Coordination without Common Knowledge

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March 2006

Introduction

- Coordination games have multiple equilibria
- Relaxing common knowledge assumptions gives uniqueness
  - Carlsson and van Damme (1993)
  - "global games"

Investment Example

two players, invest or not invest

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

- $\theta \sim N\left(y, \frac{1}{\alpha}\right)$

- $x_i \sim N\left(\theta, \frac{1}{\beta}\right)$

- unique equilibrium if and only if

$$\frac{\alpha^2}{\beta}\left(\frac{\alpha+\beta}{\alpha+2\beta}\right) \leq 2\pi$$

- each player invests if $x_i > \widehat{x}$, does not invest if $x_i < \widehat{x}$

- as $\beta \to \infty$, $\widehat{x} \to \frac{1}{2}$.

Global Game Approach

- Uniqueness if not too much "common knowledge" or "public information": nice parameterized normal examples
- Example generalizes to many players, general payoffs with strategic complementarities (Frankel, Morris and Pauzner (2003))
- Comparative statics / Policy Analysis possible

- Applications

  1. Currency Crises (Morris and Shin 1998, Guimaeres and Morris 2006)
  2. Bank Runs (Goldstein and Pauzner 2002)
  3. Financial Markets

- Information Structure relaxing common knowledge useful in other contexts (Morris and Shin 2002)

Criticisms?

Too easy to apply this methodology?

1. Endogenous Public Information (e.g., Angeletos and Werning 2005)

2. Relevance of Private Information (e.g., Sims 2006)

3. What about other ways of relaxing common knowledge assumptions (Yildiz 2006))?

To address these criticisms and understand global games, must understand higher order belief foundations of results

Outline of Talk

1. Review Investment Example and logic of Global Games

2. Discuss Investment Example with arbitrary beliefs and higher order beliefs

3. Sufficient Conditions for global game results: the importance of "approximate common knowledge" and common knowledge of "typicalness"

4. Review endogenous public signals in the light of this discussion

5. Heterogeneous Priors vs Asymmetric Information

6. Bounded Rationality

7. Learning

8. Summary

1. Investment Example

two players, invest or not invest

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
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- $\theta \sim N\left(y, \frac{1}{\alpha}\right)$
- $x_i \sim N\left(\theta, \frac{1}{\beta}\right)$

- player 1 thinks $\theta \sim N\left(\frac{\alpha y+\beta x_1}{\alpha+\beta}, \frac{1}{\alpha+\beta}\right)$

- player 1 thinks $x_2 \sim N\left(\frac{\alpha y+\beta x_1}{\alpha+\beta}, \frac{\alpha+2\beta}{\beta(\alpha+\beta)}\right)$

- suppose player 1 thinks that player 2 invests only if $x_2 \leq x^*$
  - expected payoff to investing is

$$\begin{aligned} u\left(x_1, x^*\right) &= E_1\left(\theta\right) + \Pr\left(x_2 \leq x^*\right) \\ &= \frac{\alpha y + \beta x_1}{\alpha + \beta} - \Phi\left(\sqrt{\frac{\beta\left(\alpha+\beta\right)}{\alpha+2\beta}}\left(x^* - \frac{\alpha y + \beta x_1}{\alpha+\beta}\right)\right) \end{aligned}$$

  - unique value of x_1 sets $u\left(x_1, x^*\right) = 0$

- symmetric cutoff equilibrium if $u(x^*, x^*) = 0$

$$
\begin{aligned}
u(x^*, x^*) &= \frac{\alpha y + \beta x^*}{\alpha + \beta} - \Phi\left(\sqrt{\frac{\beta(\alpha+\beta)}{\alpha+2\beta}}\left(\frac{\alpha}{\alpha+\beta}\right)(x^* - y)\right) \\
&= \frac{\alpha y + \beta x^*}{\alpha + \beta} - \Phi\left(\sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}(x^* - y)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{du(x^*, x^*)}{dx^*} &= \frac{\beta}{\alpha+\beta} - \sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}\phi\left(\sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}(x^* - y)\right) \\
&\geq \frac{\beta}{\alpha+\beta} - \sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}\frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

– thus there is only one symmetric cutoff equilibrium if and only if

$$\frac{\beta}{\alpha+\beta}-\sqrt{\frac{\beta\alpha^2}{(\alpha+\beta)(\alpha+2\beta)}}\frac{1}{\sqrt{2\pi}}\geq 0$$

or

$$\left(\frac{\alpha+\beta}{\alpha+2\beta}\right)\frac{\alpha^2}{\beta}\leq 2\pi$$

- game has strategic complementarities, so largest and smallest profile of rationalizable strategies constitute equilibria

- thus if only one symmetric cutoff equilibrium, unique strategy profile surviving iterated deletion of strictly dominated strategies

Properties of Equilibrium

- Strategic Uncertainty Intuition for Uniqueness
  - $\Pr(x_2 \leq x^* | x_1 = x^*) \approx \frac{1}{2}$ if $\frac{\alpha^2}{\beta}$ is small.
- Excess sensitivity to public signals
- in limit as $\beta \to \infty$,
  - risk dominant action is played; determinate comparative statics
  - in many player binary action games, "Laplacian" action is played
  - uniform devaluation probability in many player many action currency crisis game of Guimaeres and Morris (2006)

2. Higher Order Beliefs Analysis

- Let T_i be a finite set types of player i
- Type t_i of player i has beliefs $\pi_i(t_i) \in \Delta(T_j \times \mathbb{R})$ over the types of the other player and the state of the world.
  - write $\widetilde{\theta}_i(t_i)$ for type t_i's expectation of θ
  - write $\pi_i(t_j|t_i)$ for the probability that he assigns to player $j \neq i$ being of type t_j.

- Fix an event $E \subseteq T_1 \times T_2$. Let $\pi_i(E|t_i)$ be the probability that type t_i assigns to the event E:

$$\pi_i(E|t_i) = \sum_{\{t_j:(t_i,t_j)\in E\}} \pi_i(t_j|t_i).$$

- Let $B_i^p(E)$ be the set of states where player i assigns probability at least p to the event E:

$$B_i^p(E) = \{(t_i, t_j) : \pi_i(E|t_i) \geq p\}.$$

- We also write

$$\begin{aligned} B_*^p(E) &= B_1^p(E) \cap B_2^p(E) \\ C^p(E) &= \bigcap_{k \geq 1} [B_*^p]^k (E) \end{aligned}$$

Proposition 1. *(Monderer and Samet 1989) $\omega \in C^p(E)$ if and only if there exists $F \subseteq B_*^p(F)$ such that $\omega \in F \subseteq E$*

- For any function $f_i : T_i \to \mathbb{R}$, let $B_i^{f_i}(E)$ be the set of states where player i assigns probability at least f_i to the event E, where f_i is a random variable. Thus

$$B_i^{f_i}(E) = \{(t_i, t_j) : \pi_i(E|t_i) \geq f_i(t_i)\}.$$

- We will also write

$$\begin{aligned} B_*^{f_1,f_2}(E) &= B_1^{f_1}(E) \cap B_2^{f_2}(E) \\ C^{f_1,f_2}(E) &= \bigcap_{k\geq 1} \left[B_*^{f_1,f_2}\right]^k (E) \end{aligned}$$

Corollary 2. *$\omega \in C^f(E)$ if and only if there exists $F \subseteq B_*^f(F)$ such that $\omega \in F \subseteq E$*

Proposition 3. *Action I is rationalizable if and only if*

$$t_i \in X^I = B_i^{1-\widetilde{\theta}_i} C^{1-\widetilde{\theta}_1, 1-\widetilde{\theta}_2}(\Omega)\,;$$

Action N is rationalizable if and only if

$$t_i \in X^N = B_i^{\widetilde{\theta}_i} C^{\widetilde{\theta}_1, \widetilde{\theta}_2}(\Omega)\,.$$

Thus the game is dominance solvable if and only if $X^I \cap X^N = \varnothing$. *Equivalently, there do not exist* E^I *and* E^N *such that* $E^I \cap E^N \neq \varnothing$, $E^I \subseteq B_*^{1-\widetilde{\theta}}\left(E^I\right)$ *and* $E^N \subseteq B_*^{\widetilde{\theta}}\left(E^N\right)$.

3. Sufficient Conditions for Uniqueness

Let

$$\begin{aligned} p_i(t_i) &= \pi_i\left(\left\{(t_i, t_j) : \widetilde{\theta}_j(t_j) > \widetilde{\theta}_i(t_i)\right\} | t_i\right) \\ q_i(t_i) &= \pi_i\left(\left\{(t_i, t_j) : \widetilde{\theta}_j(t_j) = \widetilde{\theta}_i(t_i)\right\} | t_i\right) \end{aligned}$$

We will maintain the assumption that there is common knowledge that $q_i(t_i) = 0$ and $\widetilde{\theta}_i(t_i) \neq \frac{1}{2}$, for all i and t_i.

Now we have:

Proposition 4. *If there is common knowledge that* $p_1 = p_2 = \frac{1}{2}$*, then the game is dominance solvable.*

3. Sufficient Conditions for Multiplicity

Proposition 5. *If* $C^p\left(\left\{(t_1,t_2):\widetilde{\theta}_i\left(t_i\right)\geq 1-p\right\}\right)$, *then investment is rationalizable; if* $C^p\left(\left\{(t_1,t_2):\widetilde{\theta}_i\left(t_i\right)\leq p\right\}\right)$, *then not investing is rationalizable.*

4. Endogenous Public Information

- Atkeson (1999) and others: in market situations, prices act as public signals that restore multiplicity

- Explicit modelling: Tarashev (2003), Hellwig, Mukherji and Tsyvinski (2005), Angeletos and Werning (2005)

One dimensional private signals

- let $\alpha \to 0$ (no exogenous public information)
- let K individuals with private signal accuracy β be polled
- thus endogenous public signal has precision $K\beta$
- substituting into uniqueness condition, we get uniqueness if

$$\frac{(K\beta)^2}{\beta}\left(\frac{K+1}{K+2}\right) \leq 2\pi$$

or if

$$\beta \leq 2\pi \left(\frac{K+2}{K^2 (K+1)} \right)$$

- uniqueness only if private information is sufficiently inaccurate (Angeletos and Werning (2005))

two dimensional private signals

- $\theta \sim N\left(y, \frac{1}{\alpha}\right)$
- agent i observes $x_{ik} = \theta + \eta_k + \varepsilon_{ik}$, for $k = 1, 2$
- each $\eta_k \sim N\left(0, \frac{1}{\gamma_k}\right)$, $\varepsilon_{ik} \sim N\left(0, \frac{1}{\beta_k}\right)$
- each agent has one public signal (y) and two private signals (x_{i1} and x_{i2}) about θ.

2: higher order beliefs

- each $x_{ik} \sim N\left(\theta, \frac{\gamma_k \beta_k}{\gamma_k + \beta_k}\right)$
- thus

$$E_i(\theta) = \frac{\alpha y + \frac{\gamma_1+\beta_1}{\gamma_1\beta_1} x_{i1} + \frac{\gamma_2+\beta_2}{\gamma_2\beta_2} x_{i2}}{\alpha + \frac{\gamma_1+\beta_1}{\gamma_1\beta_1} + \frac{\gamma_2+\beta_2}{\gamma_2\beta_2}}$$

- take large sample $K \to \infty$, we will observe average expectation

$$\overline{E}(\theta) = \frac{\alpha y + \frac{\gamma_1+\beta_1}{\gamma_1\beta_1} \eta_1 + \frac{\gamma_2+\beta_2}{\gamma_2\beta_2} \eta_2}{\alpha + \frac{\gamma_1+\beta_1}{\gamma_1\beta_1} + \frac{\gamma_2+\beta_2}{\gamma_2\beta_2}}$$

- $\overline{E}^2(\theta)$ has no necessary relationship to θ and $\overline{E}(\theta)$
- let $\beta_1, \beta_2, \gamma_1, \gamma_2 \to \infty$ at same rate
- common knowledge of typicalness and unique equilibrium in the limit

5: difference in priors

Now suppose that player 1 has a prior expectation x_1 of θ and believes that

$$\begin{pmatrix} \theta \\ x_2 \end{pmatrix}$$

is normally distributed with mean

$$\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$$

and variance

$$\begin{pmatrix} \frac{1}{\beta} & \frac{\rho}{\sqrt{\beta\gamma}} \\ \frac{\rho}{\sqrt{\beta\gamma}} & \frac{1}{\gamma} \end{pmatrix}.$$

Suppose that in addition each player observes a conditionally independent public signal y which is normally distributed with mean θ and precision α. Observe that

$$E(\theta|x_1) = \frac{\alpha y + \beta x_1}{\alpha + \beta}$$

and

$$\Pr(x_2 \leq x_2^*|x_1) = \Phi\left(\sqrt{\frac{\gamma}{1-\rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}}\left(x_2^* - x_1 - \frac{\rho}{\sqrt{\beta\gamma}}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}(y - x_1)\right)\right)$$

Thus if player 2 follows the trigger strategy with cutoff x_2^* (i.e., investing only if $x_2 \geq x_2^*$), then player 1's expected payoff to investing is

$$
\begin{aligned}
u\left(x_1, x_2^*\right) &= E\left(\theta | -x_1\right) - \Pr\left(x_2 \leq x_2^* | x_1\right) \\
&= \frac{\alpha y + \beta x_1}{\alpha + \beta} - \Phi\left(\sqrt{\frac{\gamma}{1 - \rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}}\left(x_2^* - x_1 - \frac{\rho}{\sqrt{\beta\gamma}}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}\left(y - x_1\right)\right)\right)
\end{aligned}
$$

Now the payoff to a player from investing at the other player's cutoff is

$$
U\left(x\right) = u\left(x, x\right) = \frac{\alpha y + \beta x}{\alpha + \beta} - \Phi\left(-\sqrt{\frac{\gamma}{1 - \rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}}\frac{\rho}{\sqrt{\beta\gamma}}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}\left(y - x\right)\right)
$$

Now

$$\begin{aligned}\frac{dU}{dx} &= \frac{\beta}{\alpha+\beta}-\frac{\rho}{\sqrt{\beta\gamma}}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}\sqrt{\frac{\gamma}{1-\rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}}\phi\left(-\frac{\rho}{\sqrt{\beta\gamma}}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}(y-x)\right)\\ &\geq \frac{\beta}{\alpha+\beta}-\sqrt{\frac{\gamma}{1-\rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}}\frac{\rho}{\sqrt{\beta\gamma}}\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}\frac{1}{\sqrt{2\pi}}\end{aligned}$$

Thus we get uniqueness condition

$$\frac{1}{1-\rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}\frac{\rho^2}{\beta}\left(\frac{\alpha\beta}{\alpha+\beta}\right)^2\left(\frac{\alpha+\beta}{\beta}\right)^2\leq 2\pi$$

or

$$\frac{\alpha^2}{\beta}\left(\frac{\rho^2}{1-\rho^2\left(\frac{\alpha}{\alpha+\beta}\right)}\right) \leq 2\pi.$$

Notice that this is independent of γ, but depends on ρ.

Two cases:

- Pure differences in priors: $\rho = 0$, uniqueness for every level of public information
- The asymmetric information model: $\gamma = \frac{\beta}{2}$ and $\rho = \frac{1}{\sqrt{2}}$.

6: bounded rationality

Jehiel and Koessler (2005) "Revisiting Games of Incomplete Information with Analogy-Based Reasoning"

7: learning

Steiner and Stewart (2006) "Learning by Similarity in Global Games"

8: conclusions