

Robust Implementation: The Case of Direct Mechanisms

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Motivation

- two important limitations of mechanism design with incomplete information:
 1. analysis typically assumes common knowledge of common prior over the agents' types (ROBUSTNESS)
 2. revelation principle only establishes that direct mechanism has *an* equilibrium that achieves the social choice function, many others may not achieve it (FULL IMPLEMENTATION)
- robust implementation: require robustness *and* full implementation

Earlier Work on Robustness

- “Robust Mechanism Design”: *partial* Bayesian implementation on all type spaces
 - for social choice *function*, ex post incentive compatibility (EPIC) necessary and sufficient
 - thus partial ex post implementation = Bayesian implementation on all type spaces
- “Ex Post Implementation”: *full* implementation in ex post equilibrium:
 - EPIC and ex post monotonicity necessary and almost sufficient
 - ex post monotonicity does not imply nor is implied by either Maskin monotonicity or Bayesian monotonicity
 - ex post implementable scf f may still display Bayesian equilibria which do not coincide with scf for some type space

Paper

- Examine robust implementation (= iterative deletion procedure, Battigalli + Siniscalchi 2003) in well behaved but economically rich environment
- tight implementation results
 - scf can be robustly implemented if and only if
 - * strict ex post incentive compatibility
 - * "contraction property" - moderate value interdependence (cf Chung + Ely 2002)
 - when implementation is possible, it is possible in direct mechanism
- practical implications of robust approach?

Robust Implementation in General Environments

- general mechanisms including integer games
 - equivalence to iterated deletion of never best responses
 - ex post incentive compatibility + "robust monotonicity" necessary and almost sufficient
- robust monotonicity equals
 - Bayesian monotonicity on all type spaces
 - contraction property in single crossing aggregator environments
- compact mechanisms: strict ex post incentive compatibility necessary

Plan

- public good example
- model
- sufficient conditions for robust implementation in direct mechanism
- linear model
- necessary conditions for robust implementation
- single good auction model

Public Good Example

- I agents, agent i has payoff type $\theta_i \in [0, 1]$
- allocation $y \in \mathbb{R}_+ \times \mathbb{R}^I$
 - y_0 : public good
 - y_i : monetary transfer to i
- utility of agent i is

$$u_i(y, \theta) = (\theta_i + \gamma \sum_{j \neq i} \theta_j) y_0 + y_i,$$

- type profile enters via aggregator: $h_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$; $\gamma \geq 0$
- cost of public good is $c(y_0) = \frac{1}{2} y_0^2$.

Public Good Example

- socially optimal level of the public good is

$$f_0(\theta) = (1 + \gamma(I - 1)) \sum_{i=1}^I \theta_i.$$

- essentially unique transfers generate ex post incentive compatibility:

$$f_i(\theta) = -(1 + \gamma(I - 1)) \left(\gamma \left(\sum_{j \neq i} \theta_j \right) + \frac{1}{2} \theta_i^2 \right)$$

Iterative Procedure in Direct Mechanism

- $\beta^0(\theta_i) = [0, 1]$
- $\beta^k(\theta_i)$ is set of reports i might send for some conjecture $\lambda_i(\theta'_{-i}, \theta_{-i})$ over his opponents' types θ_{-i} and reports θ'_{-i} , with restriction on conjecture $\lambda_i(\theta'_{-i}, \theta_{-i})$ that each type θ_j of agent j sends a message in $\beta^{k-1}(\theta_j)$.

Quadratic Model

- agent i conjectures that other agents have type θ_{-i} and report θ'_{-i}
- agent i with type θ_i has best response

$$\theta'_i = \theta_i + \gamma \sum_{j \neq i} (\theta_j - \theta'_j) .$$

- linear best response allows us characterize $\beta^k(\theta_i)$:

$$\beta^k(\theta_i) = \left[\underline{\beta}^k(\theta_i), \overline{\beta}^k(\theta_i) \right]$$

Induction

- iterative step:

$$\begin{aligned} \bar{\beta}^k(\theta_i) &= \min \left\{ 1, \theta_i + \gamma \max_{\{(\theta'_{-i}, \theta_{-i}) : \theta'_j \in \beta^k(\theta_j) \text{ for all } j \neq i\}} \sum_{j \neq i} (\theta_j - \theta'_j) \right\} \\ &= \min \left\{ 1, \theta_i + \gamma \max_{\theta_{-i}} \sum_{j \neq i} \left(\theta_j - \underline{\beta}^{k-1}(\theta_j) \right) \right\} \end{aligned}$$

$$\bar{\beta}^k(\theta_i) = \min\left\{1, \theta_i + \gamma \max_{\theta_{-i}} \sum_{j \neq i} \left(\theta_j - \underline{\beta}^{k-1}(\theta_j)\right)\right\}$$

- rewriting:

$$\bar{\beta}^k(\theta_i) = \min\left\{1, \theta_i + (\gamma(I-1))^k\right\},$$

and likewise

$$\underline{\beta}^k(\theta_i) = \max\left\{0, \theta_i - (\gamma(I-1))^k\right\}.$$

- thus

$$\theta'_i \neq \theta_i \Rightarrow \theta'_i \notin \beta^k(\theta_i)$$

for sufficiently large k , provided that

$$\gamma < \frac{1}{I-1}$$

Necessity

- but now suppose that $\gamma > \frac{1}{I-1}$
- *each* type θ_i convinced that others' types are $\theta_j = \frac{1}{2} + \frac{1}{\gamma(I-1)} \left(\frac{1}{2} - \theta_i \right)$
- now $h_i(\theta_i, \theta_{-i}) = \theta_i + \gamma(I-1) \left[\frac{1}{2} + \frac{1}{\gamma(I-1)} \left(\frac{1}{2} - \theta_i \right) \right] = \frac{1}{2} [1 + \gamma(I-1)]$
- types cannot be distinguished in direct or any other mechanism....

Set-Up

- finite set of agents, $1, 2, \dots, I$
- agent i 's *payoff type* is $\theta_i \in \Theta_i$, Θ_i is a compact set
- type profile $\theta \in \Theta = \Theta_1 \times \dots \times \Theta_I$
- Y is a compact set of outcomes
- agent i 's utility: $u_i(y, \theta)$
- social choice function, $f : \Theta \rightarrow Y$

Mechanism

- $m_i \in M_i$ compact set of messages available to agent i
- outcome function: $g : M \rightarrow Y$, $g(m)$ outcome under message profile m
- mechanism is a collection:

$$\mathcal{M} = (M_1, \dots, M_I, g(\cdot)),$$

- direct mechanism: $M_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$

Deceptions

- message profile of agent i , $S_i : \Theta_i \rightarrow 2^{M_i} / \emptyset$
 - in direct mechanism, $S_i : \Theta_i \rightarrow 2^{\Theta_i} / \emptyset$
- deception: $\beta_i : \Theta_i \rightarrow 2^{\Theta_i} / \emptyset$ with $\theta_i \in \beta_i(\theta_i)$
 - truth-telling: minimal deception $\beta_i^*(\theta_i) = \{\theta_i\}$

Iterative Procedure

- belief of agent i over message and type profiles of remaining agents:

$$\lambda_i \in \Delta(M_{-i} \times \Theta_{-i})$$

- set $S_i^0(\theta_i) = M_i$ and inductively define

$$S_i^k(\theta_i) = \left\{ m_i \in M_i \left| \begin{array}{l} \exists \lambda_i \text{ s.th.:} \\ \begin{array}{ll} (1) & \text{belief } \lambda_i \text{ is consistent} \\ (2) & \text{message } m_i \text{ is best-response given } \lambda_i \end{array} \end{array} \right. \right\}$$

$$S_i^k(\theta_i) = \left\{ m_i \in M_i \mid \exists \lambda_i \text{ s.th.: } \begin{array}{l} \text{(1) belief } \lambda_i \text{ is consistent} \\ \text{(2) message } m_i \text{ is best-response given } \lambda_i \end{array} \right\}$$

consistent (1):

$$\lambda_i(m_{-i}, \theta_{-i}) > 0 \Rightarrow m_j \in S_j^{k-1}(\theta_j), \quad \forall j \neq i;$$

best response (2):

$$\begin{aligned} & \int_{m_{-i}, \theta_{-i}} \lambda_i(m_{-i}, \theta_{-i}) u_i(g(m_i, m_{-i}), (\theta_i, \theta_{-i})) \\ & \geq \int_{m_{-i}, \theta_{-i}} \lambda_i(m_{-i}, \theta_{-i}) u_i(g(m'_i, m_{-i}), (\theta_i, \theta_{-i})), \quad \forall m'_i \in M_i. \end{aligned}$$

Robust Implementation

- limit set $S^{\mathcal{M}}(\theta)$

$$S^{\mathcal{M}}(\theta) \triangleq \lim_{k \rightarrow \infty} S^k(\theta), \text{ for all } \theta \in \Theta.$$

Definition 1. [Robust Implementation]

f is robustly implemented by mechanism \mathcal{M} if

$$m \in S^{\mathcal{M}}(\theta) \Rightarrow g(m) = f(\theta).$$

Type Aggregator

- aggregator $h_i : \Theta \rightarrow \mathbb{R}$ for each i :

$$u_i(y, \theta) \triangleq v_i(y, h_i(\theta))$$

- $h_i(\theta)$ is continuous and strictly increasing in θ_i
- no restriction on behavior of $h_i(\cdot)$ in terms of θ_{-i}
- $h_i(\theta)$, possibly distinct for every i

Definition 2. [Strict Single Crossing]

The environment satisfies strict single crossing (SC) if

$$v_i(y, \phi) > v_i(y', \phi) \text{ and } v_i(y, \phi') = v_i(y', \phi') \Rightarrow v_i(y, \phi'') < v_i(y', \phi'')$$

if

$$\phi < \phi' < \phi''.$$

- only require aggregation ϕ of type profile θ to be ordered on the real line
- does not require allocation y to be one-dimensional or ordered

Understanding Single Crossing / Aggregator Condition

- two allocations, a or b
- $Y = [0, 1] \times [-K, K]^I$, where y_0 is the probability of allocation a

$$u_i(y, \theta) = y_0 v_i^a(\theta) + (1 - y_0) v_i^b(\theta) + y_i.$$

- An equivalent representation is

$$u_i(y, \theta) = y_0 [v_i^a(\theta) - v_i^b(\theta)] + y_i.$$

We can give this a monotonic aggregator representation by setting $h_i(\theta) = v_i^a(\theta) - v_i^b(\theta)$ and $v_i(y, h(\theta)) = y_0 h(\theta) + y_i$, we have

$$u_i(y, \theta) = v_i(y, h_i(\theta)),$$

and now v_i indeed satisfies the strict single crossing condition.

Understanding Single Crossing / Aggregator Condition

- two agents
- agent i 's utility depends on expected value of $\omega_0 + \omega_i$
- $\omega_0, \omega_1, \omega_2$ independently normally distributed with zero mean and variance σ_i^2 .
- agent i observes signal $\theta_i = \omega_0 + \omega_i + \varepsilon_i$, where each ε_i is independently normally distributed with zero mean and variance τ_i^2 .

- now

$$\begin{aligned} E_i(\omega_0 + \omega_i | \theta_i, \theta_j) &= \text{const} \times h_i(\theta_i, \theta_j) \\ &= \theta_i + \frac{\sigma_0^2 \tau_i^2}{\sigma_0^2 \tau_j^2 + \sigma_0^2 \sigma_i^2 + \sigma_0^2 \sigma_j^2 + \tau_j^2 \sigma_i^2 + \sigma_i^2 \sigma_j^2} \theta_j. \end{aligned}$$

$$\gamma_{ij} = \frac{\sigma_0^2 \tau_i^2}{\sigma_0^2 \tau_j^2 + \sigma_0^2 \sigma_i^2 + \sigma_0^2 \sigma_j^2 + \tau_j^2 \sigma_i^2 + \sigma_i^2 \sigma_j^2}.$$

Property 1

Definition 3. [Strict Ex Post Incentive Compatibility]

scf f satisfies strict ex post incentive compatibility (EPIC) if

$$u_i (f (\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) > u_i (f (\theta'_i, \theta_{-i}), (\theta_i, \theta_{-i}))$$

for all i , θ and θ'_i .

Property 2

Definition 4. [Strict Contraction Property]

The aggregator functions h satisfy the strict contraction property if, for all $\beta \neq \beta^*$, there exists i and $\theta'_i \in \beta_i(\theta_i)$ with $\theta'_i \neq \theta_i$, such that

$$\text{sign}(\theta_i - \theta'_i) = \text{sign}(h_i(\theta_i, \theta_{-i}) - h_i(\theta'_i, \theta'_{-i})),$$

for all θ_{-i} and $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$.

- for some agent i the direct impact of private signal θ_i on aggregator $h_i(\theta)$ is always sufficiently strong to outweigh the reported types of other agents

Sufficiency

Theorem 1. [Robust Implementation]

If strict EPIC and the strict contraction property are satisfied, then there is robust implementation in the direct mechanism.

- no conditions on how f varies with type profile
 - in particular, f does not have to respond to θ similar to the response of any of the aggregators h_i
 - no condition on the number of agents, such as $I > 2$

Intuition

- interaction of single crossing and contraction property
- fix deception β
- contraction property: there exists i and $\theta'_i \in \beta_i(\theta_i)$ with (wlog) $\theta_i > \theta'_i$ such that $h_i(\theta_i, \theta_{-i}) > h_i(\theta'_i, \theta'_{-i})$ for all $\theta'_{-i} \in \beta_i(\theta_{-i})$
- now fix true type profile: θ , reported type profile: $\theta' \in \beta(\theta)$
- choose $\theta''_i > \theta'_i$ with $h_i(\theta_i, \theta_{-i}) > h_i(\theta''_i, \theta'_{-i}) > h_i(\theta'_i, \theta'_{-i})$

Intuition

- so $h_i(\theta_i, \theta_{-i}) > h_i(\theta''_i, \theta'_{-i}) > h_i(\theta'_i, \theta'_{-i})$
- $\theta'_i \in \beta_i(\theta_i)$

Intuition

- so $h_i(\theta_i, \theta_{-i}) > h_i(\theta''_i, \theta'_{-i}) > h_i(\theta'_i, \theta'_{-i})$
- $\theta'_i \in \beta_i(\theta_i)$
- now

$$\begin{array}{rclcl}
 v_i(f(\theta'), h_i(\theta''_i, \theta'_{-i})) & < & v_i(f(\theta''_i, \theta'_{-i}), h_i(\theta''_i, \theta'_{-i})) & \text{EPIC} & h_i(\theta''_i, \theta'_{-i}) \\
 & & & & \uparrow \\
 v_i(f(\theta'), h_i(\theta')) & > & v_i(f(\theta''_i, \theta'_{-i}), h_i(\theta')) & \text{EPIC} & h_i(\theta'_i, \theta'_{-i})
 \end{array}$$

Intuition

- so $h_i(\theta_i, \theta_{-i}) > h_i(\theta''_i, \theta'_{-i}) > h_i(\theta'_i, \theta'_{-i})$
- $\theta'_i \in \beta_i(\theta_i)$
- now

$v_i(f(\theta'), h_i(\theta)) <$	$v_i(f(\theta''_i, \theta'_{-i}), h_i(\theta))$	AGG, SC	$h_i(\theta)$
			↑
$v_i(f(\theta'), h_i(\theta')) <$	$v_i(f(\theta''_i, \theta'_{-i}), h_i(\theta''_i, \theta'_{-i}))$	EPIC	$h_i(\theta''_i, \theta'_{-i})$
			↑
$v_i(f(\theta'), h_i(\theta')) >$	$v_i(f(\theta''_i, \theta'_{-i}), h_i(\theta'))$	EPIC	$h_i(\theta'_i, \theta'_{-i})$

Linear Model

- preference aggregator $h_i(\theta)$ is linear for each i :

$$h_i(\theta) = \theta_i + \sum_{j \neq i} \gamma_{ij} \theta_j.$$

- γ_{ij} represent influence of signal of j on the value of i
- interdependence matrix Γ :

$$\Gamma \triangleq \begin{bmatrix} 0 & |\gamma_{12}| & \cdots & |\gamma_{1I}| \\ |\gamma_{21}| & 0 & & \\ \vdots & & \ddots & \\ |\gamma_{I1}| & & & 0 \end{bmatrix}$$

Contraction Property

- consider the class of deceptions of the form:

$$\beta_i(\theta_i) = [\theta_i - \varepsilon c_i, \theta_i + \varepsilon c_i] \cap \Theta_i$$

Lemma 1. [Linear Aggregator]

Linear aggregator functions h satisfy the strict contraction property if and only if, for all $c \in \mathbb{R}_+^I$ with $c \neq \mathbf{0}$, there exists i such that

$$c_i > \sum_{j \neq i} |\gamma_{ij}| c_j.$$

- if the set of deceptions by i is too large, then there is an “overhang” which can be “nipped and tucked”

Duality

Lemma 2. [Duality]

The following two properties of Γ are equivalent:

1. for all $c \in \mathbb{R}_+^I$ with $c \neq \mathbf{0}$, there exists i such that:

$$c_i > \sum_{j \neq i} |\gamma_{ij}| c_j; \quad (\text{P})$$

2. there exists $d \in \mathbb{R}_+^I$ such that:

$$d_i > \sum_{j \neq i} |\gamma_{ji}| d_j, \quad (\text{D})$$

for all i .

Contraction Mapping

Theorem 2. [Contraction Property via Eigenvalue]

The interdependence matrix Γ has the contraction property if and only if its eigenvalue $\lambda < 1$.

- contraction property if and only if we can find a vector d , the eigenvector of the matrix Γ and eigenvalue $\lambda < 1$ with

$$\lambda \Gamma^T d = d$$

- for nonnegative matrices, left and right hand side eigenvalues coincide, hence also for some c

$$\lambda \Gamma c = c$$

- λ rate at which the iterative procedure contracts

Examples

- symmetric preferences: $\gamma_{ij} = \begin{cases} 1, & \text{if } j = i, \\ \gamma, & \text{if } j \neq i. \end{cases}$ and hence

$$\gamma < \frac{1}{I-1}$$

– moderate interdependence

- cyclic preferences: $\gamma_{ij} = \gamma_{(i-j) \bmod I}$ and

$$\sum_{j \neq i} \gamma_{(i-j)} < 1$$

Examples

- two bidders:

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix}$$

- sufficient condition is:

$$\gamma_{12}\gamma_{21} < 1.$$

EPIC in linear case

- public good example with general Γ :

$$1 + \sum_{j \neq i} \gamma_{ji} \geq 0$$

- single private good example

$$\gamma_{ij} \leq 1$$

Necessity

Definition 5. [Contraction Property]

The aggregator functions h satisfy the contraction property if, for all $\beta \neq \beta^*$, there exists i and $\theta'_i \in \beta_i(\theta_i)$ with $\theta'_i \neq \theta_i$, such that either

$$h_i(\theta_i, \theta_{-i}) = h_i(\theta'_i, \theta'_{-i}),$$

or

$$\text{sign}(\theta_i - \theta'_i) = \text{sign}(h_i(\theta_i, \theta_{-i}) - h_i(\theta'_i, \theta'_{-i})),$$

for all θ_{-i} and $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$.

- "nongeneric" weakening by allowing equality:

$$h_i(\theta) = h_i(\theta'_i, \theta'_{-i})$$

Necessity

Theorem 3. [Necessity]

If f is robustly implementable, then f satisfies strict EPIC and the contraction property.

- necessity holds for all mechanisms, direct or augmented
- implementation in direct mechanism cannot be improved by augmented mechanism

Robust Monotonicity

Definition 6. [Robust Monotonicity]

The social choice function satisfies robust monotonicity if, for all $\beta \neq \beta^*$, there exists i , θ_i and $\theta'_i \in \beta_i(\theta_i)$ such that, for all θ'_{-i} and $\psi_i \in \Delta(\beta_{-i}^{-1}(\theta'_{-i}))$, there exists y such that

$$\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_{-i}) u_i(y, (\theta_i, \theta_{-i})) > \sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_{-i}) u_i(f(\theta'_i, \theta'_{-i}), (\theta_i, \theta_{-i})),$$

while

$$u_i(f(\theta''_i, \theta'_{-i}), (\theta_i, \theta'_{-i})) > u_i(y, (\theta''_i, \theta'_{-i}))$$

for all $\theta''_i \in \Theta_i$.

Single Unit Auction

- agent i 's valuation of the object is

$$h_i(\theta) = \theta_i + \sum_{j \neq i} \gamma_{ij} \theta_j,$$

- efficient allocation rule if

$$q_i^*(\theta) = \begin{cases} \frac{1}{\#\{j: h_j(\theta) \geq h_k(\theta)\}}, & \text{if } h_i(\theta) \geq h_k(\theta) \text{ for all } k, \\ 0, & \text{if otherwise.} \end{cases}$$

- Dasgupta and Maskin: EPIC requires $\gamma_{ij} \leq 1$

- Chung and Ely (2002): elimination of weakly dominated strategies in direct mechanism gives efficient outcome if $\gamma_{ij} = \gamma < \frac{1}{I-1}$
- failure of strict ex post incentive compatibility
- modify VCG mechanism to symmetric ε -efficient allocation rule:

$$q_i^{**}(\theta) = \varepsilon \frac{\theta_i}{I} + (1 - \varepsilon) q_i^*(\theta)$$

- satisfies strict ex post incentive compatibility
- virtual implementation of efficient scf
- full virtual implementation if and only if not too much interdependence

Existence vs Uniqueness

- existence of efficient equilibrium in English Auction with interdependent values
 - Birulin and Izmalkov (2003): generalized single crossing condition
 - Echenique and Manelli (2004): dominant effect property
 - in symmetric model identical conclusion:

$$\gamma < 1$$

- uniqueness in second price auction

$$\gamma < \frac{1}{I - 1}$$

Conclusion

- interdependent valuations
- contraction property and moderate interdependence
- direct mechanism
- open questions:
 - can sequential mechanism improve the bound?
 - can we replace direct mechanism with detail free ascending mechanisms?
 - which type spaces achieve the bounds via Bayesian equilibrium?