq b^*, and thus could occur even if agents had common priors. A rational Beth, however, would want to take into account the information implicit in Art’s willingness to bet. At the very least, Beth should condition on the event that Art expects to gain from the bet. An example in Milgrom and Stokey (1982) showed that such "first order mutual knowledge" of rational acceptance is not enough to exclude betting even under the common prior assumption. It is necessary also that agents know that other agents are conditioning on other agents’ acceptance. An example in section 5 shows that trade can occur, with the common prior assumption holding, consistent with any finite higher order mutual knowledge of rational acceptance of trade. There is a weak sense, then, in which common knowledge of acceptance is a necessary condition for a no trade result under the common prior assumption. It is shown in section 5 (in fact, it follows directly from the definitions) that there is no common knowledge acceptance of trade if and only if the initial allocation is interim public efficient.

On the other hand, as emphasized by Geanakoplos (1989, 1992), there is an important sense in which common knowledge is far from necessary for no trade results. A weaker acceptance criterion is Nash acceptance. Could the agents bet if they were playing a game in which their strategies consisted of a decision of when to accept a trade and when to reject it? Decisions are made simultaneously and betting occurs only at those states where all agents accept in equilibrium. The interpretation of Nash equilibrium is that agents know other agents’ strategies but do not know whether they have actually accepted or not (and there is certainly no common knowledge of rational acceptance). In the example of this section, there is no Nash acceptance of any bet if and only if the no-betting situation is interim incentive compatible efficient. But, in other examples, there may be no Nash acceptance of any bet even when the initial allocation is not interim incentive compatible efficient. It is shown in section 5 that interim incentive compatible efficiency is a sufficient condition for no Nash acceptance of any trade, while interim public efficiency is a necessary condition.

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12. An additional requirement is that a_i = a_i implies b_i = a_i = b_i = b_i. 

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One interpretation of this result is that Nash acceptance of \( x \times y \) is a restricted class of all incentive compatible mechanisms depending on the entire state space (i.e. the signals \( u \) and \( d \) as well). Section 5 shows how no trade results for trades depending on private information as well are related to interim efficiency and interim incentive compatible efficiency.

2.3 Competitive Equilibrium

Consider the incomplete asset market economy where Art and Beth trade a riskless asset, paying $1 in every state, and a single risky security paying out $1 if event \( U \) occurs. Because of risk neutrality, competitive equilibria will not typically exist in the example. However, there are two cases when the no betting situation is a rational expectations equilibrium with incomplete markets (REEI) in this economy with asymmetric information. First, if beliefs are consistent and conditional beliefs differ, there is a fully revealing rational expectations equilibrium, where the price of the risky asset in terms of the riskless asset is \( a_u = b_u \) if \( U \) occurs and \( a_d = b_d \) if \( D \) occurs. Secondly, if the informed agent Art has common conditional beliefs equal to Beth’s unconditional probability of \( U \) \( [a_u = a_d = a_s = b_s = b_u] \), then there is a REEI with no revelation of information where the price is always equal to \( b_s \) (regardless of the values of \( b_u \) and \( b_d \)). Note that in the latter case, the no betting situation may be an REEI even though it is not even interim incentive compatible efficient. It is (as the general results of section 6 show it must always be) ex post public efficient.

2.4 Extension to Risk-Aversion, public and two-sided private information

This example leaves out three important aspects of the results in sections 4 to 6, and it is useful to highlight them now.

Firstly, this example has the characteristic that there is no non-trivial "public" information: \( \Omega \) is the only event which agents always know has occurred whenever it has occurred. An event can only be common knowledge in the example if it is always true.

Secondly, the information asymmetry in the example is one-sided. With many agents with different information, the distinction between interim and ex post efficiency becomes important.

Thirdly, the example treated risk-neutral agents. It is useful to consider the extension to risk-averse agents in this example. Suppose agents are risk averse expected utility maximizers, with partially state dependent utility functions: they may depend on whether \( U \) or \( D \) occur, but do not depend on whether \( u \) or \( d \) occur. "Information signals" will thus be "pure information" and not "payoff relevant". Now efficiency and other results depend on differences in utility functions and endowments motivating trade. We want to abstract from such considerations. So suppose that before Art had lunch and received his prediction, Art and Beth did not want bet about whether \( U \) or \( D \) would occur. Even if their initial probabilities of \( U \) and \( D \) differed, they must have traded to a point where relative marginal utilities on \( U \) and \( D \) are equated. Note that this "initially efficient" allocation is not necessarily ex ante efficient, because agents are not allowed to trade on the basis of different beliefs about information which has not yet arrived. Conditions for different notions of efficiency, and thus no trade and competitive equilibria, now depend only on beliefs and information. The difference from the general risk neutral case is that it is only agents’ beliefs conditional on the events \( U \) and \( D \) which matter, not their ex ante beliefs about \( U \) and \( D \) which are traded away prior to the arrival of the information.

Section 3: Pure Information Signals Framework

This section describes the model of an exchange economy with uncertainty and the dichotomy between payoff relevant uncertainty and pure information which drives results in this paper. The framework is essentially that of Milgrom and Stokey (1982). While this framework is very special in a technical sense, I will argue it captures what is important about information. Because the paper deals extensively with both incentive compatibility and common knowledge, it is useful to develop notation for both type space and partition representations of uncertainty together in this section. Because this structure is also of some independent interest, the discussion is in some detail.

There is a finite contractible uncertainty space \( Q \) (the significance of contractibility will be discussed below). There are \( H \) agents; I also write \( H \) for the set of agents. There are \( L \) commodities, and each agent has a convex, strictly increasing, twice differentiable utility function \( u_s: \mathbb{R}^L \times Q \rightarrow \mathbb{R} \). These properties for utility functions are stronger than required for most results. However, they enable all results in this paper to be given in terms of necessary and sufficient conditions that are identical for the case where utility functions are linear (implying risk-neutrality in the one good case) and utility functions with strict convexities. For the same reason, there is no non-negative consumption constraint imposed.

Suppose initially that there is an allocation of commodities contingent on contractible uncertainty \( Q \). Then agents observe pure information signals: it is assumed, firstly, that it was not possible to make contracts contingent on the new information before it arrived (thus \( Q \) was said to represent contractible uncertainty); and, secondly, that this information is not payoff relevant, so the utility functions defined above do not change. But it may be correlated with payoff relevant events. This new information is represented by each agent observing a "signal" \( t_s \in T_s \). The product of agents’ signals, \( T = T_1 \times \ldots \times T_L \),
thus incorporates all private information. I write \( t = (t_1, \ldots, t_n) \) for a typical element of \( T \). Bold type is used throughout the paper to represent vectors of agent characteristics.

Now all relevant uncertainty is described by \( Q = T \times Q \), with typical element \( \omega = (t, q) \). Each agent has a prior \( \pi_\omega : T \rightarrow \mathbb{R}_+ \). Each agent is assumed to be expected utility maximizer. Note that agents’ beliefs about events \( Q \) are arbitrary, since utility functions can depend on events in \( Q \). The results in this paper will depend only on the correlation between private signals and contriactible events. I wish to study the case where agents’ priors \( \pi \) differ. But I assume that these priors have common support i.e. agents at least agree about which combinations of private signals and contriactible events are possible. Thus there exists \( Q' \subset Q \) such that \( \{\omega \in Q \mid \pi_\omega (\omega) > 0\} = Q' \subset Q \), for all \( h \in H \).

This assumption suggests an unambiguous partition representation of uncertainty: define agents’ private information partitions, \( T_{Q, H} \), and contriactibility partition, \( Q \), on \( Q' \), by:

\[
T_Q(t, q) = \{ (t', q') \in Q' \mid t' = t \} \quad Q(t, q) = \{ (t', q') \in Q' \mid q' = q \}
\]

I can, without confusion, write \( Q \) and \( T \) also for the ranges of \( Q \) and \( t \), respectively.

Now the environment is completely described by uncertainty space \( Q' \), private information partitions \( T_{Q, H} \), contriactibility partition \( Q \) and strictly positive priors on \( Q', \pi \). I can introduce some additional notation: let partition \( I \) represent all the information the agents have observed: \( I \) is the join (coarsest common refinement) of the \( T_{Q, H} \). Let partition \( P \) represent the meet (finest common coarsening) of the \( T_{Q, H} \), which I will call the partition of public information for the following reason. An event is public if whenever it occurs, all agents know that it has occurred. It is natural to say that an agent \( h \) knows an event \( E \subset Q' \) if \( T_{Q, H}(E) \subset E \). It has been shown that an event is public if and only if it is the union of elements of \( P \), the range of the meet partition \( P \). Aumann (1976) showed that an event \( E \) is common knowledge at \( \omega \) (all agents know that all agents know etc.) \( E \) if and only if \( P(\omega) \subset E \).

What is the significance of the contriactibility partition? The formal significance is that while agents will be assumed only to be able to make trades contingent on events in \( Q \) – so markets are incomplete. The natural interpretation for this is that non-contriactible events are not verifiable ex post. One reason for this may be that private information represents agents’ hierarchies of beliefs, in the sense of Mertens and Zamir (1985) and Brandenburger and Dekel (1992). It must be assumed in what follows that agents’ priors and partitions are common knowledge. Aumann has pointed out that this need not be a restriction as long as \( Q' \) is understood to be a complete description of all relevant uncertainty: “it is not an assumption, but a theorem, a tautology; it is implicit in the model itself”.

What are the relative merits of the partition and type space representations of uncertainty? Many proofs (including some in this paper) can be more easily stated in the type space notation. But the partition representation has a number of advantages for my purposes. Firstly, having defined uncertainty in terms of the product space, it is less easy to state comparative statics results about changes in information structure, the existence of corrigible claims and, as I will discuss below, the payoff-relevance of agents’ signals. Secondly, the partition representation makes transparent the sometimes implicit assumptions about incomplete markets and payoff-relevance which drive results in the literature. Thirdly, the partition representation allows for easy comparison of different solution concepts. Fourthly, there are some concepts which are much easier to understand in the partition representation. For example, in the partition notation, we say that agent \( h \) knows event \( E \subset Q' \) if \( T_{Q, H}(E) \subset E \). In the type space notation, \( h \) knows event \( E \subset Q' \) if \( (t, q) \in E \) if \( (t, q, h^n_q) \in \bar{E} \) implies \( (t, q, h^n_q) \in E \), for all \( q \in T_h \). In the partition notation, we say \( Q \) is finer than each \( T_h \) if \( Q'(\omega) \subset T_h(\omega) \) for all \( h \in H \), \( \omega \in \Omega' \).

The equivalent claim in the type space notation is that \( (t, q) \in E \) implies \( (t', q) \notin E \) for all \( t' \neq t \). Finally, public information can be endogenously defined in terms of private information, clarifying the relation between the public and private information literatures as discussed in the introduction. On the other hand, the idea of incentive compatibility is much more easily described in the type space notation, as I will show in section 4.17.

13. Holmstrom and Myerson (1983) make this assumption in a type space representation of uncertainty. In Morris (1991) chapter IV, I relax this assumption and get results with the same interpretation. The main reason for making the assumption is the ambiguity in defining common knowledge without it.

14. Such events have been variously described as public [Milgrom (1981)], self-evident [Santet (1990)], or common truisms [Birnmore and Brandenburger (1990)].
It is useful to clarify the framework by making clear how I could have started with a partition representation as primitive and derived a type space representation. It is important to confirm that this can be done, since when I consider changing partitions on \( Q \), it will implicitly assume that the corresponding \( Q \) is also changing. Take any arbitrary finite uncertainty space \( Q \), any strictly positive priors \( \pi \), information partitions \( \{ T_1 \}_{k=1}^{\infty} \), and contractibility partition \( Q \). Let \( u : \mathbb{R}^1 \times Q \rightarrow \mathbb{R} \). Now define agent \( h \)'s "payoff-relevance" partition \( Z_h(\omega) = \{ \omega' \in Q | u_x(x,\omega') = u_x(x,\omega) \text{ for all } x \in \mathbb{R}^1 \}. \) Thus far the information partitions, the contractibility partition, and the payoff-relevance partitions are independent. You may have observed information which is not contractible and does not directly affect our ex post utilities (although it may be correlated with events that do). Tomorrow's share price may be contractible (through futures' markets), but we don't know what it will be, and it does not directly affect our ex post utilities. My utility function is payoff relevant for me (and not for you) and it may be uncontractible, and you don't know what it is.

In order to translate this back to the type space framework above, it must be assumed that the contractibility partition \( Q \) is finer than each \( Z_h \). In other words, it is assumed that it is possible to trade claims contingent on any payoff relevant event.

Now assume that the join of \( Q \) and the \( \{ T_k \}_{k=1}^{\infty} \) is the trivial partition of singletons. This is without loss of generality since if there were more than one state they would be indistinguishable in every way. Now \( Q \) can be constructed from the cross-product of the ranges of information and contractibility partitions.

Section 4: Efficiency

An allocation is said to be initially efficient in the framework of the previous section if it depends only on contractible events, and there does not exist another feasible allocation, also depending only on contractible events, which each agent prefers to it. This is thus a constrained notion of efficiency: initially efficient allocations are not typically efficient with respect to re-allocations contingent also on the information that agents will receive. Will this initially efficient allocation be efficient after the arrival of the information? Necessary and sufficient conditions are given in this section in terms of the heterogeneity of agents' beliefs.

Holmström and Myerson (1983) noted that two ambiguities about efficiency arise in the presence of asymmetries of information among agents. Firstly, if we imagine a social planner having to implement an efficient outcome, to what extent can outcomes be made contingent on the asymmetries of information? The revelation principle tells us that attention can be restricted to outcomes whose dependence on the asymmetric information is incentive compatible, so that agents have an incentive to tell the truth about their information. Secondly, should we consider agents' utility before receiving their information (ex ante), after receiving their private information but before learning others private information - if indeed they ever do (interim), or as a function of all private information (ex post)? This generates six notions of efficiency based on three different (implicit or explicit) social welfare functions, and whether or not incentive compatibility is imposed.

In this paper, it is useful to consider also another informational constraint on the social planner: outcomes can only be made contingent on public information - events which everyone knows and it is common knowledge that everyone know. This requirement is strictly stronger than incentive compatibility, as is made clear by a theorem in Holmström and Myerson (1983). Ex ante public efficiency, interim public efficiency and ex post public efficiency can be added to the toolbox of efficiency concepts. In section 5, public efficiency will be shown to underlie no common knowledge of trade results.

It is useful to explore all these concepts of efficiency in this section because these efficiency results provide a framework within which no trade results (section 5) and competitive equilibrium results (section 6) can be easily understood. In addition, this form of presentation shows how each component of the definitions of efficiency is related to a particular weakening of the common prior assumption [CPA]. Interim efficiency entails weakening the CPA to require only consistency of beliefs - agents' prior beliefs about observing their own signals no longer matter. Ex post efficiency entails weakening the CPA to require only revelation consistency - agents' prior beliefs about all agents' signals no longer matter. Incentive compatible efficiency entails weakening the CPA to require only noisy agreement among agents - agents may underestimate the significance of their own signals relative to how others perceive them. Public efficiency entails weakening the common prior assumption to require agreement only about public and contractible events. All forms of efficiency entail the concordance weakening of the CPA - agents' prior beliefs about contractible events do not matter.

16. It is natural, though not formally necessary, to assume that \( P_1 \) refines \( Z_h \) (agents know their own utility functions).


20. Wilson (1978) compares "fine efficiency" and "course efficiency" which correspond to ex ante efficiency and ex ante public efficiency in this framework.
4.1 Efficiency Definitions

If $\Omega$ has $N$ elements, an allocation or endowment is an element $e \in \mathbb{R}^N$. It is useful to represent $e$ as a collection of mappings (one for each agent) from states of the world into goods, i.e., $e = (e_1, \ldots, e_N)$, each $e_i: \Omega \to \mathbb{R}^N$. Notice that allocations are defined at zero probability states. This simplifies arguments, but does change any results. As promised in the previous section, definitions will initially be given in both partition and type space notation in order to help fix the relationship.

Definition $y$ ex ante dominates $e$ if (equivalently)

$$\sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(y(\omega), Q(\omega)) \geq \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\omega), Q(\omega)), \text{ for all } h \in H.$$  
$$\sum_{\iota \in \mathbb{T}} \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(y(\iota(\omega)), Q(\omega)) \geq \sum_{\iota \in \mathbb{T}} \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\iota(\omega)), Q(\omega)), \text{ for all } h \in H.$$  
with strict inequality for some $h \in H$.

Allocation $e$ is **contractible** if each $e_i$ is measurable with respect to $Q$. Allocation $x$ is a trade if $\Sigma_{\iota \in \mathbb{T}} e_{\iota} = 0$.

Definition $e$ is an **initially efficient** allocation if $e$ is contractible and there does not exist a contractible trade $x$ such that $e + x$ ex ante dominates $x$.

Thus $e$ is an initially efficient allocation if it is constrained ex ante efficient with respect to contractible trades only; equivalently, if it is a competitive equilibrium of the original economy with $Q$ only as the uncertainty space; equivalently, if it is a competitive equilibrium with incomplete markets of the full economy with only assets coming on contractible events traded.

I will be asking whether such initial efficient allocations are efficient allocations (this section), allocations in which no trade occurs (section 5) or competitive equilibrium (section 6) after the arrival of private and public information. There are a couple of quite different motivations for asking this question. There is the "real-time" interpretation usually given in the literature. When does the actual arrival of information lead to trade. But there is also the counterfactual question: when is it the case when there is trade in the presence of private information, when in the absence of private information, would there have been no trade? The way to answer both questions is to look at initially efficient allocations.

**Definition** $y$ interim dominates $e$ if (equivalently)

$$\sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(y, Q(\omega)) \geq \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\omega), Q(\omega)), \text{ for all } e \in \mathcal{C}_{\mathbb{T}}, h \in H.$$  
with strict inequality for some $e \in \mathcal{C}_{\mathbb{T}}(\iota \in \mathbb{T}), h \in H$.

**Definition** $y$ ex post dominates $e$ if (equivalently)

$$\sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(y(\omega), Q(\omega)) \geq \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\omega), Q(\omega)), \text{ for all } e \in \mathcal{C}_{\mathbb{T}}, h \in H.$$  
with strict inequality for some $e \in \mathcal{C}_{\mathbb{T}}(\iota \in \mathbb{T}), h \in H$.

**Definition** $e$ is **incentive compatible** if (equivalently)

$$\sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\omega), Q(\omega)) \geq \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\omega), Q(\omega)).$$  
where $e(\omega) = n_{\Omega}(e(\omega), \mathcal{T}_e(\omega)) Q(\omega)$, for all $\omega, e \in \mathcal{C}$, $h \in H$.

$$\sum_{\iota \in \mathbb{T}} \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(y(\iota(\omega)), Q(\omega)) \geq \sum_{\iota \in \mathbb{T}} \sum_{\omega \in \Omega} \pi_{\Omega}(\omega) n_{\Omega}(e(\iota(\omega)), Q(\omega)), \text{ for all } h \in H.$$  
with strict inequality for some $h \in H$.

Note that incentive compatibility is confusing to define in partition notation, and would be even more so without referring to the impossible states in $\mathcal{T}_e$.

**Definition** $e$ is public if each $e_i$ is measurable with respect to the join of public events $P$ and contractible events $Q$.

**Definition** $e$ is (ex ante) interim [ex post] [incentive compatible] <public> efficient if $e$ is (incentive compatible) <public> and there does not exist a trade $x$ such that $e + x$ is (incentive compatible) <public> and (ex ante) interim [ex post] dominates $e$.

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21. It is a thus a special case of this exchange economy with incomplete markets that general equilibria with incomplete markets [Gel] are constrained efficient (unlike in the general case, see Economopoulos and Polemarchakis (1980)).
4.2 Properties of Beliefs

Let $\Pi$ be the set of probability distributions on $\Omega^\prime$ which assign positive probability to public events $\omega^*$. Let $\Pi = \{\pi: \Omega^\prime \rightarrow [0,1] | \sum_{\omega \in \Omega^\prime} \pi(\omega) > 0 \}$.

Let $\Pi^\prime$ be the set of strictly positive probability distributions on $\Omega^\prime$. Thus $\Pi^\prime \subset \Pi$.

Thus agents' beliefs $\pi = (\pi_1, \ldots, \pi_n) \in \Pi^\prime$ is $\Pi^\prime$.

Define $C: 2^{\Omega^\prime} \rightarrow 2^{\Pi^\prime}$ by:

\[ C(\Theta) = \{\pi \in \Pi | \pi_i(\omega) = \theta_i(\omega) | \forall \omega \in \Omega^\prime, \forall \theta \in \Theta, \forall h \in H \} \]

Agent beliefs $\pi$ are consistent if there exists $\psi \in \Omega^\prime$ such that $(\psi, \ldots, \psi) = \psi^\prime \in C(\pi)$. Harsanyi (1967/68) introduced the notion of consistency in the context of games with imperfect information. Consistency requires that agents' posterior beliefs could have been derived from a common prior even if in fact they weren't. Note that Harsanyi's notion of consistency is defined using the mapping $C$ when $\Theta$ is a singleton, whereas to combine it with other weakenings of the common prior assumption, I will need a mapping on sets of probability distributions.

Define $R: 2^{\Pi^\prime} \rightarrow 2^{\Pi^\prime}$ by:

\[ R(\Theta) = \{\pi \in \Pi | \pi_i(\omega) = \theta_i(\omega) | \forall \omega \in \Omega^\prime, \forall \theta \in \Theta, \forall h \in H \} \]

Agent beliefs $\pi$ are revelation consistent if there exists $\psi \in \Omega^\prime$ such that $\psi^\prime \in R(\pi)$. Beliefs are revelation consistent if they would be consistent if all private information was publicly revealed. Thus consistency implies revelation consistency, but not vice-versa. In the example of section 2, revelation consistency and consistency were equivalent, because there was one sided private information. But consider another example:

\[ \Omega^\prime = \{a, b, c, d\} \]
\[ \pi_1 = (1/4, 1/4, 1/4, 1/4) \]
\[ P_1 = \{a, b\}, \{c, d\} \]
\[ Q = \{a, b, c, d\} \]

Here the join of agents' private information is fully informative $J(\omega) = [\omega]$, for all $\omega \in \Omega^\prime$, so beliefs would be revelation consistent, whatever the priors. But agents disagree about the unconditional probability of every event in $\Omega$ that it is possible to disagree about 22. To see why agents' beliefs are nonetheless consistent, consider prior $\psi = (2/5, 2/5, 1/10, 1/10)$. Each agent's beliefs, conditional on his prior $T_i$, would be the same if his prior was replaced by $\psi$.

Define $I: 2^{\Theta} \rightarrow 2^{\Pi^\prime}$ by:

\[ I(\Theta) = \{\pi \in \Pi | \pi_i(\omega) = \theta_i(\omega) | \forall \omega \in \Omega^\prime, \forall \theta \in \Theta, \forall h \in H \} \]

Beliefs $\pi$ are concordant if there exists $\psi \in \Pi$ such that $\psi^\prime \in I(\pi)$. Hakansson et al. (1982) called this property homogenous beliefs in a closely related context. Milgrom and Stokey (1982) called it concordance. It captures the idea that it is the interpretation of new information which matters, not agents' beliefs about events they have already been able to make contracts contingent on.

Define $N: 2^{\Pi^\prime} \rightarrow 2^{\Pi^\prime}$ by:

\[ N(\Theta) = \{\pi \in \Pi | \pi_i(\omega) = \theta_i(\omega) | \forall \omega \in \Omega^\prime, \forall \theta \in \Theta, \forall h \in H \} \]

Beliefs $\pi$ are noisy if there exists $\psi \in \Pi$ such that $\psi^\prime \in N(\pi)$. The intuition behind noisy beliefs is that agents are more conservative in interpreting their own signals than other agents are. They update their beliefs in the same direction as other agents, but not as much as they do. This is exactly what the example of section 2 illustrated. If we think of $\psi$ as a "true prior," such behavior is consistent with some early findings in the psychological literature on choice under uncertainty; in simple urn problems, agents were indeed found to use information to update their posteriors in the right direction, but less than Bayes rule would imply 23.

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22. They agree about the probability of the empty set and universal set.
23. See Phillips and Edwards (1966) and Slovic and Lichtenstein (1971). On the hand, more recent empirical work is more ambiguous: the "representativeness" heuristic of Kahneman and Tversky (see Kahneman, Slovic and Tversky 1982 part II) suggests that agents' overvalue information which is felt to be more "representative", regardless of the numerical implications of Bayes law. If the new information is representative, agent beliefs will not be noisy. On the other hand, if background information (reflected in the prior) is felt to be more representative than new information, beliefs will be noisy.
Define $A : 2^n \rightarrow 2^\mathbb{N}$ by

$$A(\theta) = \left\{ \pi \in \Pi \mid \sum_{\omega \in \Omega^*} \pi_\omega = \sum_{\omega \in \Omega^*} \theta_\omega, \text{ for all } \omega \in \Omega^*, \ h \in H \right\}$$

Beliefs $\pi$ are public if there exists $\psi \in \Pi$ such that $\phi \in A(\psi)$. Beliefs are public, or in public agreement, if agents agree on the probability of public and contractible events. If beliefs are noisy, they must be public. To see why algebraically, notice that the $\mu$ terms disappear in the definition of $N$ if you sum across the set of $\eta$, contained in some public event. Intuitively, incentive compatibility constraints cannot bind between public events.

Concordance, consistency, and noisiness represent three independent kinds of weakening the common prior assumption. Revelation consistency is strictly weaker than consistency and public is strictly weaker than noisy. These five weakenings of the common prior assumption will be combined to give the efficiency results which follow. The concordance weakening is always going to be in force if we consider initially efficient allocations (prior differences of opinion about contractible events are going to be traded away at the margin before information arrives). The consistency and revelation consistency weakenings correspond to interim and post efficient concepts, respectively. Prior probabilities of observing your own signals, or anyone's signals, are going to be irrelevant for interim and post efficiency, respectively. The noisy weakening is always going to be force when incentive compatibility is in force. The social planner is not able to exploit differences in beliefs due to the noisy weakening in making ex anto / interim / ex post utility Pareto-improvements because to do so he would have to reward agents if their signals have less than the impact implied by the common prior assumption. But this would inevitably give them incentives to lie about their signals. Finally, the public beliefs weakening corresponds to the public efficiency concept because with only public re-allocations of $Q$ contingent contracts available, only beliefs about public and contractible events matter.

These weakenings can be simply combined. Beliefs are consistent concordant if there exists $\psi \in \Pi$ such that $\phi \in C(\psi)$. In other words, there exist $\psi \in \Pi$ and $\pi' \in \Pi$ such that $\phi \in C(\psi)$ and $\pi' \in C(\pi')$. Beliefs are noisy consistent concordant if there exists $\psi \in \Pi$ such that $\phi \in N(\psi)$. The same terminology applies for any combination of properties. What do such combinations imply about beliefs? Consistent concordance of beliefs implies that agents' prior probabilities of observing the signals they have observed, or of contractible events, do not matter. But any two agents must agree about the correlation between their signals and contractible events, and they must agree about the probability of a third agent observing a certain signal, conditional on their own signals and verifiable events. Rather than giving detailed motivation of each combination of these properties, however, I believe the best intuition about combining properties comes from thinking of them as the sum of each of the different kinds of weakening.

**Theorem 1 (Efficiency):** The following are true of any initially efficient $\phi$:

1. $\phi$ is ex ante (public) [incentive compatible] efficient if and only if beliefs are (public) [noisy] consistent concordant.
2. A sufficient condition for $\phi$ to be interim (public) [incentive compatible] efficient is that beliefs are (public) [noisy] consistent concordant. A necessary condition for $\phi$ to be interim (public) [incentive compatible] efficient is that beliefs are (public) [noisy] weakly consistent concordant.
3. $\phi$ is ex post (public) [incentive compatible] efficient if and only if beliefs are (public) [noisy] revelation consistent concordant.

All proofs in the paper are in an appendix. To state the theorems in more readable form: all forms of efficiency are entail the concordance weakening of the common prior assumption; interim efficiency entails the consistency weakening; ex post efficiency entails the revelation consistency weakening; incentive compatible efficiency entails the noisy weakening; and public efficiency entails the public weakening.

There are two caveats to that summary: conditions for ex post incentive compatible efficiency are more complex to state in the language of weakenings, and were excluded from the theorem since this efficiency concept is not required elsewhere in the paper. For the interim efficiency concept, it turns out that a slight weakening of consistency is required to get a necessary condition for efficiency. This has to do with the shape of the utility functions: if utility functions are strictly concave, the necessary condition is also sufficient. If utility functions are linear (and thus agents are risk-averse), the sufficient condition is also necessary.

So now the weak consistency condition is described, and an example is given which illustrates both the theorem and gap between necessary and sufficient conditions.

**Definition** Define $C_m : 2^n \rightarrow 2^\mathbb{N}$ by:

$$C_m(\theta) = \left\{ \pi \in \Pi \mid \pi_\omega | T_m(\omega) = \theta_\omega | T_m(\omega), \text{ for some } \theta \in \theta, \text{ for all } \omega \in \Omega^* \text{ such that } \theta_\omega | T_m(\omega) > 0 \text{ and } \pi_\omega | T_m(\omega) > 0, \ h \in H \right\}$$

24. Lemma 2 in the proof in the appendix does give conditions.
an event \( E \) is common knowledge at \( \omega \) if and only if there exists a public event \( F \) such that each agent knows \( F \) and \( F \in E \).

Would each agent act? Each agent knows that each agent knows that no agent acts. (n times), or each agent is accepting the trade? Formally, say that trade \( x \) is nth order mutual knowledge accepted at \( \omega \) if \( \omega \in A(x) \cap K\alpha(x) \). A trade is common knowledge accepted at \( \omega \) if it is n-th mutual knowledge accepted for all \( n \), or, equivalently, if \( \omega \in A(x) \) and \( P(\omega) \in A(x) \).

The following example, whose information structure is closely related to that in the co-ordinated attack problem (Gray (1976), Rubinstein (1989)) illustrates why even with the common prior assumption, even nth order mutual knowledge is insufficient to preclude trade.

**Example**

\[ \Omega' = \{1, 2, \ldots, 2N\} \quad \mathbb{N} \in \mathbb{N} \]

\[ \pi_1 = \pi_2 = \{1, 2N\} \]

\[ P_i = \{ \{1, 2\}, \{3, \ldots, \}, \{2N-1, 2N\} \} \]

\[ x_i = -x_i = (1, 1+e, -(1+2e), \ldots, \{1+(2N-2)e\}, \{1+(2N-1)e\} \text{ for some } 0 < e < 1/N \]

Observe that the set of states where the trade is weakly accepted equals the set of states where the trade is accepted, \( A(x) = A(x) = \{1, 2, 3, \ldots, 2N\} \). \( K\alpha(x) = \{e+2, e+3, \ldots, 2N-1\} \). Thus there is nth order knowledge of acceptance at state \( N \) and state \( N+1 \) and if only if \( N \leq 2N \). Thus there is always nth order acceptance of trade \( x \) at any \( \omega \in \Omega' \), for any \( n \), for \( N \) sufficiently large.

In this example, and, we will see, in general, common knowledge of acceptance is sufficient (under the CPA) to preclude trade. Is common knowledge of acceptance necessary for the CPA no trade result? This is certainly not the case. In many economic situations, you may have to decide whether to accept a trade without knowing whether other agents have accepted. You decide simultaneously. What matters is that you condition on the fact that the other agent is pursuing a rational acceptance rule, not that you know if he has accepted. One way to model this is to consider the game where each agent's strategy specifies whether he accepts or rejects the trade as a function of his information. This occurs only if all agents accept, so in the Nash equilibrium of that game, agents are conditioning on other agents' acceptance. Thus the strategy space \( S \) and payoffs \( v \) of the game are defined by:

\[ v(A) = \sum_{n=1}^{N} \sum_{q=1}^{2N} \pi_n(t,q) w_n(t,q) + \sum_{n=1}^{N} \sum_{q=1}^{2N} \pi_n(t,q) w_n(t,q) \]

where \( A = (A_1, \ldots, A_N) \in X_{n \in N} S_i \) represent "actions" or "acceptance sets" of the agents.

Each agent \( h \) chooses a mixed strategy \( v_h \in \Delta(S_h) \). Trade \( x \) is Nash accepted if there exists a Nash equilibrium of the game, \( v \in X_{n \in N} \Delta(S_n) \), such that for some \( v_i \in \Omega', \) for all \( h \in H, \pi_h(A_h) > 0 \) for some \( A_h \) such that \( t_i \in A_h \).

The following example illustrates this acceptance rule and the difference from common knowledge acceptance. Agents are risk neutral and there is a single good.

**Example**

\[ \Omega' = \{a,b,c,d\} \]

\[ Q = \{a,b,c,d\} \]

\[ x_i = \{1/6, 1/3, 1/3, 1/6\} \]

\[ P_i = \{a,b,c,d\} \]

\[ A_i^* = \{a,b\} \]

\[ x_i = (-1, 1, 1, 1) \]

Here not only is the trade \( x \) shown not common knowledge accepted, but nor is any trade which is contractible (measurable with respect to \( Q \)). The agents here have diametrically opposite interpretations of the information: agent 1 thinks that observing \( a, b \) makes \( a, c \) less likely, while 2 thinks it makes \( a, c \) more likely. So there cannot be common knowledge acceptance of a trade where agent 1's private information becoming public. However, \( (A_i^*, A_i^*) \) form a pure Nash equilibrium.

**Lemma (Acceptance Rules)**

1. Common knowledge acceptance implies Nash acceptance and nth order mutual knowledge acceptance.
2. nth order mutual knowledge acceptance implies naive acceptance.
3. \( e + x \) interim dominates \( e \) implies common knowledge acceptance everywhere.

**5.2 Static no trade results**

The acceptance rules apply for a given trade. The next question to ask is whether there exists a trade of a given type that is accepted under one or all of the acceptance rules. It is natural, for
comparison with the initial state, to consider trades which depend only on contractible events, not on agents' information.

To characterize naive acceptance, I need additional properties on beliefs. These conditions say when the private (i.e. not public) part of an agent's information does not change his posterior beliefs.

**Definition**  Private information is *valueless* if \( \pi_0[Q(\omega') \cap T_{\omega}(\omega') \cap T_{\omega}(\omega') \cap Q(\omega')] = \pi_0[Q(\omega')] \cap T_{\omega}(\omega') \cap T_{\omega}(\omega'), \) for all \( \omega' \in \Omega, \omega' \in P(\omega) \) and \( h \in H. \)

Notice that this is the first property of beliefs considered in this paper which can fail even if the common prior is satisfied. It requires that the private part of agent's signals does not change their posterior beliefs about contractible events.

**Theorem 2 (No Trade)** The following are true for any initially efficient allocation:
1. There is no common knowledge accepted contractible trade if and only if the allocation is interim public efficient.
2. There is no Nash accepted contractible trade if the allocation is interim incentive compatible efficient.
3. There is no na"ive accepted contractible trade if and only if the allocation is interim public efficient and private information is valueless.

Morris (1991) gives an example that shows that interim incentive compatible inefficiency does not imply the existence of a contractible trade which is Nash accepted.

Notice that if private information is valueless, and the CPA holds, there is no trade. This is true even for naive acceptance. It is only when private information is valuable that anything stronger than naive acceptance is needed to preclude trade.

What happens if trades are allowed to depend on private information? Morris (1991) (chapter II) shows that in this case there exists an (incentive compatible) trade which is Nash accepted or common knowledge accepted if the initial allocation is interim (incentive compatible) efficient. Holmstrom and Myerson (1983) proposed the notion of *durability* to capture the idea of allocations from which agents could not agree to trade. This is essentially the same as Nash acceptance of incentive compatible trades. For durability, however, agents decide simultaneously whether to accept and what signal to send. But in this environment, that will not make any difference. Because the initial allocation does not depend on private information, durability is equivalent to interim incentive compatible efficiency.

5.3 Dynamic no trade results

The above results asked whether there was trade, for given information structures, and necessary sufficient conditions for no trade are given in terms of those information structures. But we imagine that information structures change during the course of the trading process (agents learn). What results can be given which are independent of the final information? In the example above, where there was Nash acceptance of trade, but not common knowledge acceptance, it was clear that in fact the uninformed agent would become informed during the course of trading.

One approach is to explicitly model the process by which information becomes revealed. But it is also possible to draw some general conclusions for all dynamic processes.

First, notice that there is no Nash or common knowledge acceptance of trade for every information structure if and only if beliefs are concordant (i.e. the initial allocation is ex ante efficient). If not, it would always be possible to give the agents information which exploits the difference in their beliefs. Thus Milgrom and Stokey’s (1982) sufficient condition for no trade is in fact necessary for no trade, if the result must hold true for all information structures.

But suppose agents start with information \( T_{\omega}^{(h)} \). Without explicitly modelling the learning process, we can impose natural restrictions on what they can learn. Say that \( T_{\omega}^{(h)} \) is a *feasible revelation* if \( T(\omega) \subseteq T(\omega), \) for all \( \omega \in \Omega, h \in H. \) Agents can only learn things which other agents know, and they cannot forget things they knew before. Say that \( T_{\omega}^{(h)} \) is a *feasible public revelation* if, for some partition \( R \) over \( j, T(\omega) = T(\omega) \cap R(\omega), \) for all \( \omega \in \Omega, h \in H. \) Now agents can only learn things which other agents know, they cannot forget things they knew before and if one agent learns something he did not know before, then all agents learn it.

With these notions, we can extend the idea of common knowledge acceptance. When is it the case, for some given initial information, that trade occurs with common knowledge of acceptance at the end of some feasible revelation process? A number of results follow directly from the static framework:
1. If \( e \) is an interim efficient allocation, there can be no common knowledge acceptance of contractible trades for any feasible revelation, because \( e \) will always be interim public efficient after any feasible revelation.

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2. If there is Nash acceptance in pure strategies of some contractible trade, then there is common knowledge acceptance of that trade after some feasible revelation. That feasible revelation could simply be that agents were publicly told when the intersection of their acceptance sets were true. This result makes clear why in the example above illustrating the difference between Nash and common knowledge acceptance, common knowledge acceptance would occur eventually.

3. If there is common knowledge acceptance of some contractible trade after some feasible revelation, then there is common knowledge acceptance of that trade after a feasible public revelation.

Section 6: Competitive Outcomes

Efficient and "no trade" allocations are closely related to competitive equilibria, although asymmetric information also complicates this relationship. For completeness, I briefly review the connection in this framework.

Suppose competitive markets open after information is observed. There are incomplete markets in the sense that only bundles of commodities contingent on contractible events can be traded. Agents are able to infer information from prices. Then I have:

Definition: Allocation e is a rational expectations equilibrium with incomplete markets (REE1) if there exists a price vector \( \pi : \Omega \rightarrow \mathbb{R}^n \), such that there does not exist an allocation \( x \), each \( x \) measurable with respect to the join of h's information partition \( \mathcal{T}_h \), the contractibility partition \( Q \) and information partition revealed by prices \( S \); such that \( s(\omega) = \{ \omega' \in \Omega | s(\omega') = s(\omega) \} \); and such that \( e + x \) ante dominates \( e \) and

\[
\sum_{w' \in \mathcal{G}(\omega) \setminus \mathcal{Q}(\omega)} \frac{e(\omega')x(\omega')}{\mathcal{Q}(\omega')} \leq 0, \text{ for all } \omega \in \Omega', \ h \in \mathcal{H},
\]

where \#(C) is the number of elements of set C; this is to avoid double counting.

Theorem 3 (Rational Expectations Equilibria) The following are true of any initially efficient allocation \( e \): (1) e is a RE1 if e is interim efficient; (2) if e is RE1, then e is ex post public efficient.

29. The restriction to pure strategies is binding. Morris (1991) gives an example where there is no pure Nash acceptance, but there is mixed Nash acceptance.

Efficiency is defined here with respect to the original information partitions.

It is not possible to give tighter characterization of the relation between RE1 and efficiency. In the example of section 2, an example was given with a (non-revealing) RE1 which was not incentive compatible interim (or ex post) efficient. Interim public efficiency is not sufficient for e to be a RE1 since it may not be possible to exploit interim gains from trade before revelation of prices.

It would be possible to give a tighter characterization of RE1 in terms of "public revelations of information" as discussed in section 53. e is an RE1 if there exists a public revelation of information of some partition \( S \) such that after the revelation, there is no naïve acceptance of contractible trade, and that partition \( S \) is informative about \( Q \) (conditional probabilities of events in \( Q \) vary across events in \( S \)).

It is useful to compare this notion of competitive equilibrium with others. If agents did not learn from prices, and we looked at the naïve competitive equilibria after the arrival of information, e would be an equilibrium if there was no naïve acceptance of contractible trades, as discussed in the previous section.

It is possible to use first order conditions to characterize how, given an initially efficient allocation, prices change, independently of endowments and utility functions, after the arrival of new information. Such results are given in Milgrom and Stokey (1982) for the case of concordant beliefs, and by Varian (1989) in a more special model with more general heterogeneous prior beliefs.

Section 7: Applications and Conclusion

"Anything can happen if agents have heterogeneous prior beliefs." This is the ultimate, pragmatic reason for accepting the common prior assumption. One contribution of this paper has been to identify many differences in beliefs which do not lead to trade, so that perhaps not quite anything can happen. It might equally be argued that "anything can happen if agents have heterogeneous utility functions and endowments"; yet this argument is not universally accepted as a justification for representative agent models. I believe that what really underlies suspicion of the common prior assumptions is that (unlike for other heterogeneity of utility functions and endowments) there are no standards for what constitutes reasonable heterogeneity to posit in explaining a given phenomenon. The challenge is to develop such standards.

29. For discussions of the common prior assumption, see Aumann (1967), Bergstrom (1986), Morris (1991) Chapter V.
Consider first public information. It is differences in interpretation of information that lead to trades contingent on public information (a lack of public consistent concordance). There are some classes of information where past experience of such information might be expected to have a lead to uniformity of interpretation (actuarial table, accounting data about old firms in stable industries), whereas there may not be sufficient experience of other types of information to have generated common interpretation (the break-up to the Soviet Union, new firms in industries using new technologies). We expect to observe more trading volume following the release of the latter kind of information.

One problem with this argument is that even if it is not the case that all differences in posterior beliefs are explained by information, it is presumably typically the case that I may at least want to take into account your different interpretation of some public announcement before making my judgement. As long as I want to change my posterior beliefs in the light of your posterior beliefs, your posterior beliefs have some information content. In other words, our analysis must always take into account the presence of some private information, even if it is only implicit as in this case.

The results of this paper showed how the "different interpretation of information" explanation extends to the case of private information. Common knowledge acceptance of trade requires differences in interpretation of public information. But under natural weaker acceptance rules for trade (sections 5.1 and 5.2), under dynamic mechanisms leading to common knowledge (section 5.3) and rational expectations equilibrium (section 6), difference in beliefs about initially private information can to trade. The equivalence between the noisy weakening of a common prior assumption and incentive compatible efficiency (section 4) suggests that private information leads to trade when agents overvalue their own information, but not vice versa.

31. Varian (1989) has a discussion of the "credibility" or implicit information content of agents beliefs.


Simpson J (1990) Beliefs, information and trading intensity. M.I.T.


Appendix: Proofs

Theorem 1 (Efficiency)

Lemma 1 (Farkas’ Lemma) Suppose I and 2 are finite sets and \( C \) is a collection of subsets of I (i.e. C is a subset of the power set of I, \( C \subseteq 2^I \)). Then there exist real numbers \( \lambda_C \) solving

\[
\sum_{i \in C} \alpha_i x_i \leq 0, \text{ for all } C, \text{ strict for all } I \in C, \text{ for some } x \in C.
\]

iff there exist \( \lambda = (\lambda_C) \geq 0 \) such that

\[
(1) \sum_{i \in C} \lambda_i x_i = 0, \text{ for all } j \in J, \text{ and (2) } \sum_{i \in C} \lambda_i > 0, \text{ for all } J \subseteq C.
\]

Proof A standard argument shows this is true when \( C = 2^I \) is singletons. So for each \( S \subseteq C \) there exists \( \lambda(S) \) satisfying (1) and \( \lambda(S) > 0 \) for some \( I \in C \). Now \( \lambda = \lambda(C(S)) \) satisfies the conditions above.

The proof of theorem 1 will make extensive use of the betting case, when \( C = 2^I \) is only one good (\( I = 1 \)) and each agent is risk averse (\( \lambda_1(S) = \lambda, \text{ for all } x \in R, \text{ and } \theta \subseteq \emptyset \)).

Lemma 2 (Betting Case: Efficiency - linear-algebraic representation) The following is true of any allocation \( \rho \) in the betting case:

\( \rho \) is ex post efficient iff there exists \( \theta \subseteq \emptyset \) and, for each \( h, \lambda_T \in R \), such that

\[
\lambda_T(1) \rho(1) = \theta(1), \text{ for all } (1,1) \in C, \text{ for } \theta, \text{ heft.}
\]

\( \rho \) is ex post incentive compatible iff there exists \( \theta \subseteq \emptyset \) and, for each \( h, \lambda_T \in R \), such that

\[
\lambda_T(1) \rho(1) = \sum_{u \in C(\rho)} \mu(u) \rho(1)(u) - \sum_{u \in C(\rho)} \mu(u) \rho(1)(u) = \theta(1), \text{ for all } (1,1) \in C, \text{ heft.}
\]

\( \rho \) is ex post public efficient iff there exists \( \theta \subseteq \emptyset \) and, for each \( h, \lambda_T \in R \), such that

\[
\sum_{(1,1) \in C(\rho)} \rho(1)(u) = \sum_{u \in C(\rho) \setminus \emptyset} \mu(u) \rho(1)(u), \text{ for all } (1,1) \notin C, \text{ heft.}
\]

For interim efficiency, substitute \( \lambda_T \in R \), in the above statements. For ex ante efficiency, substitute \( \lambda_T \in R \), for \( \lambda_T \in R \), in the above statements.

Proof In the betting case, interim rationality, incentive-compatibility and feasibility constraints are linear constraints. There exist \( x \) satisfying these linear constraints, with at least one interim incentive constraint holding with strict inequality. Now lemma 1 implies lemma 2 where the \( \lambda_t \) and \( \theta \) are multipliers for the interim rationality, incentive compatibility and feasibility constraints respectively. \( \theta \) is normalized to sum to one without loss of generality.

Lemma 3 (Betting Case: Efficiency) In the betting case, an allocation is (ex ante) interim incentive compatible iff there exists \( \theta \subseteq \emptyset \) such that \( \theta \) is ex post efficient, for all \( \lambda_T \in R \), in the above statements.

Proof It must be shown that the conditions in lemmas 2 and 3 are equivalent. In the lemmas 2 conditions for interim incentive compatible efficiency, let \( x^* \) be defined by \( \rho(x^*)(1) = \theta(1) \). This is equivalent to \( x^* \in C(\theta) \).

\[
\rho(1)(u) = \sum_{(1,1) \in C(\theta)} \frac{\mu(1,1) \rho(1)(u)}{\mu(1,1) \rho(x^*)(u)} = \theta(1), \text{ for some } \mu.
\]

But algebraic manipulation shows this is equivalent to \( \theta \subseteq C \subseteq N(x^*) \). Similar algebraic manipulations show equivalence in other cases.

To relate the general case to the betting case, some stricter conditions will be required.

Definition 2 y strongly interim dominates e if

\[
\sum_{u \in C(\theta)} \rho(1)(u) \theta(1)(u) > 0, \text{ for all } \theta \subseteq \emptyset, \text{ for all } \lambda_T \in R, \text{ for } \theta \subseteq C \subseteq N(x^*).
\]

with strict inequality for some \( \theta \), for all \( \lambda_T \in R \), for some \( \theta \subseteq C \subseteq N(x^*).

Now it must be a public event that some agent strictly prefers y to e. This is thus a stronger requirement than interim dominance.

Definition 3 e is strongly incentive compatible if

\[
\sum_{u \in C(\theta)} \rho(1)(u) \theta(1)(u) > 0, \text{ for all } \theta \subseteq \emptyset, \text{ for all } \lambda_T \in R, \text{ for } \theta \subseteq C \subseteq N(x^*).
\]

with strict inequality for all \( \theta \), for \( \lambda_T \) such that \( \sum_{u \in C(\theta)} \theta(1)(u) \theta(1)(u) > 0, \) for all \( \theta, \lambda_T \) for some \( \theta \subseteq C \subseteq N(x^*) \).

Strict incentive compatibility requires that incentive compatibility constraints hold strictly unless posterior beliefs are the same, given two signals, so that the constraint can never hold strictly.

Definition 4 e is strongly incentive compatible (publicly efficient if e is incentive compatible) if it is incentive compatible (publicly efficient if it is incentive compatible) and there does not exist a trade \( x \) such that \( e \gg x \) is (strongly) incentive compatible (publicly efficient) and interim dominates \( e \).

Lemma 4 (Strong interim efficiency in the betting case) In the betting case, an allocation is strongly (interim incentive compatible) publicly efficient if beliefs are (noisy) (publicly) weakly consistent.

Proof First, suppose a strong incentive compatibility was not required in the definition of strong incentive compatible efficiency. The argument then goes through as for lemma 2, but now the \( \lambda_T \) need only satisfy \( \lambda_T(1)^{<\theta} > 0 \) for some \( \theta \subseteq C \subseteq N(x^*) \) for all \( \theta \subseteq C \subseteq N(x^*) \). This leads to the replacement of consistency by weak consistency.

Now suppose there does not exist a strongly incentive compatible trade \( x \) that strongly interim dominates \( e \). Then for each \( h \in R \), \( h \in R \), there exists \( \theta \subseteq C \subseteq N(x^*) \), for each \( h \in R \), \( h \in R \), and, for each \( h \in R \), \( h \in R \), with \( \mu(h) > 0 \) such that

\[
\lambda_T(1) \rho(1)(u) = \sum_{u \in C(\theta)} \rho(1)(u) - \sum_{u \in C(\theta)} \rho(1)(u) = \theta(1), \text{ for all } (1,1) \in C, \text{ heft.}
\]

But suppose there does exist a incentive compatible trade \( x \) such that \( e \ll x \) strongly interim dominates \( e \). Then \( \theta = 0 \) in the above expression. This is possible if posterior beliefs given \( u \) and \( u' \) are the same, contradicting non-existence of strongly incentive compatible \( x \) strongly interim dominating \( e \).

Lemma 5 (Conditions for interim efficiency) An allocation is (ex ante) interim efficient if there exists \( e \in R \), and prices \( x : Q \subseteq R \), such that

\[
\frac{\partial}{\partial a} \sum_a \rho(a)(1) - x(1) = 0, \text{ for all } x \subseteq Q \text{ (1)}
\]

Proof Standard argument from first order conditions.

Notice that \( x \) allocation is efficient in the betting case iff agents have common beliefs about contractible events \( \rho(a) = \rho(a) \theta(1) \), for all \( h, \lambda_T \in R \), \( h \subseteq C \subseteq N(x^*) \).

Proof of Theorem 1

Consider an L good initially efficient allocation \( e \), with agent beliefs \( \theta \), satisfying equation (1) above. We will compare this with the betting case where beliefs are given by \( \rho(a) = \rho(a) \theta(1) \), for all \( h \in R \), \( h \subseteq C \subseteq N(x^*) \).

For any L good trade \( e \), consider the L good trade y where

\[
\rho(1)(1) = \sum_{a \subseteq \emptyset} \rho(a)(1) = \sum_{a \subseteq \emptyset} \rho(a)(1) = \lambda_T(1).
\]
Theorem 1

In an efficient market, the price of an asset is determined by its fundamental value. The agent's expected utility is maximized when the market price equals the asset's fundamental value. This is because the market price reflects all available information, which is already incorporated into the asset's price. Therefore, an investor cannot earn abnormal profits by trading based on the asset's price.

Corollary 2

If the market is efficient, then the market price of an asset is equal to its fundamental value at all times. This implies that the asset's price adjusts instantaneously to any new information, making it impossible for investors to exploit any pricing anomalies.

Theorem 3

In an incomplete market, the price of a derivative security is determined by the underlying asset's price and the risk-neutral probability measure. The agent's expected utility is maximized when the derivative security's price reflects the risk-adjusted value of the underlying asset. This is because the derivative security's price incorporates the market's perception of the underlying asset's future value, which is already discounted at the risk-free rate.

Corollary 4

If the market is incomplete, then the derivative security's price is influenced by the investor's risk aversion and the market's risk-neutral probability measure. This implies that the derivative security's price is not solely determined by the underlying asset's price, but also by the investor's risk preferences and the market's perception of future uncertainties.

Beliefs are used to predict the future value of the underlying asset, which is then used to determine the derivative security's price. The agent's expected utility is maximized when the derivative security's price reflects the market's risk-neutral probability measure and the investor's risk preferences.

For any 1-poded trade, consider not Good news x future

It is not possible to predict the future value of the underlying asset accurately, and hence, the derivative security's price cannot be determined precisely. This implies that the agent's expected utility cannot be maximized by trading in the derivative security, and hence, any 1-poded trade is not profitable.

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