Common Belief Foundations of Global Games

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Introduction

- Coordination games have multiple equilibria
- Relaxing common knowledge assumptions gives uniqueness

Carlsson & van Damme "Global Games and Equilibrium Selection" *Econometrica 1993*

"Regime Change" Game

- Continuum of players "invest" or "not invest"
- Payoffs
  - Cost of investing: $c \in (0, 1)$
  - Return to investing
    - 1 if proportion investing is at least $1 - \theta$
    - 0 otherwise

Equilibrium

1. Common knowledge of θ
   - Multiple equilibria if $\theta \in (0, 1)$

Equilibrium

1. Common knowledge of θ
   - Multiple equilibria if $\theta \in (0, 1)$
2. Lack of common knowledge of θ
   - $\theta \sim N\left(y, \frac{1}{\alpha}\right)$, $x_i \sim N\left(\theta, \frac{1}{\beta}\right)$
   - unique equilibrium if and only if $\alpha^2 \leq 2\pi\beta$
   - each player invests if $x_i > \widehat{x}$, does not invest if $x_i < \widehat{x}$
   - as $\beta \to \infty$, $\widehat{x} \to 1 - c$.

History of "Regime Change" Example

- "Coordination Risk and Price of Debt"
  - 1999 Working Paper
  - plenary talk at 1999 Santiago de Compostela Econometric Society European Meetings
  - published (eventually) in the *European Economic Review 2004*

"Regime Change" Example

- "Coordination Risk and Price of Debt"
- Static
  - Metz "Private and Public Information in Self-Fulfilling Currency Crises" *JE 2002*
  - Hellwig "Public Information, Private Information and the Multiplicity of Equilibria in Coordination Games" *JET 2002*

"Regime Change" Example Applications

- Dynamic
  - without signalling
    - Morris & Shin "Coordination Risk and Price of Debt" *EER 2004*
    - Rochet & Vives "Coordination Failures and Lender of Last Resort" *JEEA 2004*
    - Dasgupta "Coordination and Delay in Global Games" *JET forthcoming*
    - Morris & Shin "Liquidity Black Holes" *RF 2004*
    - Morris & Shin "Catalytic Finance" *JIE 2006*

- Dynamic
  - with signalling
    - Angeletos, Hellwig & Pavan "Signalling in a Global Game" *JPE 2006*
    - Angeletos, Hellwig & Pavan "Policy in a Global Coordination Game" *this session*
  - with informational externality only
    - Angeletos, Hellwig & Pavan "Dynamic Global Games of Regime Change" *Econometrica 2006*
    - Angeletos & Werning "Crises and Prices" *AER 2006*

"Regime Change" Example

- "Coordination Risk and Price of Debt"
- Static
- Dynamic
- Static Generalizations
  - Corsetti, Dasgupta, Morris & Shin "Does one Soros Make a Difference?" *REStud 2004*
  - Guimaeres & Morris "Risk and Wealth in a Model of Self-Fulfilling Currency Attacks" *this session*

Usefulness of Methodology?

Catchy name, cute model but is it relevant?

1. Robustness to Endogenous Public Information (e.g., Angeletos & Werning 2005 etc...)

2. Relevance of Private Information (e.g., Sims 2006 etc...)

3. What about other ways of relaxing common knowledge assumptions (Weinstein & Yildiz 2006)?

Common belief foundations may help address these questions....

Common Belief Foundations

1. Carlsson & van Damme 1993
2. Morris & Shin "Informational Events that Trigger Currency Attacks" Philly Fed Working Paper 1995
3. Morris & Shin "Global Games: Theory and Applications" Econometric Society World Congress 2000
4. Hellwig 2002

Model

1. Background

- Players $\mathcal{I} = \{1, ..., I\}$
- Finite "payoff states" θ

Model

1. Background
   - Players $\mathcal{I} = \{1, ..., I\}$
   - Finite "payoff states" θ
2. Type Space $\mathcal{T} = (T_i, \pi_i)_{i=1}^{I}$
   - i's types: T_i
   - i's belief: $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$

Model

1. Background

2. Type Space $\mathcal{T} = (T_i, \pi_i)_{i=1}^{I}$

3. Binary Action Game with Strategic Complementarities $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^{I}$

   - i chooses $a_i \in \{0, 1\}$
   - $\lambda_i (Z, \theta)$ is payoff gain to action 1 over 0 in state θ if Z is the set of opponents choosing 1, i.e.

$$u_i (1, a_{-i}, \theta) - u_i (0, a_{-i}, \theta) = \lambda_i (\{j \neq i | a_j = 1\}, \theta)$$

   - $\lambda_i : 2^{\mathcal{I}/\{i\}} \times \Theta \rightarrow \mathbb{R}$, increasing in Z

Question

What joint restriction on higher order beliefs ($\mathcal{T}$) and payoffs ($\boldsymbol{\lambda}$) gives uniqueness rationalizable outcomes?

Generalized Belief Operators I

- "Simple" event
  - $F = \underset{i=1,I}{\times} F_i$
  - each $F_i \subseteq T_i$
  - F is an event in $T = \underset{i=1,I}{\times} T_i$

Generalized Belief Operators II

- Belief Operator

$$B_i^{\lambda_i}(F) = \left\{ t_i \left| \sum_{t_{-i},\theta} \pi_i(t_{-i},\theta)\, \lambda_i(\{j \neq i \,| t_j \in F_j\}, \theta) \geq 0 \right. \right\}$$

  - $t_i \in B_i^{\lambda_i}(F)$ means that type t_i thinks it likely that $t_j \in F_j$ for many $j \neq i$
  - monotonic: $F \subseteq F' \Rightarrow B_i^{\lambda_i}(F) \subseteq B_i^{\lambda_i}(F')$
  - $B^{\boldsymbol{\lambda}}(F) = \underset{i=1,I}{\times} B_i^{\lambda_i}(F)$

Generalized Common Belief

DEFINITION: There is *common* $\boldsymbol{\lambda}$-*belief* of F at t if

$$t \in C^{\boldsymbol{\lambda}}(F) \equiv \bigcap_{k \geq 1} \left[B^{\boldsymbol{\lambda}}\right]^k (F).$$

DEFINITION: Event F is $\boldsymbol{\lambda}$-*evident* if

$$F \subseteq B^{\boldsymbol{\lambda}}(F).$$

PROPOSITION (cf, Aumann 1976, Monderer and Samet 1989): Event F is common $\boldsymbol{\lambda}$-belief at t $(t \in C^{\boldsymbol{\lambda}}(F))$ if and only if there exists a $\boldsymbol{\lambda}$-evident event F' such that $t \in F' \subseteq F$.

Generalized Common Belief

- Fix $X \subseteq \Theta$ and let

$$\lambda_i^{X,p}(Z,\theta) = \begin{cases} 1-p, \text{ if } Z = \mathcal{I}/\{i\} \text{ and } \theta \in X \\ -p, \text{ otherwise} \end{cases}$$

- Now $C^{\boldsymbol{\lambda}}(T)$ is the event that

1. everyone believes X with probability at least p
2. everyone believes with probability at least p that everyone believes X with probability at least p
3. et cetera....

Rationalizability

DEFINITION: Action a_i is rationalizable for type t_i if $a_i \in R_i^*(\boldsymbol{\lambda}, t_i)$, where

$$\begin{aligned} R_i^0(\boldsymbol{\lambda}, t_i) &= \{0,1\} \\ R_i^{k+1}(\boldsymbol{\lambda}, t_i) &= \left\{ a_i \left| \begin{array}{l} \text{there exists } \mu_i \in \Delta(T_{-i} \times \Theta \times \{0,1\}) \text{ such that} \\ (1)\ \mu_i(t_{-i}, \theta, a_{-i}) > 0 \Rightarrow a_j \in R_j^k(\boldsymbol{\lambda}, t_j) \text{ for all } j \neq i \\ (2)\ \sum\limits_{a_{-i}} \mu_i(t_{-i}, \theta, a_{-i}) = \pi_i(t_{-i}, \theta \,|t_i) \\ (3)\ a_i \in \arg\max\limits_{a_i'} \sum\limits_{t_{-i}, \theta, a_{-i}} \mu_i(t_{-i}, \theta, a_{-i})\, u_i((a_i', a_{-i}), \theta) \end{array} \right. \right\} \\ R_i^*(\boldsymbol{\lambda}, t_i) &= \bigcap_{k \geq 1} R_i^k(\boldsymbol{\lambda}, t_i) \end{aligned}$$

Characterization

PROPOSITION: Action 1 is rationalizable for type t_i if and only if $t_i \in B_i^{\lambda_i}\left(C^{\boldsymbol{\lambda}}(T)\right)$.

Inverse operator:

$$\widetilde{\lambda}_i(Z,\theta) = -\lambda_i(\mathcal{I}/Z,\theta)$$

PROPOSITION: Action 0 is rationalizable for type t_i if and only if $t_i \in B_i^{\widetilde{\lambda}_i}\left(C^{\widetilde{\lambda}}(T)\right)$.

Two Player Linear Example

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

$$\lambda_i (Z, \theta) = \left\{ \begin{array}{l} \theta, \text{ if } Z = \{3 - i\} \\ \theta - 1, \text{ if } Z = \varnothing \end{array} \right\}$$

Two Player Linear Example

Action 1 if rationalizable for player 1 only if

1. Player 1's expectation of θ is at least 0....
   - $E_1(\theta) \geq 0$
2. Player 1's expectation of θ is at least 1 minus player 1's probability that player 2's expectation of θ is at least 0...
   - $E_1(\theta) \geq 1 - \Pr_1(E_2(\theta) \geq 0)$
3. Player 1's expecation of θ is at least 1 minus player 1's probability that player 2's probability that player 1's expectation of θ is at least 0....

- $E_1(\theta) \geq 1 - \Pr_1(E_2(\theta) \geq 1 - \Pr_2(E_1(\theta) \geq 0))$

4. Et Cetera.....

Regime Change Example

- Each player invests or not invests
- Cost of investing: $c \in (0,1)$
- Return to investing: 1 if proportion investing is at least $1-\theta$, 0 otherwise

$$\lambda_i(Z,\theta) = \begin{cases} 1-c, \text{ if } \frac{\#Z}{I-1} \geq 1-\theta \\ -c, \text{ otherwise} \end{cases}$$

Regime Change Example

Action 1 if rationalizable for player 1 only if

1. Player 1's probability that $\theta \geq 0$ is at least c....
   - $\Pr_1(\theta \geq 0) \geq c$
2. Player 1's probability that [the proportion of other players with probability that $\theta \geq 0$ is at least c] is at least c
   - $\Pr_1\left(\frac{\#\left\{j\neq 1 \middle| \Pr_j(\theta\geq 0)\geq c\right\}}{I-1} \geq 1-\theta\right) \geq c$
3. Et Cetera.....

A Uniqueness Condition

PROPOSITION: Game $\boldsymbol{\lambda}$ is dominance solvable if $C^{\boldsymbol{\lambda}}(T) \cap C^{\widetilde{\lambda}}(T) = \varnothing$.

Sufficient Condition for Uniqueness

Let $t_i \equiv \left(E_i\left(\theta|t_i\right), \widetilde{\psi}_i\left(t_i\right)\right) \in \mathbb{R} \times \Psi_i$

DEFINITION: i's *beliefs in differences*, $\xi_i : T_i \rightarrow \Delta\left(\mathbb{R}^{I-1} \times \Psi_{-i}\right)$, are

$$\xi_i\left(t_i\right)\left[\delta_{-i} \times \psi_{-i}\right] = \sum_{\theta \in \Theta} \pi_i\left(t_i\right)\left[\left(E_i\left(\theta|t_i\right) + \delta_j, \psi_j\right)_{j \neq i}, \theta\right]$$

DEFINITION: No $\boldsymbol{\lambda}$-*ties* holds at t if for all F.

$$\sum_{t_{-i}, \theta} \pi_i\left(t_{-i}, \theta\right) \lambda_i\left(\left\{j \neq i \left| t_j \in F_j\right.\right\}, \theta\right) \neq 0$$

Sufficient Condition for Uniqueness

PROPOSITION: If there is common knowledge of beliefs about differences, $\boldsymbol{\xi} = (\xi_i)_{i=1}^{I}$, and no $\boldsymbol{\lambda}$-ties, then $\boldsymbol{\lambda}$ is dominance solvable.

Announcement

The Stony Brook Game Theory Festival of the Game Theory Society

July 9-20, 2007

http://www.gtcenter.org/

Workshop on Global Games

July 18 - 20, 2007

Organizers: **Stephen Morris** (Princeton University)

Amil Dasgupta (London School of Economics)

Alessandro Pavan (Northwestern University)