Robust Virtual Implementation

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An Older Title

Strategic Distinguishability with an application to Robust Virtual Implementation
Interdependent Preferences

- preferences are frequently assumed to be interdependent for informational or psychological reasons
- what are the observable implications of interdependent preferences?
- “revealed preference” is well-developed theory to understand individual choice with independent preferences
- an approach to “strategic revealed preference” is suggested to understand interdependent preferences
Strategic Distinguishability

- each agent’s preference depends on the “payoff types” of all agents
- two types of an agent are “strategically indistinguishable” if in every game there exists some common action which each type might rationally choose given some beliefs and higher-order beliefs
- two types of an agent are “strategically distinguishable” if there exists a game such that those types must rationally choose different actions whatever their beliefs and higher-order beliefs
- we characterize strategic distinguishability for general environments:
  - basic idea: types are strategically distinguishable if there is not too much interdependence of preferences
Robust Virtual Implementation

- social choice function maps payoff type profiles to outcomes
- "robust implementation": there exists a mechanism such that every equilibrium delivers the socially desired outcome whatever players’ beliefs and higher order beliefs about others’ types
- "robust virtual implementation": there exists a mechanism such that every equilibrium delivers the socially desired outcome with probability at least $1 - \varepsilon$ whatever players’ beliefs and higher order beliefs about others’ types
- necessary conditions:
  1. ex post incentive compatibility
  2. robust measurability: strategically indistinguishable always receive same allocation
- sufficiency: extending an argument of Abreu-Matsushima 1992
Auction Example

- $l$ agents with quasilinear utility
- agent $i$ has type $\theta_i \in \Theta_i = [0, 1]$
- agent $i$’s valuation of a single object is $\nu_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$
Auction Example

- $l$ agents with quasilinear utility
- agent $i$ has type $\theta_i \in \Theta_i = [0, 1]$
- agent $i$'s valuation of a single object is $v_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$
- suppose
  1. $\gamma \geq \frac{1}{l-1}$
  2. every low $\theta_i$ valuation agent was convinced that other agents were high $\theta_j$ agents, and vice versa
  3. in particular, each payoff type $\theta_i$ is convinced that his opponents are types $\theta_j = \frac{1}{2} + \frac{1}{\gamma(l-1)} \left( \frac{1}{2} - \theta_i \right)$
Auction Example

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- each payoff type $\theta_i$ is convinced that his opponents are types $\theta_j = \frac{1}{2} + \frac{1}{\gamma (I-1)} (\frac{1}{2} - \theta_i)$
- so

$$v_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$$
$$= \theta_i + \gamma \sum_{j \neq i} \left( \frac{1}{2} + \frac{1}{\gamma (I-1)} \left( \frac{1}{2} - \theta_i \right) \right)$$
$$= \frac{1}{2} \left( 1 + \gamma (I - 1) \right)$$
Auction Example

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- suppose
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  2. every low $\theta_i$ valuation agent was convinced that other agents were high $\theta_j$ agents, and vice versa
  3. in particular, each payoff $\theta_i$ is convinced that his opponents are types $\theta_j = \frac{1}{2} + \frac{1}{\gamma(l-1)} \left( \frac{1}{2} - \theta_i \right)$
- then common knowledge that everyone’s valuation of the object is $\frac{1}{2} \left( 1 + \gamma(l-1) \right)$
Auction Example

- So if $\gamma \geq \frac{1}{l-1}$, all pairs of types are strategically indistinguishable.
- It will turn out that if $\gamma < \frac{1}{l-1}$, all distinct pairs of types are strategically distinguishable.
Robust Virtual Implementation Results in Auction Example

- if $\gamma < \frac{1}{t-1}$, robust virtual implementation of the efficient outcome can be achieved (in the example, using a simple direct mechanism)
- if $\gamma \geq \frac{1}{t-1}$, inefficient multiple equilibria in ALL mechanisms
Robust Virtual Implementation Results in General Environments

Necessary and Sufficient Conditions:

1. Ex Post Incentive Compatibility
   - in example, $\gamma \leq 1$

2. "Robust Measurability" or Not Too Much Interdependence
   - in example, $\gamma < \frac{1}{I-1}$
Section 3:
ENVIRONMENT AND SOLUTION CONCEPTS
Environment

- $l$ agents
- lottery outcome space $Y = \Delta(X)$, $X$ finite
- finite ”payoff” types $\Theta_i$
- vNM utilities: $u_i : Y \times \Theta \to \mathbb{R}$
Mechanism

A mechanism $\mathcal{M}$ is a collection $((M_i)_{i=1}^I, g)$

- each $M_i$ is a finite message set
- outcome function $g : M \rightarrow Y$
Rationalizable Messages

- initialize at \( S_i^{M,0}(\theta_i) = M_i \), inductive step:
- \( S_i^{M,k+1}(\theta_i) = \)
  \[
  \exists \mu_i \in \Delta (\Theta_{-i} \times M_{-i}) \quad \text{s.t.:} \\
  \begin{align*}
  (1) \ & \mu_i (\theta_{-i}, m_{-i}) > 0 \Rightarrow m_{-i} \in S_i^{M,k}(\theta_i) \\
  (2) \ & m_i \in \arg \max_{m'_i} \sum_{\theta_{-i}, m_{-i}} \mu_i (\theta_{-i}, m_{-i}) u_i (g(m'_i, m_{-i}), \theta)
  \end{align*}
  \]
- limit set
  \[
  S_i^M(\theta_i) = \bigcap_{k \geq 0} S_i^{M,k}(\theta_i).
  \]
- \( S_i^M(\theta_i) \) are rationalizable actions of type \( \theta_i \) in mechanism \( M \)
Epistemic Foundations: Framework

- Type Space $\mathcal{T} = \left( T_i, \hat{\pi}_i, \hat{\theta}_i \right)_{i=1}^I$
  
  1. $T_i$ countable types of agent $i$
  2. $\hat{\pi}_i : T_i \rightarrow \Delta (T_{-i})$ (belief type component)
  3. $\hat{\theta}_i : T_i \rightarrow \Theta_i$ (payoff type component)

- incomplete information game $(\mathcal{T}, \mathcal{M})$
  
  - $i$’s strategy: $\sigma_i : T_i \rightarrow \Delta (M_i)$
  - strategy profile $\sigma$ is an equilibrium if $\sigma_i (m_i | t_i) > 0$ implies $m_i$ is in

$$\arg \max_{m'_i \in \mathcal{T}_{-i,m_{-i}}} \sum_{t_{-i},m_{-i}} \hat{\pi}_i [t_i] (t_{-i}) \left( \prod_{j \neq i} \sigma_j (m_j | t_j) \right) u_i \left( g (m'_i, m_{-i}), \hat{\theta} (t) \right)$$
Epistemic Foundations: Result

PROPOSITION. \( m_i \in S_i^M (\theta_i) \) if and only if there exist

1. a type space \( T \),
2. an equilibrium \( \sigma \) of \((T, M)\) and
3. a type \( t_i \in T_i \), such that
   3.1 \( \sigma_i (m_i|t_i) > 0 \) and
   3.2 \( \hat{\theta}_i (t_i) = \theta_i \).

Section 4:
STRATEGIC DISTINGUISHABILITY
DEFINITION. Types $\theta_i$ and $\theta'_i$ are strategically indistinguishable if

$$S^\mathcal{M} (\theta_i) \cap S^\mathcal{M} (\theta'_i) \neq \emptyset$$

for every $\mathcal{M}$.
Preference Relation

DEFINITION. $R_{\theta_i, \lambda_i}$ is a preference relation of agent $i$ with payoff type $\theta_i$ and conjecture $\lambda_i \in \Delta (\Theta_{-i})$ about types of others:

$$y R_{\theta_i, \lambda_i} y' \iff \sum_{\theta_{-i} \in \Theta_{-i}} \lambda_i (\theta_{-i}) u_i (y, \theta) \geq \sum_{\theta_{-i} \in \Theta_{-i}} \lambda_i (\theta_{-i}) u_i (y', \theta)$$

- write $\Psi_i \subseteq \Theta_j$ for subset and $\Psi_{-i} = \{ \Psi_j \}_{j \neq i}$ for profile of subsets
- possible preference profiles if $i$ assigns probability 1 to his opponents’ types to be $\theta_{-i} \in \Psi_{-i}$:

$$R_i (\theta_i, \Psi_{-i}) = \{ R \in \mathcal{R} | R = R_{\theta_i, \lambda_i} \text{ for some } \lambda_i \in \Delta (\Psi_{-i}) \}$$
Defining Separability

- with:
  \[ \mathcal{R}_i (\theta_i, \Psi_{-i}) = \{ R \in \mathcal{R} \mid R = R_{\theta_i, \lambda_i} \text{ for some } \lambda_i \in \Delta (\Psi_{-i}) \} \]

DEFINITION. Type set profile \( \Psi_{-i} \) separates \( \Psi_i \) if

\[
\bigcap_{\theta_i \in \Psi_i} \mathcal{R}_i (\theta_i, \Psi_{-i}) = \emptyset.
\]

- \( \Psi_{-i} \) separates \( \Psi_i \) if whatever realized preference of \( i \), we can rule out at least one possible type of \( i \).
Iterative Definition of Separability

- iteratively delete type sets of $i$ that are separated by some type set profile $\Psi_i$

\[ \Xi_i^0 = 2^{\Theta_i} \]
\[ \Xi_i^{k+1} = \left\{ \Psi_i \in 2^{\Theta_i} \left| \Psi_i \text{ doesn’t separate } \Psi_i \text{ for some } \Psi_{-i} \in \Xi_{-i}^k \right. \right\} \]

and limit type set profile is

\[ \Xi_i^* = \bigcap_{k \geq 0} \Xi_i^k \]
DEFINITION. Types $\theta_i$ and $\theta'_i$ are pairwise inseparable if

$$\{\theta_i, \theta'_i\} \in \mathbb{E}^*_i,$$

and we write $\theta_i \sim \theta'_i$.

- note $\sim$ is reflexive, symmetric, but not necessarily transitive
Back to the Auction Example

- $l$ bidders
- bidder $i$ has type $\theta_i \in \Theta_i = [0, 1]$
- bidder $i$’s valuation is $v_i(\theta, m_i) = \theta_i + \gamma \sum_{j 
eq i} \theta_j - m_i$
- set of possible preferences $= \text{set of possible valuations}$

$$V_i(\theta_i, \Psi_{-i}) = \left[ \theta_i + \gamma \sum_{j \neq i} \min \Psi_j, \quad \theta_i + \gamma \sum_{j \neq i} \max \Psi_j \right]$$
Separability in the Auction Example I

- now $\Psi_{-i}$ separates $\Psi_i$ if and only if

$$\bigcap_{\theta_i \in \Psi_i} V_i (\theta_i, \Psi_{-i}) = \emptyset$$

- suppose $\theta_i, \theta'_i \in \Psi_i$ and $\theta_i < \theta'_i$;

- there exist $\lambda_i, \lambda'_i \in \Delta (\Psi_{-i})$ such that $R_{\theta_i, \lambda_i} = R_{\theta'_i, \lambda'_i}$ iff

$$\theta_i + \gamma \sum_{j \neq i} \max \Psi_j \geq \theta'_i + \gamma \sum_{j \neq i} \min \Psi_j$$
Separability in the Auction Example II

- $\Psi_i$ is separable given $\Psi_{\neg i}$ if and only if

\[
\max \Psi_i - \min \Psi_i > \gamma \left( \sum_{j \neq i} \max \Psi_j - \min \Psi_j \right)
\]

- thus

\[
\Xi^1_i = \{ \Psi_i \mid \max \Psi_i - \min \Psi_i \leq [\gamma (l - 1)] \}
\]

and iteratively:

\[
\Xi^k_i = \{ \Psi_i \mid \max \Psi_i - \min \Psi_i \leq [\gamma (l - 1)]^k \}
\]
Pairwise Inseparability in the Auction Example

- If $\gamma \geq \frac{1}{i-1}$, all $\theta_i, \theta'_i$ are pairwise inseparable.
- If $\gamma < \frac{1}{i-1}$, $\theta_i \neq \theta'_i \Rightarrow \theta_i$ and $\theta'_i$ are pairwise separable.
- Pairwise separability requires “not too much interdependence.”
Fixed Point Characterization

Consider a collection of sets $\Xi = (\Xi_i)_{i=1}^I$, each $\Xi_i \subseteq 2^\Theta_i$.

**DEFINITION.** A collection $\Xi$ is mutually inseparable if, for each $i$ and $\Psi_i \in \Xi_i$, there exists $\Psi_-i \in \Xi_-i$ such that $\Psi_-i$ does not separate $\Psi_i$.

**LEMMA.** Types $\theta_i$ and $\theta'_i$ are pairwise inseparable if and only if there exists mutually inseparable $\Xi$ such that $\{\theta_i, \theta'_i\} \subseteq \Psi_i$ for some $\Psi_i \in \Xi_i$. 
Strategic Distinguishability

DEFINITION. Types $\theta_i$ and $\theta'_i$ are strategically indistinguishable if

$$S^\mathcal{M}(\theta_i) \cap S^\mathcal{M}(\theta'_i) \neq \emptyset$$

for every $\mathcal{M}$.

THEOREM 1. Types $\theta_i$ and $\theta'_i$ are strategically indistinguishable if and only if they are pairwise inseparable.
Sufficiency of Pairwise Separability I

PROPOSITION 1: If $\theta_i$ and $\theta'_i$ are indistinguishable, then

$$S_i^M (\theta_i) \cap S_i^M (\theta'_i) \neq \emptyset$$

in any mechanism $M$.

Suppose $\Xi$ is mutually inseparable

Fix any finite mechanism.
Sufficiency of Pairwise Separability II

By induction on $k$, for each $k$, $i$ and $\Psi_i \in \Xi_i$, there exists a common action $m_i^k (\Psi_i)$ such that $m_i^k (\Psi_i) \in S_i^k (\theta_i)$ for each $\theta_i \in \Psi_i$

1. True by definition for $k = 0$.

2. Suppose true for $k - 1$

   - fix any $i$ and $\Psi_i \in \Xi_i$
   - since $\Xi$ is mutually inseparable, $\exists \Psi_{-i} \in \Xi_{-i}$, $R_i$ and, for each $\theta_i \in \Psi_i$, $\lambda_{i}^{\theta_i} \in \Delta (\Psi_{-i})$ such that $R_{\theta_i, \lambda_{i}^{\theta_i}} = R_i$
   - $m_i^{k} (\Psi_i)$ be any element of the argmax under $R_i$ of $g \left( m_{i}, m_{-i}^{k-1} (\Psi_{-i}) \right)$
   - by construction, $m_i^{k} (\Psi_i) \in S_i^{\mathcal{I},k} (\theta_i)$ for all $\theta_i \in \Psi_i$. 
Necessity of Pairwise Separability

PROPOSITION 2: There exists a mechanism $\mathcal{M}^*$ such that if $\theta_i \sim \theta'_i$, then

$$S_i^{\mathcal{M}^*}(\theta_i) \cap S_i^{\mathcal{M}^*}(\theta'_i) = \emptyset.$$ 

PROOF: By construction of “maximally revealing mechanism”.
Construction of Maximally Revealing Mechanism I

uniform lottery \( \tilde{y} : \tilde{y}(x) \triangleq 1/|X| \)

KEY LEMMA:
Type set profile \( \Psi_i \) separates \( \Psi \) iff there exists \( \tilde{y} : \Psi_i \rightarrow Y \) such that

\[
\sum_{\theta_i \in \Psi_i} (\tilde{y}(\theta_i) - \bar{y}) = 0
\]

and, for each \( \theta_i \in \Psi_i \) and \( \lambda_i \in \Delta(\Psi_i) \),

\[
\tilde{y}(\theta_i) P_{\theta_i, \lambda_i} \bar{y}.
\]
Construction of Maximally Revealing Mechanism II

LEMMA (Morris 1994, Samet 1998): Let $V_1, ..., V_L$ be closed, convex, subsets of the $N$-dimensional simplex $\Delta^N$. These sets have an empty intersection if and only if there exist $z_1, ..., z_L \in \mathbb{R}^N$ such that

$$\sum_{l=1}^{L} z_l = 0$$

and

$$\nu \cdot z_l > 0$$

for each $l = 1, ..., L$ and $\nu \in V_i$.

Key lemma follows from this duality lemma, letting $\Theta_i = \{1, ..., L\}$ and $V_i$ be the set of possible utility weights of type $\theta_i = l$ with any $\lambda_i \in \Delta(\Psi_{-i})$. 
Construction of Maximally Revealing Mechanism III

- let $B^Y(\theta_i, \lambda_i)$ be the agents most preferred lotteries in the set $Y$ given type $\theta_i$ and belief $\lambda_i$:

$$B^Y_i(\theta_i, \lambda_i) = \{y \in Y \mid y R_{\theta_i, \lambda_i} y' \text{ for all } y' \in Y\}$$

TEST SET LEMMA. There exists a finite set $Y^* \subseteq Y$ such that

1. for each $i$, $\theta_i$ and $\lambda_i \in \Delta(\Theta_{-i})$, $B^Y_i(\theta_i, \lambda_i) \neq Y^*$
2. for each $i$, $\Psi_i$ and $\Psi_{-i}$, if $\Psi_{-i}$ separates $\Psi_i$, then for each $\theta_i \in \Psi_i$ and $\lambda_i \in \Delta(\Psi_{-i})$, there exists $\theta'_i \in \Psi_i$ such that

$$B^Y_i(\theta_i, \lambda_i) \cap B^Y_i(\theta'_i, \Psi_{-i}) = \emptyset.$$
Mechanism in Words

- each player makes $K$ simultaneous announcements:
  1. an element of test set $Y^*$
  2. a profile of first round announcements of other players he thinks possible, plus an element of $Y^*$
  3. a profile of second round announcements of other players he thinks possible, plus an element of $Y^*$
  4. ..... 

- all chosen outcomes selected with positive probability, with much higher weight on "earlier" announcements
Mechanism in Formulae

mechanism $\mathcal{M}^{K,\varepsilon} = \left( \left( M^K_i \right)_{i=1}^I, g^{K,\varepsilon} \right)$ parameterized by

1. $\varepsilon > 0$
2. integer $K$

- $i$'s message set is $M^K_i$ where
  - $M^0_i = \{ \overline{m}^0_i \}$
  - $M^{k+1}_i = M^k_i \times M^{k-1}_{-i} \times Y^*$
- typical element $m^k_i = \{ \overline{m}^0_i, r^1_i, y^1_i, \ldots, r^k_i, y^k_i \}$
- allocation rule:

$$g^{K,\varepsilon}(m) = \overline{y} + \frac{1 - \varepsilon^K}{1 - \varepsilon} \frac{1}{I} \sum_{k=1}^K \varepsilon^{k-1} \sum_{i=1}^I \mathbb{I} \left( r^k_i, m^{k-1}_{-i} \right) \left( y^k_i - \overline{y} \right)$$

where

$$\mathbb{I} \left( r^k_i, m^{k-1}_{-i} \right) = \begin{cases} 1, & \text{if } r^k_i = m^{k-1}_{-i} \\ 0, & \text{otherwise} \end{cases}$$
Conclusion of Proof of Proposition 2

1. Let

\[
\overline{\Theta}_i^k \left( m_i^k \right) = \overline{\Theta}_i^k \left( \left( m_i^{k-1}, r_i^k, y_i^k \right) \right) = \left\{ \theta_i \mid \theta_i \in \overline{\Theta}_i^{k-1} \left( m_i^{k-1} \right), \overline{\Theta}_i^{k-1} \left( r_i^k \right) \neq \emptyset, y_i^k \in B_i \left( \theta_i, \overline{\Theta}_i^{k-1} \left( r_i^k \right) \right) \right\}
\]

2. There exists \( \bar{\epsilon} > 0 \) such that

\[
\left\{ \theta_i \in \Theta_i \left| m_i^k \in S_i^{M_i^k, \epsilon} \left( \theta_i \right) \right\} \subseteq \overline{\Theta}_i^k \left( m_i^k \right)
\]

for all \( \epsilon \leq \bar{\epsilon} \) and \( m_i^k \in M_i^k \).

3. There exists \( \bar{\epsilon} > 0 \) and \( K \) such that

\[
\left\{ \theta_i \in \Theta_i \left| m_i^K \in S_i^{M_i^K, \epsilon} \left( \theta_i \right) \right\} \in \Xi_i^*
\]

for all \( \epsilon \leq \bar{\epsilon} \) and \( m_i^K \in M_i^K \).
Section 5:
ROBUST VIRTUAL IMPLEMENTATION
Definitions Reminder

- "implementation": requires ALL equilibria deliver the right outcome, a.k.a. full implementation
- "robust": same mechanism works independent of agents’ beliefs and higher order beliefs about the environment
- "virtual": enough if correct outcome arises with probability $1 - \varepsilon$
DEFINITION: A social choice function $f : \Theta \rightarrow Y$. Write $\|y - y'\|$ for the Euclidean distance between a pair of lotteries $y$ and $y'$, i.e.,

$$
\|y - y'\| = \sqrt{\sum_{x \in X} (y(x) - y'(x))^2}.
$$

DEFINITION: Social choice function $f$ is robustly $\varepsilon$-implementable if there exists a mechanism $M$ such that

$$
m \in S^M(\theta) \Rightarrow \|g(m) - f(\theta)\| \leq \varepsilon.
$$

DEFINITION: Social choice function $f$ is robustly virtually implementable if, for every $\varepsilon > 0$, $f$ is robustly $\varepsilon$-implementable.
RESULT:

DEFINITION: Social choice function $f$ satisfies ex post incentive compatibility if, for all $i$, $\theta_i$, $\theta_{-i}$ and $\theta'_i$:

$$u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) \geq u_i(f(\theta'_i, \theta_{-i}), (\theta_i, \theta_{-i})) .$$

DEFINITION: Social choice function $f$ satisfies robust measurability if $\theta_i \sim \theta'_i \Rightarrow f(\theta_i, \theta_{-i}) = f(\theta'_i, \theta_{-i})$, $\forall \theta_{-i}$.

THEOREM 2. Social choice function $f$ is robustly virtually implementable if and only if $f$ satisfies ex post incentive compatibility and robust measurability.
Intermediate Notions of Robustness

- Artemov-Kunimoto-Serrano (2008)
- given finite payoff types $\theta_i \in \Theta_i$
- every type puts probability $(1 - \delta)$ on uniform beliefs over opponents’ types, but could put remaining $\delta$ anywhere
- robust virtual implementation if $\gamma \delta < \frac{1}{I-1}$
Second Price Auction

- private values $\gamma = 0$ so $v_i = \theta_i$
- second price sealed bid auction
  - object goes to highest bidder
  - winner pays second highest bid
- truth-telling is a dominant strategy, but there are inefficient equilibria
Approximate Second Price Auction

- with probability $1 - \varepsilon$,
  - allocate object to highest bidder
  - winner pays second highest bid
- for each $i$, with probability $\frac{\varepsilon b_i}{I}$
  - $i$ gets object
  - pays $\frac{1}{2} b_i$
- truth-telling is a strictly dominant strategy so we can guarantee Robust Virtual Implementation
Modified Second Price Auction

- $\gamma > 0$, $v_i = \theta_i + \gamma \sum_{j \neq i} \theta_j$
- generalized second price sealed bid auction
  - object goes to highest bidder
  - winner pays $\max_j b_j + \gamma \sum_{j \neq i} b_j$
- if $\gamma \leq 1$, truth-telling is an "ex post" equilibrium but there are inefficient ex post equilibria ("ex post incentive compatibility")
Modified Second Price Auction

- with probability $1 - \varepsilon$,
  - allocate object to highest bidder $i$
  - winner pays $\max_{j \neq i} b_j + \gamma \sum_{j \neq i} b_j$

- for each $i$ with probability $\frac{\varepsilon b_i}{i}$,
  - $i$ gets object
  - pays $\frac{1}{2} b_i + \gamma \sum_{j \neq i} b_j$

truth telling is a strict ex post equilibrium
Abreu-Matsushima (1992) Incomplete Information

- Standard "Bayesian" incomplete information setting, i.e., common knowledge of common prior on type space
- Necessary conditions for virtual implementation
  - Bayesian incentive compatibility
  - Abreu-Matsushima measurability: types are iteratively distinguishable
  - reduces to "value distinction" in private values case
Adding Robustness

- with robustness, full implementation equivalent to belief free version of iterated deletion of strictly dominated strategies
- generalizing Abreu-Matsushima, necessary conditions become:

1. Ex post incentive compatibility (instead of Bayesian IC)
   - Bergemann-Morris "Robust Mechanism Design"

2. robust measurability as belief free version of AM measurability
Intermediate Notions of Robustness

Artemov-Kunimoto-Serrano (2008) consider type space with

- given finite payoff types $\theta_i \in \Theta_i$;
- given finite first-order beliefs $q_i (\theta_i | \theta_{-i})$

and general type space $T_i$ is assumed to be consistent with payoff types and first-order beliefs

- in the presence of a type diversity condition, incentive compatibility and AM measurability is necessary and sufficient for robust virtual implementation
- some tension between rich type space and type diversity
Exact Implementation I

following Maskin methods, necessary and sufficient conditions for exact robust implementation - using ANY mechanism:
(Bergemann-Morris "Robust Implementation in General Mechanisms" (2008))

1. ex post incentive compatibility
2. "robust monotonicity": not too much interdependence
in large class of economically interesting ”monotonic aggregator” environments:
(Bergemann-Morris ”Robust Implementation in Direct Mechanisms” (2007))

1. robust monotonicity = robust measurability
2. natural generalization of $\gamma < \frac{1}{t-1}$ condition
3. if robust virtual implementation is possible, it arises in modified direct mechanism
Conclusion

- strategic distinguishability: information revelation through choice in some game
- strategic distinguishability $\equiv$ not too much interdependence
- information revelation in maximally revealing mechanism
- virtual implementation via maximally revealing mechanism
- robust virtual implementation leads to sharp possibility but also impossibility results
Back to the Interdependent Auction Example

- $I$ agents with quasilinear utility
- agent $i$ has type $\theta_i \in \Theta_i = [0, 1]$
- agent $i$'s valuation of a single object is $v_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$
- efficient allocation:
  1. object to individual with highest valuation
  2. winner pays $\max_{j \neq i} b_j + \gamma \sum_{j \neq i} b_j$
Abreu-Matsushima (1992)

- Fix finite $\overline{\Theta}_i \subseteq \Theta_i = [0, 1]$
- Impose common prior $\pi \in \Delta (\overline{\Theta})$, where $\overline{\Theta} = \overline{\Theta}_1 \times \ldots \times \overline{\Theta}_I$
- Let $v_i(\theta_i) = \theta_i + \gamma \sum_j \sum_{j \neq i} \pi (\theta_j | \theta_i) \theta_j$
- With either independent priors or generic priors, we get type diversity

$$\theta_i, \theta'_i \in \overline{\Theta}_i, \theta_i \neq \theta'_i \Rightarrow v_i (\theta_i) \neq v'_i (\theta'_i)$$

- Thus all types can be distinguished on the basis of preferences over ”constant” or ”unconditional” lotteries
- Abreu-Matsushima measurability is trivially satisfied.
Artemov, Kunimoto and Serrano (2008)

- Fix finite ”payoff type” $\overline{\Theta}_i \subseteq \Theta_i = [0, 1]$
- A type space
  - agent $i$’s (epistemic) ”types” $T_i$
  - $\hat{\theta}_i : T_i \rightarrow \overline{\Theta}_i$
  - $\hat{\pi}_i : T_i \rightarrow \Delta (T_{-i})$
- First order belief restriction: fix finite ”pseudo-types” $PT_i \subseteq \overline{\Theta}_i \times \Delta (\overline{\Theta}_{-i})$
Type space consistent with first order belief restriction if

\[ \hat{\theta}_i(t_i) = \theta_i \]
and

\[ \sum_{\{t_{-i}|\hat{\theta}_{-i}(t_{-i})=\theta_{-i}\}} \hat{\pi}_i(t_{-i}|t_i) = q_i(\theta_{-i}) \text{ for all } \theta_{-i} \]

\[ \Rightarrow (\theta_i, q_i) \in PT_i \]

Can we find a mechanism that equilibrium fully implements an outcome within \( \varepsilon \) of the efficient on every type space consistent with the first order beliefs restriction?
Relation to Bergemann and Morris (2008)

- We provide a solution to the case with no restrictions on beliefs

<table>
<thead>
<tr>
<th></th>
<th>Equilibrium</th>
<th>$\Delta$-Rat.</th>
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</thead>
<tbody>
<tr>
<td>No Restrictions of FOB</td>
<td>$\cdot$</td>
<td>BM</td>
</tr>
<tr>
<td>Restrictions on FOB</td>
<td>AKS</td>
<td>$\cdot$</td>
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</tbody>
</table>

- Fact that we have no restrictions on beliefs allows compact representation of the relevant measurability condition: in particular expressed in terms of preferences over constant lotteries, not lotteries contingent on other agents’ types
First Order Beliefs Restriction in Auction Example

- $t_i \equiv (\theta_i, q_i)$
- $v_i(\theta_i, q_i) = \theta_i + \gamma \sum_{j \neq i} \sum \theta_j q_i (\theta_j|\theta_i)$
- Pseudo-Type Diversity (AKS): $(\theta_i, q_i), (\theta'_i, q'_i) \in PT_i, (\theta_i, q_i) \neq (\theta'_i, q'_i) \Rightarrow v_i(\theta_i, q_i) \neq v_i(\theta'_i, q'_i)$
- Generically satisfied for finite $PT_i$?
- Weaker Pseudo-Type Diversity (BM): $(\theta_i, q_i), (\theta'_i, q'_i) \in PT_i, \theta_i \neq \theta'_i \Rightarrow v_i(\theta_i, q_i) \neq v_i(\theta'_i, q'_i)$
- Pseudo-Type Diversity is sufficient for virtual robust implementation
- Weaker pseudo-type diversity also sufficient
Continuum Payoff Types / First Order Belief Restrictions

- $\Theta_i = [0, 1]$
- Pseudo-Types $PT_i = \Theta_i \times Q_i$, where $Q_i \subseteq \Delta\left([0, 1]^{l-1}\right)$
- Suppose $\gamma \in \left(0, \frac{1}{l-1}\right]$. Pseudo Type Diversity only if each $Q_i$ is a singleton.
- Suppose $Q_i$ puts mass $1 - \delta$ on uniform distribution, $\delta$ elsewhere.

$$Q_i^\delta = \left\{ q_i \in [0, 1]^{l-1} \mid q_i(E) \geq (1 - \delta) \text{Leb}(E) \right\}$$

- Pseudo-Types $PT_i = \Theta_i \times Q_i^\delta$
CLAIM: If $\gamma \delta > \frac{1}{I-1}$, then virtual robust implementation is impossible.

- The valuation of the object for agent with payoff type 0 could be
  $0 + (1 - \delta) \gamma (I - 1) \frac{1}{2} + \delta \gamma (I - 1) (1)$

- The valuation of the object for agent with payoff type 1 could be
  $1 + (1 - \delta) \gamma (I - 1) \frac{1}{2} + \delta \gamma (I - 1) (0)$

- The former exceeds the latter.
Discrete Approximation

- \( \Theta_i = [0, \frac{1}{K}, \frac{2}{K}, \ldots, 1] \)
- Suppose \( Q_i \) puts mass \( 1 - \delta \) on uniform distribution, \( \delta \) elsewhere

\[
\overline{Q}_i^\delta = \left\{ q_i \in \Theta_{-i} \, \middle| \, q_i(\theta_{-i}) \geq \frac{1 - \delta}{(K+1)^{i-1}} \right\}
\]

- Pseudo-Types \( T_i = \Theta_i \times \overline{Q}_i^\delta \)

CLAIM: If \( \gamma \delta > \frac{1}{i-1} \), then virtual robust implementation is impossible.

- The valuation of the object for agent with payoff type 0 could be
  \[
  0 + (1 - \delta) \gamma (i - 1) \frac{1}{2} + \delta \gamma (i - 1) (1)
  \]
- The valuation of the object for agent with payoff type 1 could be
  \[
  1 + (1 - \delta) \gamma (i - 1) \frac{1}{2} + \delta \gamma (i - 1) (0)
  \]
- The former exceeds the latter.