Limit Uniqueness in Global Games with General Prior and Noise Distributions
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1 Introduction

Carlsson and van Damme (1993) has shown that for a fixed smooth prior over payoff parameters, as noise goes to zero, there is a unique equilibrium. The noise removes common knowledge away from the limit. Intuitively, this suggests that "enough lack of common knowledge" is required for uniqueness.

In earlier work (Morris and Shin (1999, 2001, 2003), we showed that in the particular case of normally distributed priors and noise, we can get an exact characterization of uniqueness conditions. For example, consider two players playing the following game:

<table>
<thead>
<tr>
<th></th>
<th>Invest</th>
<th>Not Invest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invest</td>
<td>1, 1</td>
<td>0, \theta</td>
</tr>
<tr>
<td>Not Invest</td>
<td>\theta, 0</td>
<td>\theta, \theta</td>
</tr>
</tbody>
</table>

Let \( \theta \) be normally distributed with mean \( y \) and variance \( \tau^2 \). Let each player observe a signal \( x_i = \theta + \varepsilon_i \), where the \( \varepsilon_i \) are independently normally distributed with mean 0 and precision \( \sigma^2 \). Then there is a unique equilibrium if and only if

\[
\frac{\sigma^2}{\tau^4} \left( \frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2} \right) \leq 2\pi. \tag{2}
\]

This implies that if both \( \sigma \) and \( \tau \) converge to zero, it is necessary that \( \sigma \) goes to zero faster than \( \tau \), i.e. \( \frac{\sigma}{\tau} \leq 2\pi \) in the limit. The same condition applies to a many player or continuum player game where a player’s payoff to investing is linear in the proportion of others investing. Similar conditions apply to general global games with different payoffs.

In these notes, we make three observations.

First, we show (in an example) that if the prior is continuous and strictly positive but not smooth then even as noise goes to zero, there may be multiple equilibria.

Second, we show that for differentiable priors, if \( \sigma = c\tau \), there is multiplicity as \( \tau \to 0 \).

Third, we show that for differentiable priors and uniform noise, the square root rule is general, i.e., as \( \tau \to 0 \) and \( \sigma \to 0 \), there is uniqueness if and only if \( \frac{\sigma}{\tau} \) is less than some constant in the limit.

2 Model

Two players are playing the game described in (1). The true state \( \theta = y + \tau\eta \), where \( \eta \sim g(\cdot) \). Each player observes \( x_i = \theta + \sigma\varepsilon_i \), where each \( \varepsilon_i \) is distributed
according to density $f(\cdot)$. Thus $\tau$ and $\sigma$ are thus parameters that allow as to change the distribution. Up to a constant, $\tau$ measures the standard deviation of the public signal and $\sigma$ measures the standard deviation of the private signal. We assume that $f$ satisfies a monotone likelihood ratio property (MLRP), i.e., $\theta > \theta' \Rightarrow \frac{f(x_i - \theta)}{f(x_i - \theta')}$ is increasing in $x_i$.

3 A First Characterization of Uniqueness

Suppose player 1 expects his opponent to follow the threshold strategy with cutoff $x^*$, i.e.,

$$s_2(x_2) = \begin{cases} \text{Invest, if } x \leq x^* \\ \text{Not Invest, if } x > x^* \end{cases}$$

If player 1 observes signal $x^*$, his expected payoff to investing is:

$$\Pr(x_2 \leq x^* | x^*) = \int_{-\infty}^{\infty} g \left( \frac{1}{\tau} (x^* - y - \sigma \varepsilon) \right) f(\varepsilon) F(\varepsilon) d\varepsilon$$

His expected payoff to not investing is:

$$E(\theta | x^*) = x^* - \sigma \int_{-\infty}^{\infty} g \left( \frac{1}{\tau} (x^* - y - \sigma \varepsilon) \right) f(\varepsilon) d\varepsilon$$

Net gain to investing at $x^*$ if opponent has threshold $x^*$ is

$$U(x^*, y) = \Pr(x_2 \leq x^* | x^*) - E(\theta | x^*)$$

Lemma 1 If $\frac{\partial U(x^*, y)}{\partial x^*} > 0$ for some $(x^*, y)$, then there are multiple equilibria. If $\frac{\partial U(x^*, y)}{\partial x^*} \leq 0$ for all $(x^*, y)$, then there is a unique equilibrium.

We sketch the argument.
We first claim that there is a threshold equilibrium of the game with the \( x^* \) cutoff strategy if and only if \( U(x^*, y) = 0 \). The "if" part follows from the fact that individual observing \( x^* \) is indifferent between investing or not investing. By MLRP, anyone observing a higher (lower) signal will strictly prefer to (not) invest. The "only if" part is true by contradiction: if the \( x^* \) cutoff strategy were an equilibrium and \( U(x^*, y) > 0 \), then by continuity there would be a neighbourhood of \( x^* \) where player would have a strict gain to investing.

If there are multiple values of \( x \) solving \( U(x; y) = 0 \), then clearly there are multiple equilibria for that \( y \). If there is a unique value of \( x \) solving \( U(x; y) = 0 \), then a standard iterative deletion argument ensures that the equilibrium is the threshold equilibrium with that threshold.

Now observe that for any \( y \), as \( x^* \to -\infty \), \( U(x^*, y) \to -\infty \) and as \( x^* \to \infty \), \( U(x^*, y) \to -\infty \). Thus for any fixed \( y \), a necessary and sufficient condition for multiple equilibria is that there exists \( x^* \) with \( \frac{\partial U(x^*, y)}{\partial x} > 0 \) and \( U(x^*, y) = 0 \). Thus a sufficient condition for uniqueness is that \( \frac{\partial U(x^*, y)}{\partial x} \leq 0 \) for all \((x^*, y)\).

But now suppose that \( \frac{\partial U(x^*, y)}{\partial x} > 0 \). Then set \( x^* = x^* + U(x^*, y) \) and \( \bar{y} = y + U(x^*, y) \). Now \( \frac{\partial U(x^*, \bar{y})}{\partial x} > 0 \) and \( U(x^*, \bar{y}) = U(x^*, y) - U(x^*, y) = 0 \). So there are multiple equilibria.

Now we can exploit a convenient change of variables. Let \( z = \frac{x}{y} (x^* - y) \) and \( c = \frac{x}{y} \) (so \( x^* = y + rz \)). Let

\[
V(z, y, c, \tau) = \int_{\varepsilon=-\infty}^{\varepsilon=\infty} g(z - c\varepsilon) f(\varepsilon) F(\varepsilon) d\varepsilon - \int_{\varepsilon=-\infty}^{\varepsilon=\infty} g(z - c\varepsilon) f(\varepsilon) c d\varepsilon
\]

**Corollary 2** A necessary and sufficient condition for uniqueness is that \( \frac{\partial V}{\partial z} \) is never positive.

We will be using this condition.

### 4 Exponential Example

Let

\[
f(\varepsilon) = \begin{cases} 1, & \text{if } \varepsilon \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases}
\]

\[
F(\varepsilon) = \begin{cases} 0, & \text{if } \varepsilon \leq -\frac{1}{2} \\ \varepsilon + \frac{1}{2}, & \text{if } -\frac{1}{2} \leq \varepsilon \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq \varepsilon \end{cases}
\]

\[
g(\eta) = \frac{1}{2} e^{-|\eta|}
\]
4.1 Some Properties of Exponential Distribution

\[
\int_{a}^{b} e^{-\lambda x} dx = \left[ -\frac{1}{\lambda} e^{-\lambda x} \right]_{a}^{b} = \frac{1}{\lambda} (e^{-\lambda a} - e^{-\lambda b})
\]

\[
\int_{a}^{b} xe^{-\lambda x} dx = \left[ -\frac{1}{\lambda} \left(x + \frac{1}{\lambda}\right) e^{-\lambda x} \right]_{a}^{b}
\]

\[
\int_{a}^{b} xe^{\lambda x} dx = \left[ \frac{1}{\lambda} \left(x - \frac{1}{\lambda}\right) e^{\lambda x} \right]_{a}^{b}
\]

4.2 Evaluating V

4.2.1 Case 1: \( z \geq \frac{1}{2}c \)

\[
\int_{\varepsilon=-\infty}^{\infty} g(z-c\varepsilon) f(\varepsilon) d\varepsilon = \int_{\varepsilon=-\frac{1}{2}}^{\frac{1}{2}} g(z-c\varepsilon) d\varepsilon
\]

\[
= \int_{\varepsilon=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} e^{-(z-c\varepsilon)} d\varepsilon
\]

\[
= \frac{1}{2} e^{-z} \int_{\varepsilon=-\frac{1}{2}}^{\frac{1}{2}} e^{c\varepsilon} d\varepsilon
\]

\[
= \frac{1}{2} e^{-z} \left[ \frac{1}{c} e^{c\varepsilon} \right]_{-\frac{1}{2}}^{\frac{1}{2}}
\]

\[
= \frac{1}{2c} e^{-z} \left( e^{\frac{1}{2}c} - e^{-\frac{1}{2}c} \right)
\]

\[
\int_{\varepsilon=-\infty}^{\infty} g(z-c\varepsilon) f(\varepsilon) \varepsilon d\varepsilon = \frac{1}{2} e^{-z} \int_{\varepsilon=-\frac{1}{2}}^{\frac{1}{2}} e^{c\varepsilon} \varepsilon d\varepsilon
\]

\[
= \frac{1}{2} e^{-z} \left[ \frac{1}{c} \left( \varepsilon - \frac{1}{c} \right) e^{c\varepsilon} \right]_{-\frac{1}{2}}^{\frac{1}{2}}
\]

\[
= \frac{1}{2c} e^{-z} \left( \left( \frac{1}{2} - \frac{1}{c} \right) e^{\frac{1}{2}c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}c} \right)
\]
Thus

\[
\psi(z, c) = \frac{\int_{c}^{\infty} g(z - ce) f(\varepsilon) \varepsilon d\varepsilon}{\int_{c}^{\infty} g(z - ce) f(\varepsilon) d\varepsilon}
\]

\[
= \frac{\frac{1}{2}e^{-z} \left( \left( \frac{1}{2} - \frac{1}{c} \right) e^{\frac{1}{2}c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}c} \right) - \frac{1}{2}e^{-z} \left( e^{\frac{1}{2}c} - e^{-\frac{1}{2}c} \right)}{e^{\frac{1}{2}c} - e^{-\frac{1}{2}c}}
\]

Thus

\[
V(z, y; c, \tau) = \frac{\int_{c}^{\infty} g(z - ce) f(\varepsilon) F(\varepsilon) d\varepsilon}{\int_{c}^{\infty} g(z - ce) f(\varepsilon) d\varepsilon} - y - \tau z + \tau c \int_{c}^{\infty} g(z - ce) f(\varepsilon) d\varepsilon
\]

\[
= \frac{1}{2} - y - \tau z + (1 + \tau c) \psi(z, c)
\]

\[
= \frac{1}{2} - y - \tau z + (1 + \tau c) \left( \frac{\left( \frac{1}{2} - \frac{1}{c} \right) e^{\frac{1}{2}c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}c}}{e^{\frac{1}{2}c} - e^{-\frac{1}{2}c}} \right)
\]

4.2.2 Case 2: \( z \leq -\frac{1}{2}c \)

\[
\int_{c}^{\infty} g(z - ce) f(\varepsilon) d\varepsilon = \int_{-\frac{1}{2}}^{\infty} g(z - ce) d\varepsilon
\]

\[
= \int_{-\frac{1}{2}}^{\infty} \frac{1}{2} e^{-\varepsilon} c d\varepsilon
\]

\[
= \frac{1}{2} e^{-z} \int_{-\frac{1}{2}}^{\infty} e^{-\varepsilon} d\varepsilon
\]

\[
= \frac{1}{2} e^{-z} \left[ \left( -\frac{1}{c} e^{-c\varepsilon} \right)^{\frac{1}{2}} \right]_{-\frac{1}{2}}^{\infty}
\]

\[
= \frac{1}{2} e^{-z} \left( e^{\frac{1}{2}c} - e^{-\frac{1}{2}c} \right)
\]
\[ \int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) f(\varepsilon) \varepsilon d\varepsilon = \frac{1}{2} e^{-z} \int_{\varepsilon = -\frac{1}{2}}^{\varepsilon = \frac{1}{2}} e^{-\varepsilon} \varepsilon d\varepsilon \]

\[ = \frac{1}{2} e^{-z} \left[ -\frac{1}{e} \left( \varepsilon + \frac{1}{e} \right) e^{\varepsilon} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \]

\[ = \frac{1}{2} e^{-z} \left( \left( -\frac{1}{2} + \frac{1}{e} \right) e^{\frac{1}{2}e} - \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}e} \right) \]

Thus

\[ \psi(z, c) = \frac{\int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) f(\varepsilon) \varepsilon d\varepsilon}{\int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) f(\varepsilon) d\varepsilon} \]

\[ = \frac{\frac{1}{2} e^{-z} \left( \left( -\frac{1}{2} + \frac{1}{e} \right) e^{\frac{1}{2}e} - \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}e} \right)}{\frac{1}{2} e^{-z} \left( e^{\frac{1}{2}e} - e^{-\frac{1}{2}e} \right)} \]

\[ = \frac{\left( -\frac{1}{2} + \frac{1}{e} \right) e^{\frac{1}{2}e} - \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}e}}{e^{\frac{1}{2}e} - e^{-\frac{1}{2}e}} \]

Thus

\[ V(z, y; c, \tau) = \frac{\int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) F(\varepsilon) d\varepsilon}{\int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) f(\varepsilon) d\varepsilon} - y - \tau z + \tau \psi(z, c) \]

\[ = \frac{1}{2} - y - \tau z + (1 + \tau c) \psi(z, c) \]

\[ = \frac{1}{2} - y - \tau z + (1 + \tau c) \left( -\frac{1}{2} + \frac{1}{e} \right) e^{\frac{1}{2}e} - \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2}e} \]

\[ e^{\frac{1}{2}e} - e^{-\frac{1}{2}e} \]

6
4.2.3 Case 3: \(-\frac{1}{2}c \leq z \leq \frac{1}{2}c\)

\[
\int_{\varepsilon=\infty}^{\infty} g(z - \varepsilon c) f(\varepsilon) \, d\varepsilon = \int_{\varepsilon=\frac{1}{2}}^{\frac{1}{2}} g(z - \varepsilon c) \, d\varepsilon
\]

\[
= \int_{\varepsilon=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} e^{-z+\varepsilon c} \, d\varepsilon + \int_{\varepsilon=\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} e^{z-\varepsilon c} \, d\varepsilon
\]

\[
= \frac{1}{2} e^{-z} \left[ \frac{1}{c} e^{\varepsilon c} \right]_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2} e^{z} \left[ -\frac{1}{c} e^{-\varepsilon c} \right]_{\frac{1}{2}}^{\frac{1}{2}}
\]

\[
= \frac{1}{2} e^{-\varepsilon} \left( \frac{z}{c} - \frac{1}{c} e^{\varepsilon c} \right) + \frac{1}{2} e^{z} \left( \frac{1}{2} + \frac{1}{c} e^{-\varepsilon c} \right) e^{-\varepsilon c} \left( -\frac{1}{2} c + \frac{1}{2} c e^{-\varepsilon c} + \frac{z}{2} \right)
\]

Thus

\[
\psi(z, c) = \frac{1}{c} e^{-\varepsilon} \left( \frac{2z}{c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2} c} (e^{-z} - e^z) \right)
\]

\[
= \frac{1}{c} \left( -e^{-\frac{1}{2} c} \left( e^{-z} + e^z \right) \right)
\]

\[
= \frac{2z}{c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2} c} (e^{-z} - e^z)
\]

\[
= \frac{2z}{c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2} c} (e^{-z} - e^z)
\]

\[
= \frac{2z}{c} + \left( \frac{1}{2} + \frac{1}{c} \right) e^{-\frac{1}{2} c} (e^{-z} - e^z)
\]
Thus

\[ V(z, \tau; c, \sigma) = \int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) f(\varepsilon) d\varepsilon - y - \tau z + \tau c \int_{\varepsilon = -\infty}^{\infty} g(z - c\varepsilon) f(\varepsilon) \varepsilon d\varepsilon \]

\[ = \frac{1}{2} - y - \tau z + (1 + \tau c) \psi(z, c) \]

\[ = \frac{1}{2} - y - \tau z + (1 + \tau c) \left( \frac{2z \varepsilon + (1 + \frac{1}{2}c) e^{-\frac{1}{2}c}(e^{-\frac{1}{2}c} - e^{\frac{1}{2}c})}{2 - e^{-\frac{1}{2}c}(e^{-\frac{1}{2}c} + e^{\frac{1}{2}c})} \right) \]

### 4.3 What the multiple equilibria look like....

In this section, we give a heuristic description of the multiple equilibria that exist when \( y = \frac{1}{2}, \tau < \frac{1}{2} \) and \( \sigma \) (and thus \( c \)) is very small. First observe that \( V(0, \frac{1}{2}; c, \tau) = 0 \) for all \( c \) and \( \tau \). Thus there is an equilibrium corresponding to \( z = 0 \) for all \( c, \tau \).

Observe that the second order Taylor series expansion for \( e^x \) is \( 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \). Thus

\[ e^{\frac{1}{2}c} \approx 1 + \frac{1}{2}c + \frac{1}{8}c^2 + \frac{1}{48}c^3 \]

\[ e^{-\frac{1}{2}c} \approx 1 - \frac{1}{2}c + \frac{1}{8}c^2 - \frac{1}{48}c^3 \]

for small \( c \). Thus if \( z \geq \frac{1}{2}c \),

\[ \psi(z, c) = \frac{(1 - \frac{1}{2}c) e^{\frac{1}{2}c} + (1 + \frac{1}{2}c) e^{-\frac{1}{2}c}}{e^{\frac{1}{2}c} - e^{-\frac{1}{2}c}} \approx \frac{(1 - \frac{1}{2}c) (1 + \frac{1}{2}c + \frac{1}{8}c^2 + \frac{1}{32}c^3) + (1 + \frac{1}{2}c) (1 - \frac{1}{2}c + \frac{1}{8}c^2 - \frac{1}{32}c^3)}{(1 + \frac{1}{2}c + \frac{1}{8}c^2 + \frac{1}{32}c^3) - (1 - \frac{1}{2}c + \frac{1}{8}c^2 - \frac{1}{32}c^3)} \]

\[ = \frac{1}{12} c \]

\[ \approx \frac{1}{12} c \]

Thus if \( z \geq \frac{1}{2}c \) and \( c \) is small,

\[ V\left(z, \frac{1}{2}; c, \tau\right) \approx \frac{1}{2} - \frac{1}{2} - \tau z + (1 + \tau c) \frac{1}{12} c \]

\[ \approx -\tau z + \frac{1}{12} c. \]

This has a zero with \( z = \frac{\tau}{12\tau} \). Observe that if \( \tau < \frac{1}{6} \Rightarrow z > \frac{1}{2}c. \)
Also if \( z \leq -\frac{1}{2} c \),

\[
\psi(z, c) = \frac{\left(-\frac{1}{2} + \frac{1}{c}\right) e^{\frac{1}{2} c} - \left(\frac{1}{2} + \frac{1}{c}\right) e^{-\frac{1}{2} c}}{c_{2}^{\frac{1}{2} c} - e^{-\frac{1}{2} c}}
\]

\[
\approx \frac{\left(-\frac{1}{2} + \frac{1}{c}\right) \left(\frac{1}{2} + \frac{3}{8} c^{2} + \frac{1}{35} c^{3}\right) - \left(\frac{1}{2} + \frac{1}{c}\right) \left(1 - \frac{1}{2} c + \frac{1}{5} c^{2} - \frac{1}{35} c^{3}\right)}{\left(\frac{1}{2} + \frac{3}{8} c^{2} + \frac{1}{35} c^{3}\right) - \left(1 - \frac{1}{2} c + \frac{1}{5} c^{2} - \frac{1}{35} c^{3}\right)}
\]

\[
= -\frac{1}{12} c^{2}
\]

\[
= -\frac{1}{12} c
\]

Thus if \( z \leq \frac{1}{2} c \) and \( c \) is small,

\[
V\left(z, \frac{1}{2} c, \tau\right) \approx \frac{1}{2} - \frac{1}{2} - \tau z - (1 + \tau c) \frac{1}{12} c
\]

\[
\approx -\tau z - \frac{1}{12} c.
\]

This has a zero with \( z = -\frac{1}{12} \). Observe that if \( \tau < \frac{1}{6} \Rightarrow z < -\frac{1}{2} c \).

Thus we have three zeros with \( z = -\frac{c}{12} \), \( 0 \) and \( \frac{c}{12} \). This corresponds to equilibria with thresholds \( x^{*} = y - \frac{c}{12}, y \) and \( y + \frac{c}{12} \).

### 4.4 Evaluating \( \frac{\partial V}{\partial z} \)

As \( \tau \) moves above \( \frac{1}{2} \), the multiple solutions to \( V(z, y; c, \tau) = 0 \) will all arise for \(-\frac{1}{2} \leq z \leq \frac{1}{2} \). Rather than explicit calculate equilibria, we can use the restriction on \( \frac{\partial V}{\partial z} \) to check if there are multiple equilibria.

In cases 1 and 2, \( \frac{\partial V}{\partial z} = -\tau \) (since \( \psi(z, c) \) does not depend on \( z \)). So the interesting case is case 3. In this case,

\[
\frac{\partial V}{\partial z} = -\tau + (1 + \tau c) \left(\frac{1}{2} - e^{-\frac{1}{2} c} (e^{-z} + e^{z})\right) \left(\frac{2}{2 - e^{-\frac{1}{2} c} (e^{-z} + e^{z})} - \left(\frac{2}{2 e^{-\frac{1}{2} c} (e^{-z} + e^{z})} - \frac{1}{2} \right) e^{-\frac{1}{2} c} (e^{-z} + e^{z})\right)
\]

This expression is maximized at \( z = 0 \) (this can be seen by simulating the function and is pretty intuitive, but an analytic proof would be nice). Now

\[
\left. \frac{\partial V}{\partial z} \right|_{z=0} = -\tau + (1 + \tau c) \left(\frac{1}{1 - e^{-\frac{1}{2} c}}\right) \left(\frac{1}{c} - \left(\frac{1}{2} + \frac{1}{c}\right) e^{-\frac{1}{2} c}\right)
\]

\[
= -\tau + (1 + \tau c) \left(\frac{1}{1 - e^{-\frac{1}{2} c}}\right) \left(\frac{1}{c} - \left(1 - e^{-\frac{1}{2} c}\right) - \frac{1}{2} e^{-\frac{1}{2} c}\right)
\]

\[
= -\tau + (1 + \tau c) \left(\frac{1}{c} - \frac{1}{2} \left(1 - e^{-\frac{1}{2} c}\right)\right)
\]

\[
= \frac{1}{c} - \frac{1}{2} (1 + \tau c) \left(\frac{1}{1 - e^{-\frac{1}{2} c}} - 1\right)
\]
Thus the uniqueness condition is that we must have

\[
\frac{1}{c} - \frac{1}{2} \left( 1 + \tau c \right) \left( \frac{1}{1 - e^{-\frac{1}{2}c}} - 1 \right) \leq 0.
\]

This requires that

\[
\tau \geq \frac{1}{c} \left[ \frac{2}{c \left( \frac{1}{1 - e^{-\frac{1}{2}c}} - 1 \right)} - 1 \right]
\]
\[
= \frac{1}{c} \left[ \frac{2 \left( 1 - e^{-\frac{1}{2}c} \right)}{ce^{-\frac{1}{2}c}} - 1 \right]
\]
\[
= \frac{1}{c^2} \left[ 2e^{\frac{1}{2}c} - 2 - c \right]
\]

and this is the exact uniqueness condition. For small \( c \), the right hand side is approximately equal to

\[
\frac{1}{c^2} \left[ 2e^{\frac{1}{2}c} - 2 - c \right] \approx \frac{1}{c^2} \left[ 2 \left( 1 + \frac{1}{2}c + \frac{1}{8}c^2 + \frac{1}{48}c^3 \right) - 2 - c \right]
\]
\[
= \frac{1}{4} + \frac{1}{24}c.
\]

In summary, there is a unique equilibrium if and only

\[
\tau \geq \frac{1}{c^2} \left[ 2e^{\frac{1}{2}c} - 2 - c \right]
\]

For small \( c \), this is equivalent to the requirement that

\[
\tau \geq \frac{1}{4} + \frac{1}{24}c.
\]

5 General Distributions

Now consider the case where \( g \) is a smooth density.

**Proposition 3** Fix \( c > 0 \). For \( \sigma = c\tau \) and sufficiently small \( \tau \), there are multiple equilibria.

Fix \( c^* > 0 \). Suppose player 1 expects his opponent to follow the threshold strategy with cutoff \( x^* \), i.e.,

\[
s_2 (x_2) = \begin{cases} 
\text{Invest, if } x \leq x^* \\
\text{Not Invest, if } x > x^*
\end{cases}.
\]
Recall that if player 1 observes signal $x^*$, his expected payoff to investing is:

$$
\Pr(x_2 \leq x^* | x^*) = \frac{\int_{\varepsilon=-\infty}^{\infty} g \left( \frac{1}{\tau} (x^* - y - \sigma \varepsilon) \right) f(\varepsilon) F(\varepsilon) d\varepsilon}{\int_{\varepsilon=-\infty}^{\infty} g \left( \frac{1}{\tau} (x^* - y - \sigma \varepsilon) \right) f(\varepsilon) d\varepsilon}.
$$

With the change of variables $z = \frac{1}{\tau} (x^* - y)$ and $\sigma = \tau c^*$, this equals

$$
h(z, c) = \frac{\int_{\varepsilon=-\infty}^{\infty} g(z - \sigma \varepsilon) f(\varepsilon) F(\varepsilon) d\varepsilon}{\int_{\varepsilon=-\infty}^{\infty} g(z - \sigma \varepsilon) f(\varepsilon) d\varepsilon}.
$$

This is a differentiable function of $z$. It is not possible for $\frac{dh(z, c^*)}{dz}$ to be non-decreasing everywhere (I need a formal proof of this). Pick $\hat{z}$ such that

$$
\frac{dh(\hat{z}, c^*)}{dz} < 0.
$$

Now observe that the expected payoff to not investing is:

$$
E(\theta|x^*) = x^* - \sigma \int_{\varepsilon=-\infty}^{\infty} \frac{\int_{\varepsilon=-\infty}^{\infty} g \left( \frac{1}{\tau} (x^* - y - \sigma \varepsilon) \right) f(\varepsilon) \varepsilon d\varepsilon}{\int_{\varepsilon=-\infty}^{\infty} g \left( \frac{1}{\tau} (x^* - y - \sigma \varepsilon) \right) f(\varepsilon) d\varepsilon}.
$$

With the change of variables, this equals

$$
y + \tau z - \tau c \int_{\varepsilon=-\infty}^{\infty} \frac{\int_{\varepsilon=-\infty}^{\infty} g(z - \sigma \varepsilon) f(\varepsilon) \varepsilon d\varepsilon}{\int_{\varepsilon=-\infty}^{\infty} g(z - \sigma \varepsilon) f(\varepsilon) d\varepsilon}.
$$

Now let

$$
q(z, c) = \frac{\int_{\varepsilon=-\infty}^{\infty} g(z - \sigma \varepsilon) f(\varepsilon) \varepsilon d\varepsilon}{\int_{\varepsilon=-\infty}^{\infty} g(z - \sigma \varepsilon) f(\varepsilon) d\varepsilon}.
$$
Choose $\tau^*$ sufficiently small such that
\[
\left| \frac{dh(\tilde{z}, c^*)}{dz} \right| > \tau^* \left| 1 + c^* \frac{dq(\tilde{z}, c^*)}{dz} \right|
\]

Now observe that the payoff gain to investing at $z$ is
\[
V(z; y; c^*, \tau^*) = \int_{-\infty}^{\infty} g(z - c^*\varepsilon) f(\varepsilon) d\varepsilon - y - \tau^* z + \tau^* c^* q(z, c^*)
\]
\[
= h(z, c^*) - y - \tau^* z + \tau^* c^* q(z, c^*)
\]

Now setting $y^* = h(\tilde{z}, c^*) - \tau^* \tilde{z} + \tau^* c^* q(\tilde{z}, c^*)$. We have
\[
V(\tilde{z}, y^*; c^*, \tau^*) = h(\tilde{z}, c^*) - y^* - \tau^* \tilde{z} + \tau^* c^* q(\tilde{z}, c^*) = 0
\]
and
\[
\frac{\partial V(\tilde{z}, y^*; c^*, \tau^*)}{\partial z} = -\frac{dh(\tilde{z}, c^*)}{dz} - \tau^* - \tau^* c^* \frac{dq(\tilde{z}, c^*)}{dz} > 0.
\]

6 General Prior and Uniform Noise

Let
\[
f(\varepsilon) = \begin{cases} 
1, & \text{if } \varepsilon \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\
0, & \text{otherwise}
\end{cases}
\]
\[
F(\varepsilon) = \begin{cases} 
0, & \text{if } \varepsilon \leq -\frac{1}{2} \\
\varepsilon + \frac{1}{2}, & \text{if } -\frac{1}{2} \leq \varepsilon \leq \frac{1}{2} \\
1, & \text{if } \frac{1}{2} \leq \varepsilon
\end{cases}
\]

We allow for a general differentiable prior with a uniform bound on $\frac{d}{dz} \left( \frac{g'(z)}{g(z)} \right)$.

**Proposition 4** As both $\sigma \to 0$ and $\tau \to 0$, there is uniqueness only if
\[
\tau^2 > \frac{\sigma}{12} \left| \min_{z} \left( \frac{d}{dz} \left( \frac{g'(z)}{g(z)} \right) \right) \right|
\]
\[
\psi(z,c) = \frac{1}{12} \int_{-1/2}^{1/2} \left( g(z - c\varepsilon) - g(z) \right) \varepsilon \, d\varepsilon
\]

So for small \( c \),
\[
\frac{dV}{dz} = -\tau + \frac{d\psi}{dz}
\]

So there is uniqueness for small \( c \) if and only if
\[
\tau > \frac{c}{12} \left| \min_z \left( \frac{d}{dz} \left( \frac{g'(z)}{g(z)} \right) \right) \right|
\]