Lectures on Robust Mechanism Design at BU

Stephen Morris

January 2009
Introduction

➤ Mechanism Design and Implementation literatures are theoretical successes
➤ mechanisms seem to complicated to use in practise...
➤ successful applications of theory include ad hoc restrictions.....
➤ simplicity, parametric, detail free, ex post equilibrium, Wilson doctrine...
Wilson Doctrine

“Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent that it assumes other features to be common knowledge, such as one agent’s probability assessment about another’s preferences or information. I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality.” Wilson (1987)
Relaxing Common Knowledge Assumptions

- in game theory, Harsanyi (1967/68), Mertens and Zamir (1985) established that relaxing common knowledge assumptions is equivalent to adding types....
- environments with incomplete information can be modeled as a Bayesian game where wlog there is common knowledge among players of (i) each player’s type spaces (ii) each type’s beliefs over types of other players
- economic analysis assumes smaller type spaces than universal type space yet maintains common knowledge of (i) and (ii)
- are the implicit common knowledge assumptions that come from working with small type spaces problematic? perhaps especially in mechanism design (Neeman (2004))
Our Agenda (circa 1998)

- Introduce rich (higher order belief) types into mechanism design literature.
- Relax (implicit) common knowledge assumptions by comparative statics on the type space, going from "naive" type space to "universal" type space.
- In particular
  1. Briefly establish a few easy benchmark "abstract" results.
  2. Develop a close link between this robust approach and applications.
- Ten years and seven papers/notes later, we are kind of done with (1).
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
Allocating a Single Object

- $I$ agents
- Agent $i$ has type $\theta_i \in [0, 1]$
- Agent $i$’s valuation of the "object" is $v_i(\theta_1, \ldots, \theta_I)$
- All agents have quasi-linear utility
- Don’t know anything about agent $i$’s beliefs and higher order beliefs about $\theta_{\neg i}$
Private Values

\[ v_i (\theta) = \theta_i \]

- "Second Price Sealed Bid Auction"
  - Agents bids \( b_i \in [0, 1] \)
  - Object allocated to agent with the highest bid
  - Winner pays the second highest bid
Private Values

- This is "direct mechanism"
- Truthful reporting leads to efficient allocation of the object:

\[ q_i^* (\theta) = \begin{cases} 
\frac{1}{\#\{j: \theta_j \geq \theta_k \text{ for all } k\}}, & \text{if } \theta_i \geq \theta_k \text{ for all } k \\
0, & \text{otherwise}
\end{cases} \]

- Dominant strategy to truthfully report type
Interdependent Values

- Simplest example, Maskin 92 etc....

\[ v_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j \]

- "Generalized VCG Mechanism"
  - Agents bids \( b_i \in [0, 1] \)
  - Object allocated to agent with the highest bid
  - Winner pays the second highest bid PLUS \( \gamma \) times the bid of others, i.e., if \( i \) wins with bid \( b_i \), he pays

\[ \max_{j \neq i} b_j + \gamma \sum_{j \neq i} b_j \]
Interdependent Values

- This is "direct mechanism"
- Truthful reporting leads to allocation $q^*$, which is efficient if $\gamma \leq 1$.
- Truthful reporting is an ex post equilibrium of the direct mechanism if $\gamma \leq 1$. 
Detour for Definitions

- "Ex post equilibrium": each type of each agent has an incentive to tell truth \( if \) he expects all other agents to tell the truth.

- Under private values, ex post equilibrium is equivalent to dominant strategies equilibrium

- If truthtelling is an ex post equilibrium of the direct mechanism for an allocation rule (\textit{including} transfers), then the allocation rule "ex post incentive compatible" [EPIC]
Algebra

- Suppose your type is \( \theta_i \), others’ types are \( \theta_{-i} \) and you report (bid) \( b_i \):
  - Your value of the object is \( \theta_i + \sum_{j \neq i} \theta_j \)
  - You get the object (for sure) only if \( b_i \geq \max_{j \neq i} \theta_j \)
  - If you get the object, you pay \( \max_{j \neq i} \theta_j + \sum_{j \neq i} \theta_j \)

- Thus payoff is

\[
u_i (\theta_i, \theta_{-i}, b_i) = \begin{cases} 
\theta_i - \max_{j \neq i} \theta_j, & \text{if } b_i > \max_{j \neq i} \theta_j \\
\theta_i - \max_{j \neq i} \theta_j, & \text{if } b_i = \max_{j \neq i} \theta_j \\
\# \{ k : \theta_k = \max_{j \neq i} \theta_j \}, & \text{if } b_i = \max_{j \neq i} \theta_j \\
0, & \text{if } b_i < \max_{j \neq i} \theta_j
\end{cases}
\]
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 1 (Robust Mechanism Design)

- When does there exist a mechanism such that, whatever each player’s beliefs and higher order beliefs about other players’ types, there is an equilibrium with outcomes consistent with a social choice correspondence (SCC)?
- In example, consider "efficient correspondence" generated by efficient allocation of the good $q^*$ (and any transfers).
- Truth telling is an equilibrium under the generalized VCG mechanism for any beliefs and higher order beliefs.
- Thus the answer is "yes" in our single object environment.
More generally.....

- [1] verifies that anytime there is an ex post incentive compatible allocation consistent with SCC, the answer is "yes"
- But is this necessary? [1] shows yes in the case of "separable" environments (e.g., the one above), not necessarily in non-separable environments, e.g., efficient level of a public good with budget balance.
Seven Papers

1. Robust Mechanism Design
2. **Ex Post Implementation**
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 2 (Ex Post Implementation)

- When does there exist a mechanism such that truthtelling is the unique ex post equilibrium of some mechanism? If yes, we say there is "ex post implementation".
- [2] shows that a social choice function must be EPIC and "ex post monotonic"; these conditions are "almost" sufficient.
- In example, with three or more agents, truthtelling is the only ex post equilibrium in the generalized VCG mechanism (if $\gamma \neq 0$)
- Birulin (2003) showed not true with two agents.
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 3 (Robust Implementation)

When does there exist a mechanism such that, whatever each player’s beliefs and higher order beliefs about other players’ types, every equilibrium outcome is consistent with a social choice function? If yes, we say there is "robust implementation".
Robust implementation fails \textit{even in the private values case}, since truth-telling is only a weak best response and there are many equilibria leading to inefficient outcomes in second price sealed bid auctions.

Robust implementation of the efficient allocation is not possible in the single object example (with private or independent values) even if augmented (but well-behaved) mechanisms are allowed.

But robust implementation is achievable for a nearly efficient allocation under additional restrictions....
The Modified Sealed Bid Second Price Auction

- with probability $1 - \varepsilon$,
  - allocate object to highest bidder
  - winner pays second highest bid
- for each $i$, with probability $\frac{eb_i}{I}$
  - $i$ gets object
  - pays $\frac{1}{2} b_i$
- truth-telling is a strictly dominant strategy
- thus this $\varepsilon$-efficient allocation is robustly implemented
Algebra

- Suppose your type is $\theta_i$, others report their types to be (bid) $b_{-i}$ and you report (bid) $b_i$:
- Your payoff is $u_i(\theta_i, \theta_{-i}, \theta'_i) =$

\[
\begin{cases}
\frac{\varepsilon}{I} \left( b_i \left( \theta_i - \frac{1}{2} b_i \right) \right) + (1 - \varepsilon) \left( \theta_i - \max_{j \neq i} b_j \right), & \text{if } b_i > \max_{j \neq i} b_j \\
\frac{\varepsilon}{I} \left( b_i \left( \theta_i - \frac{1}{2} b_i \right) \right) + (1 - \varepsilon) \frac{\theta_i - \max_{j \neq i} b_j}{\# \left\{ k : b_k = \max_{j \neq i} b_j \right\}}, & \text{if } b_i = \max_{j \neq i} b_j \\
\frac{\varepsilon}{I} \left( b_i \left( \theta_i - \frac{1}{2} b_i \right) \right) + (1 - \varepsilon), & \text{if } b_i < \max_{j \neq i} b_j
\end{cases}
\]

$\theta_i - b_i = 0$
The Modified Generalized VCG Mechanism

- with probability $1 - \varepsilon$,
  - allocate object to highest bidder $i$
  - winner pays $\max_{j \neq i} b_j + \gamma \sum_{j \neq i} b_j$

- for each $i$ with probability $\frac{\varepsilon b_i}{I}$,
  - $i$ gets object
  - pays $\frac{1}{2} b_i + \gamma \sum_{j \neq i} b_j$

- truth telling is a strict ex post equilibrium
The Modified Generalized VCG Mechanism

- but existence of strict ex post equilibrium does not imply robust implementation
- in fact, this mechanism robustly implements the efficient outcome if $|\gamma| < \frac{1}{i-1}$
- and no mechanism robustly implements the efficient outcome if $|\gamma| \geq \frac{1}{i-1}$
- c.f. Chung and Ely (2001)
A Key Epistemic Result

A message $m_i$ can be sent by an agent with payoff type $\theta_i$ in an equilibrium on some type space if and only if $m_i$ is "incomplete information rationalizable" in the following sense:

1. First, suppose that every payoff type $\theta_i$ could send any message $m_i$;
2. Delete those messages $m_i$ that are a best response to some conjecture over payoff type - message pairs of the opponents that have not yet been deleted.
3. Repeat step 2 until you converge
Illustrate this notion of rationalizability in the modified generalized VCG mechanism

- $\beta^0(\theta_i) = [0, 1]$
- $\beta^k(\theta_i)$ is set of reports $i$ might send for some conjecture $\lambda_i(\theta'_i, \theta_i)$ over his opponents' types $\theta_i$ and reports $\theta'_i$, with restriction on conjecture $\lambda_i(\theta'_i, \theta_i)$ that each type $\theta_j$ of agent $j$ sends a message in $\beta^{k-1}(\theta_j)$. 
Rationalizability

- agent $i$ conjectures that other agents have type $\theta_{-i}$ and report $\theta'_{-i}$.
- agent $i$ with type $\theta_i$ has best response

$$\theta'_i = \theta_i + \gamma \sum_{j \neq i} (\theta_j - \theta'_j).$$

Linear best response allows us characterize $\beta^k(\theta_i)$:

$$\beta^k(\theta_i) = \begin{bmatrix} \beta^k(\theta_i) & \bar{\beta}^k(\theta_i) \end{bmatrix}$$
Rationalizability

\[
\bar{\beta}^k (\theta_i) = \min \left\{ 1, \theta_i + \gamma \max_{(\theta_i', \theta_j): \theta_j \in \beta^k(\theta_j)} \sum_{j \neq i} (\theta_j - \theta_j') \right\}
\]

\[
= \min \left\{ 1, \theta_i + \gamma \max_{\theta_j} \sum_{j \neq i} \left( \theta_j - \bar{\beta}^{k-1}(\theta_j) \right) \right\}
\]

\[
\bar{\beta}^k (\theta_i) = \min\left\{1, \theta_i + \gamma \max_{\theta_j} \sum_{j \neq i} \left( \theta_j - \bar{\beta}^{k-1}(\theta_j) \right) \right\}
\]
Rationalizability

rewriting:

\[ \bar{\beta}^k (\theta_i) = \min \left\{ 1, \theta_i + (\gamma (l - 1))^k \right\}, \]

and likewise

\[ \underline{\beta}^k (\theta_i) = \max \left\{ 0, \theta_i - (\gamma (l - 1))^k \right\}. \]

thus

\[ \theta'_i \neq \theta_i \Rightarrow \theta'_i \not\in \beta^k (\theta_i) \]

for sufficiently large \( k \), provided that

\[ \gamma < \frac{1}{l - 1} \]
On the other hand...

- but now suppose that $\gamma \geq \frac{1}{l-1}$
- each type $\theta_i$ convinced that others’ types are
  $\theta_j = \frac{1}{2} + \frac{1}{\gamma(l-1)} \left( \frac{1}{2} - \theta_i \right)$
- now $\theta_i + \gamma (l - 1) \left[ \frac{1}{2} + \frac{1}{\gamma(l-1)} \left( \frac{1}{2} - \theta_i \right) \right] = \frac{1}{2} \left[ 1 + \gamma (l - 1) \right]$
- types cannot be distinguished in direct or any other mechanism....
In the example:

- robust implementation possible (using the modified generalized VCG mechanism) if $|\gamma| < \frac{1}{i-1}$
- robust implementation impossible (in any mechanism) if $\gamma \geq \frac{1}{i-1}$
In general....

- Each $\Theta_i$ is a compact subset of the real line
- Agent $i$’s preferences depend on $\theta$ through $h_i : \Theta \to \mathbb{R}$
- Preferences are single crossing in $h_i(\theta)$
- Robust implementation is possible in the *direct* mechanism if strict EPIC and the "contraction property" hold.
- Robust implementation is impossible in *any* mechanism if either strict EPIC or the "contraction property" fails.
Contraction Property

- "deception": \( \beta = (\beta_1, \ldots, \beta_I); \beta_i : \Theta_i \rightarrow 2^{\Theta_i} / \emptyset \) with \( \theta_i \in \beta_i(\theta_i) \)
- "truth-telling": \( \beta^* = (\beta_1^*, \ldots, \beta_I^*) \)

The aggregator functions \( h \) satisfy the strict contraction property if, for all \( \beta \neq \beta^* \), there exists \( i \) and \( \theta'_i \in \beta_i(\theta_i) \) with \( \theta'_i \neq \theta_i \), such that

\[
\text{sign} (\theta_i - \theta'_i) = \text{sign} (h_i (\theta_i, \theta_{-i}) - h_i (\theta'_i, \theta'_{-i}))
\]

for all \( \theta_{-i} \) and \( \theta'_{-i} \in \beta_{-i}(\theta_{-i}) \).

- this is equivalent to \( |\gamma| < \frac{1}{i-1} \)
Contraction Property 2

- if \( h_i(\theta) = \theta_i + \sum_{j \neq i} \gamma_{ij} \theta_j \)

- contraction property satisfied if largest eigenvalue of

\[
\Gamma \triangleq \begin{bmatrix}
0 & |\gamma_{12}| & \cdots & |\gamma_{11}|
|\gamma_{21}| & 0 & \cdots & \\
|\gamma_{j1}| & 0 & \cdots & \\
\vdots & & & \\
|\gamma_{11}| & & \cdots & 0
\end{bmatrix}
\]

is less than 1.
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 4: Can we do better with dynamic mechanisms?

- Static Cournot Oligopoly
- Linear Demand $P = 2 - Q$
- Zero Cost
- If there are three or more firms, every outcome between 0 and monopoly output 1 is rationalizable
  - best response if you think all opponents will choose 1 is 0
  - best response if you think all opponents will choose 0 is 1
  - therefore every outcome between 0 and 1 can be played in subjective correlated equilibrium
Dynamic Mechanisms 2

- Dynamic Cournot Oligopoly
- Auctioneer has continuous clock from 0 and 1
- Each firm decides when to drop out from auction; he is then committed to the quantity at the time he dropped out
- Uniquely sequentially rationalizable action: stay until symmetric Nash output of \( \left( \frac{i}{i+1} \right)^2 \)
Dynamic Mechanisms 3

- But with $0 < \gamma < 1$, direct mechanism in single object example is a linear best response game with strategic substitutes:

$$b_i = \theta_i + \gamma \sum_{j \neq i} (\theta_j - b_j).$$

- If $\gamma \geq \frac{1}{i-1}$, many rationalizable outcomes
- But truthful revelation in increasing clock mechanism.
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. **The Role of the Common Prior Assumption in Robust Implementation**
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 5: What if we know the common prior assumption holds?

- Cournot Oligopoly example has unique correlated equilibrium (≡ Nash equilibrium)
- If $\gamma \geq \frac{1}{l-1}$, direct mechanism has many rationalizable actions
- But every "incomplete information correlated equilibrium" has each player following the unique ex post equilibrium and telling the truth.
- With strategic complementarities, there are multiple equilibria if and only if there are multiple rationalizable actions (famous general observation)
- Strategic complementarities in direct mechanism if "negative interdependence," i.e., $\gamma < 0$
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 6: But what about implementation in general environments using general mechanisms?

- "Maskin monotonicity" required to rule out bad equilibria in complete information settings
- In (classical) incomplete information settings, Bayesian incentive compatibility sufficient for existence of a desirable equilibrium (by revelation principle, truth-telling in the direct mechanism); additional "Bayesian monotonicity" condition required to rule out bad equilibria (Jackson etc...)
- We show EPIC and "robust monotonicity" necc. and (almost) suff. for robust implementation
- robust monotonicity = contraction property $= |\gamma| < \frac{1}{I-1}$ = Bayesian monotonicity on all type spaces
- like earlier literature, must use badly behaved infinite mechanisms
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
General Question 7: Does it help if "virtual" is enough?

- Abreu-Matsushima papers interpreted as very permissive results if "approximate" implementation is enough.
- they also did it using iterated deletion of strictly dominated strategies and finite (or well-behaved mechanisms)
- We show EPIC and "robust measurability" necc. and (almost) suff. for virtual robust implementation (with finite mechanism)
- In example, efficient allocation can be virtually robustly implemented
- robust measurability = contraction property in special environment
- more tomorrow!
Environment

- agents $i \in \mathcal{I} = \{1, 2, \ldots, I\}$
- $i$’s "payoff type" $\theta_i \in \Theta_i$
- payoff type profile $\theta \in \Theta = \Theta_1 \times \ldots \times \Theta_I$
- social outcome $y \in Y$
- utility function $u_i : Y \times \Theta \rightarrow \mathbb{R}$
- social choice correspondence $F : \Theta \rightarrow 2^Y / \{\emptyset\}$
Back to the Example

- agents \( i \in \mathcal{I} = \{1, 2, \ldots, I\} \)
- \( i \)'s "payoff type" \( \theta_i \in \Theta_i \equiv [0, 1] \)
- payoff type profile \( \theta \in \Theta = \Theta_1 \times \ldots \times \Theta_I \)
- social outcome \( y = (q, x) \in \Delta(\emptyset, 1, \ldots, I) \times \mathbb{R}^I = Y \)
- utility function \( u_i(q, x) = q_i \left( \theta_i + \gamma \sum_{j \neq i} \theta_j \right) + x_i \)
- efficient social choice correspondence

\[
F(\theta) = \left\{ (q, x) \left| q_i(\theta) > 0 \Rightarrow \theta_i = \max_j \theta_j \right. \right\}
\]
Type Space

- richer type spaces than set of payoff types $\Theta$
- $i$’s type is $t_i \in T_i$
- $t_i$ includes description of
  - payoff type: $\hat{\theta}_i : T_i \rightarrow \Theta_i$
    - $\hat{\theta}_i(t_i)$ is $i$’s payoff type of $t_i$
  - beliefs about types of other players: $\hat{\pi}_i : T_i \rightarrow \Delta(T_{-i})$
    - $\hat{\pi}_i(t_i)$ is $i$’s belief type of $t_i$

*type space* is a collection

$$\mathcal{T} = \left\{ T_i, \hat{\theta}_i, \hat{\pi}_i \right\}_{i=1}^l$$
Properties of Type Spaces

- Type Space $\mathcal{T}$ is “payoff type space” if $T_i = \emptyset_i$ and each $\hat{\theta}_i$ is the identity map.

- Finite Type Space $\mathcal{T}$ satisfies the common prior assumption if there exists $\pi \in \Delta (T)$ such that

  $$\sum_{t_{-i}} \pi (t_i, t_{-i}) > 0 \text{ for all } i \text{ and } t_i$$

  and

  $$\hat{\pi}_i (t_i) [t_{-i}] = \frac{\pi (t_i, t_{-i})}{\sum_{t'_{-i}} \pi (t_i, t'_{-i})} > 0.$$  

- Type Space $\mathcal{T}$ is universal if $T_i \approx \Delta (T_{-i})$

- Universal type space $\approx \text{union of all type spaces}$. 

Robust Mechanism Design
Mechanism

A mechanism $\mathcal{M}$ is a collection $\left((M_i)_{i=1}^I, g\right)$

- each $M_i$ is a message set (finite/compact/anything)
- outcome function $g : M \to Y$
Equilibrium

incomplete information game \((\mathcal{T}, \mathcal{M})\)

- \(i\)'s strategy: \(\sigma_i : T_i \rightarrow \Delta(M_i)\)
- strategy profile \(\sigma\) is an equilibrium if \(\sigma_i(m_i|t_i) > 0\) implies \(m_i\) is in

\[
\arg\max_{m'_{i}} \sum_{t_{-i}, m_{-i}} \hat{\pi}_i [t_i] (t_{-i}) \left( \prod_{j \neq i} \sigma_j (m_j | t_j) \right) u_i \left( g (m'_i, m_{-i}), \hat{\theta} (t) \right)
\]
Questions

Does there exist a mechanism \( M \) (with property...) such that for all type spaces \( T \) (satisfying property...), and for all (some) equilibrium \( \sigma \) of \((T, M)\),

\[
\sigma (m|t) > 0 \Rightarrow g (m) \in F \left( \hat{\theta} (t) \right).
\]
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
Robust Mechanism Design

DEFINITION. Social choice function $f : \Theta \rightarrow Y$ is ex post incentive compatible [EPIC] if

$$u_i (f (\theta_i, \theta_{-i}) , (\theta_i, \theta_{-i})) \geq u_i (f (\theta'_i, \theta_{-i}) , (\theta_i, \theta_{-i}))$$

for all $i$, $\theta_i$, $\theta_{-i}$ and $\theta'_i$.

DEFINITION. Social choice correspondence $F$ is ex post weakly implementable if there exists social choice function $f$ such that $f$ is EPIC and $f (\theta) \in F (\theta)$ for all $\theta$. 
Robust Mechanism Design

PROPOSITION. If $F$ is ex post weakly implementable, then there exists a mechanism $\mathcal{M}$ such that for all type spaces $\mathcal{T}$, there exists an equilibrium $\sigma$ of $(\mathcal{T}, \mathcal{M})$, with

$$\sigma (m|t) > 0 \Rightarrow g (m) \in F \left( \hat{\theta} (t) \right).$$

PROOF. Direct mechanism i.e., each agent announces a type. Consider "truth-telling strategy" profile $\sigma_i^* (\theta_i|t_i) = 1$ if $\theta_i = \hat{\theta}_i (t_i)$

CONVERSE IS FALSE.
Example 2 in Paper

- Two agents, 1 and 2
- $\Theta_1 = \{\theta_1, \theta_1', \theta_1''\}$
- $\Theta_2 = \{\theta_2, \theta_2'\}$
- $Y = \Delta (\{a, b, c, d, a', b', c', d'\})$. 
(Private Values) Payoffs

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$a'$</th>
<th>$b'$</th>
<th>$c'$</th>
<th>$d'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>1</td>
<td>$-1$</td>
<td>$\frac{1}{2}$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\theta_1'$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_1''$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$u_2$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$a'$</th>
<th>$b'$</th>
<th>$c'$</th>
<th>$d'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\theta_2'$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Social Choice Correspondence

<table>
<thead>
<tr>
<th>$F$</th>
<th>$\theta_2$</th>
<th>$\theta'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>${a, b}$</td>
<td>${a', b'}$</td>
</tr>
<tr>
<td>$\theta'_1$</td>
<td>${c}$</td>
<td>${c'}$</td>
</tr>
<tr>
<td>$\theta''_1$</td>
<td>${d}$</td>
<td>${d'}$</td>
</tr>
</tbody>
</table>
NOT dominant strategies implementable

<table>
<thead>
<tr>
<th>$F$</th>
<th>$\theta_2$</th>
<th>$\theta'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>{a, b}</td>
<td>{a', b'}</td>
</tr>
<tr>
<td>$\theta'_1$</td>
<td>{c}</td>
<td>{c'}</td>
</tr>
<tr>
<td>$\theta''_1$</td>
<td>{d}</td>
<td>{d'}</td>
</tr>
</tbody>
</table>

- write $q$ for prob. $a$ chosen at $(\theta_1, \theta_2)$
- write $q'$ for prob. $a'$ is chosen at $(\theta_1, \theta'_2)$
NOT dominant strategies implementable

- **EPIC for** $\theta_1$ **on** $\theta'_1$ **given** $\theta_2$

  \[ q - (1 - q) \geq \frac{1}{2} \iff q \geq \frac{3}{4}. \]

- **EPIC for** $\theta_1$ **on** $\theta''_1$ **given** $\theta_2$

  \[ -q' + (1 - q') \geq \frac{1}{2} \iff q' \leq \frac{1}{4}. \]

- **EPIC for** $\theta_2$ **on** $\theta'_2$ **given** $\theta_1$

  \[ 1 - q \geq 1 - q' \implies q' \geq q \]

- **these constraints give contradiction**
But we can achieve desirable outcome on every type space in equilibrium....

Mechanism

<table>
<thead>
<tr>
<th></th>
<th>$m_2^1$</th>
<th>$m_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1^1$</td>
<td>a</td>
<td>$a'$</td>
</tr>
<tr>
<td>$m_1^2$</td>
<td>b</td>
<td>$b'$</td>
</tr>
<tr>
<td>$m_1^3$</td>
<td>c</td>
<td>$c'$</td>
</tr>
<tr>
<td>$m_1^4$</td>
<td>d</td>
<td>$d'$</td>
</tr>
</tbody>
</table>
But we can achieve desirable outcome on every type space in equilibrium....

Strategies

- agent 1
  - type $\theta_1$ of agent 1 sends message $m^1_1$ if he believes agent 2 is type $\theta_2$ with probability at least $\frac{1}{2}$ and message $m^2_1$ if he believes agent 2 is type $\theta_2$ with probability less than $\frac{1}{2}$;
  - type $\theta'_1$ always sends message $m^3_1$; and type $\theta''_1$ always sends message $m^4_1$.

- type $\theta_2$ of agent 2 sends message $m^1_2$ and type $\theta'_2$ sends message $m^2_2$
An Old Debate (with Private Values)

- If social choice correspondence must be implemented without knowing the prior, dominant strategies is required [Dasgupta, Hammond + Maskin (1979), Ledyard (1978, 1979), Groves + Ledyard (1987)]

- Counterargument: if the agents’ know the prior, you can always ask them to reveal it as part of the mechanism (e.g., Choi and Kim (1999))

Our response

- "Wrong" question, since common knowledge of a common prior being maintained NOT wlog
- With "right" question....
  - dominant strategies is not always required
  - but it is in many special cases
Separability

DEFINITION. Environment and SCF can be written as

\[ Y = Y_0 \times Y_1 \times \ldots \times Y_I; \]
\[ \tilde{u}_i : Y_0 \times Y_i \times \Theta \rightarrow \mathbb{R} \]
\[ u_i ((y_0, y_1, \ldots, y_I), \theta) = \tilde{u}_i (y_0, y_i, \theta) \text{ for all } i \text{ and } y \]
\[ f_0 : \Theta \rightarrow Y_0 \]
\[ F(\theta) = f_0(\theta) \times F_1(\theta) \times \ldots \times F_I(\theta) \]

Includes:

- \( F \) single valued
- quasi-linear environment where transfers do not effect \( F \)
Result

In separable environments, the following are equivalent:

1. $F$ is ex post weakly implementable
2. There exists a mechanism $\mathcal{M}$ such that for all type spaces $\mathcal{T}$, there exists an equilibrium $\sigma$ of $(\mathcal{T}, \mathcal{M})$, with

   \[
   \sigma(m|t) > 0 \Rightarrow g(m) \in F\left(\hat{\theta}(t)\right).
   \]
Idea of Proof

Consider type space where agents other than $i$ are known to be $\theta_{-i}$.
There must exist $g^{i, \theta_{-i}} : \Theta_i \rightarrow Y_i$ such that

$$\tilde{u}_i (f_0 (\theta_i, \theta_{-i}), g (\theta_i), (\theta_i, \theta_{-i})) \geq \tilde{u}_i (f_0 (\theta'_i, \theta_{-i}), g (\theta'_i), (\theta_i, \theta_{-i}))$$

for all $i, \theta_i, \theta'_i$.
Now let

$$f (\theta) = \left( f_0 (\theta), g_1^{1, \theta_{-1}} (\theta_{-1}), \ldots, g_i^{i, \theta_{-i}} (\theta_{-i}), \ldots, g_l^{l, \theta_{-l}} (\theta_{-l}) \right)$$
Seven Papers

1. Robust Mechanism Design
2. Ex Post Implementation
3. Robust Implementation in Direct Mechanisms
4. An Ascending Auction for Interdependent Values
5. The Role of the Common Prior Assumption in Robust Implementation
6. Robust Implementation in General Mechanisms
7. Robust Virtual Implementation
Rationalizable Messages

- initialize at $S_{i}^{M,0} (\theta_i) = M_i$, inductive step:

  $S_{i}^{M,k+1} (\theta_i) = \begin{cases} \exists \mu_i \in \Delta (\Theta_i \times M_i) \text{ s.t.:} \\ (1) \mu_i (\theta_{-i}, m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^{M,k} (\theta_{-i}) \\ (2) m_i \in \arg \max_{m_i'} \sum_{\theta_{-i}, m_{-i}} \mu_i (\theta_{-i}, m_{-i}) u_i (g(m_i', m_{-i}), \theta) \end{cases}$

- limit set

  $S_{i}^{M} (\theta_i) = \bigcap_{k \geq 0} S_{i}^{M,k} (\theta_i) .$

- $S_{i}^{M} (\theta_i)$ are rationalizable actions of type $\theta_i$ in mechanism $M$
Proposition. \( m_i \in S_i^\mathcal{M} (\theta_i) \) if and only if there exist

1. a type space \( \mathcal{T} \),
2. an equilibrium \( \sigma \) of \((\mathcal{T}, \mathcal{M})\) and
3. a type \( t_i \in T_i \), such that
   3.1 \( \sigma_i(m_i|t_i) > 0 \) and
   3.2 \( \hat{\theta}_i(t_i) = \theta_i \).

Epistemic Foundations: Proof

If $m_i \in S_i^M (\theta_i)$, then $\exists \mu_i^{m_i, \theta_i} \in \Delta (\Theta - \theta_i \times M_i)$ s.t.

$$\mu_i (\theta_{-i}, m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^M (\theta_{-i})$$

$$m_i \in \arg \max_{m'_i} \sum_{\theta_{-i}, m_{-i}} \mu_i (\theta_{-i}, m_{-i}) u_i (g (m'_i, m_{-i}), \theta)$$

Construct type space:

- $T_i = \{ (\theta_i, m_i) \in \Theta_i \times M_i \mid m_i \in S_i^M (\theta_i) \}$
- $\hat{\theta}_i (\theta_i, m_i) = \theta_i$
- $\hat{\pi}_i ((\theta_i, m_i)) (\theta_j, m_j)_{j \neq i} = \mu_i^{m_i, \theta_i} (\theta_{-i}, m_{-i})$

Construct equilibrium:

- Type $(\theta_i, m_i)$ sends message $m_i$
Environment

- finite set of agents, 1, 2, ..., \( I \)
- agent \( i \)'s payoff type is \( \theta_i \in \Theta_i \), \( \Theta_i \) is a compact set
- type profile \( \theta \in \Theta = \Theta_1 \times \cdots \times \Theta_I \)
- \( Y \) is a compact set of outcomes
- agent \( i \)'s utility: \( u_i(y, \theta) \)
- social choice function, \( f : \Theta \rightarrow Y \)
Mechanism

- \( m_i \in M_i \) compact set of messages available to agent \( i \)
- outcome function: \( g : M \rightarrow Y \), \( g(m) \) outcome under message profile \( m \)
- mechanism is a collection:
  \[
  \mathcal{M} = (M_1, \ldots, M_I, g(\cdot)) ,
  \]
- direct mechanism: \( M_i = \Theta_i \) for all \( i \) and \( g(\theta) = f(\theta) \)
Deceptions

- message profile of agent $i$, $S_i : \Theta_i \rightarrow 2^{M_i} / \emptyset$
  - in direct mechanism, $S_i : \Theta_i \rightarrow 2^{\Theta_i} / \emptyset$
- deception: $\beta_i : \Theta_i \rightarrow 2^{\Theta_i} / \emptyset$ with $\theta_i \in \beta_i(\theta_i)$
  - truth-telling: minimal deception $\beta_i^* (\theta_i) = \{\theta_i\}$
Rationalizable Messages

- belief of agent $i$ over message and type profiles of remaining agents:
  \[ \lambda_i \in \Delta (M_{-i} \times \Theta_{-i}) \]
- set $S_i^0 (\theta_i) = M_i$ and inductively define

\[
S_i^k (\theta_i) = \left\{ m_i \mid \exists \lambda_i \text{ s.th.:} \begin{align*}
(1) & \quad \text{belief } \lambda_i \text{ is consistent} \\
(2) & \quad \text{message } m_i \text{ is best-response}_i
\end{align*} \right\}
\]
Rationalizable Messages

\[ S_i^k (\theta_i) = \left\{ m_i \in \left| \exists \lambda_i \text{ s.th.:} \begin{align*} & (1) \text{ belief } \lambda_i \text{ is consistent} \\ & (2) \text{ message } m_i \text{ is best-response} \end{align*} \right\} \]

consistent (1):

\[ \lambda_i (m_{-i}, \theta_{-i}) > 0 \Rightarrow m_j \in S_j^{k-1} (\theta_j), \ \forall j \neq i; \]

best response (2):

\[
\int_{m_{-i}, \theta_{-i}} \lambda_i (m_{-i}, \theta_{-i}) u_i (g (m_i, m_{-i}), (\theta_i, \theta_{-i})) \\
\geq \int_{m_{-i}, \theta_{-i}} \lambda_i (m_{-i}, \theta_{-i}) u_i (g (m'_i, m_{-i}), (\theta_i, \theta_{-i})), \ \forall m'_i \in M_i.
\]
Rationalizable Messages

- limit set $S^M(\theta)$

$$S^M(\theta) \triangleq \lim_{k \to \infty} S^k(\theta), \text{ for all } \theta \in \Theta.$$ 

Definition (Robust Implementation)

$f$ is robustly implemented by mechanism $M$ if

$$m \in S^M(\theta) \Rightarrow g(m) = f(\theta).$$
Aggregator Single Crossing

- aggregator $h_i : \Theta \rightarrow \mathbb{R}$ for each $i$:
  \[
  u_i (y, \theta) \triangleq v_i (y, h_i (\theta))
  \]
- $h_i (\theta)$ is continuous and strictly increasing in $\theta_i$
- no restriction on behavior of $h_i (\cdot)$ in terms of $\theta_{-i}$
- $h_i (\theta)$, possibly distinct for every $i$
Aggregator Single Crossing

The environment satisfies strict single crossing (SC) if

\[ v_i (y, \phi) > v_i (y', \phi) \quad \text{and} \quad v_i (y, \phi') = v_i (y', \phi') \]

\[ \Rightarrow \quad v_i (y, \phi'') < v_i (y', \phi'') \]

if

\[ \phi < \phi' < \phi''. \]

- only require aggregation \( \phi \) of type profile \( \theta \) to be ordered on the real line
- does not require allocation \( y \) to be one-dimensional or ordered
Example

- two allocations, a or b
- \( Y = [0, 1] \times [-K, K]^l \), where \( y_0 \) is the probability of allocation a

\[
    u_i (y, \theta) = y_0 v_i^a (\theta) + (1 - y_0) v_i^b (\theta) + y_i.
\]

- An equivalent representation is

\[
    u_i (y, \theta) = y_0 \left[ v_i^a (\theta) - v_i^b (\theta) \right] + y_i.
\]

We can give this a monotonic aggregator representation by setting \( h_i (\theta) = v_i^a (\theta) - v_i^b (\theta) \) and \( v_i (y, h (\theta)) = y_0 h (\theta) + y_i \), we have

\[
    u_i (y, \theta) = v_i (y, h_i (\theta)),
\]

and now \( v_i \) indeed satisfies the strict single crossing condition.
Informational Example

- two agents
- agent $i$’s utility depends on expected value of $\omega_0 + \omega_i$
- $\omega_0, \omega_1, \omega_2$ independently normally distributed with zero mean and variance $\sigma_i^2$.
- agent $i$ observes signal $\theta_i = \omega_0 + \omega_i + \epsilon_i$, where each $\epsilon_i$ is independently normally distributed with zero mean and variance $\tau_i^2$.

Now

$$E_i(\omega_0 + \omega_i | \theta_i, \theta_j) = \text{const} \times h_i(\theta_i, \theta_j)$$

$$= \theta_i + \frac{\sigma_0^2 \tau_i}{\sigma_0^2 \tau_j^2 + \sigma_0^2 \sigma_i^2 + \sigma_0^2 \sigma_j^2 + \tau_j^2 \sigma_i^2 + \sigma_i^2 \sigma_j^2} \theta_j.$$

$$\gamma_{ij} = \frac{\sigma_0^2 \tau_i^2}{\sigma_0^2 \tau_j^2 + \sigma_0^2 \sigma_i^2 + \sigma_0^2 \sigma_j^2 + \tau_j^2 \sigma_i^2 + \sigma_i^2 \sigma_j^2}.$$
Definition (Strict Ex Post Incentive Compatibility)

scf \( f \) satisfies strict ex post incentive compatibility (EPIC) if

\[
    u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) > u_i(f(\theta'_i, \theta_{-i}), (\theta_i, \theta_{-i}))
\]

for all \( i, \theta \) and \( \theta'_i \neq \theta_i \).
Definition (Strict Contraction Property)

The aggregator functions $h$ satisfy the strict contraction property if, for all $\beta \neq \beta^*$, there exists $i$ and $\theta'_i \in \beta_i(\theta_i)$ with $\theta'_i \neq \theta_i$, such that

$$\text{sign} \left( \theta_i - \theta'_i \right) = \text{sign} \left( h_i(\theta_i, \theta_{-i}) - h_i(\theta'_i, \theta'_{-i}) \right),$$

for all $\theta_{-i}$ and $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$.

- for some agent $i$ the direct impact of private signal $\theta_i$ on aggregator $h_i(\theta)$ is always sufficiently strong to outweigh the reported types of other agents.
THEOREM. If strict EPIC and the strict contraction property are satisfied, then there is robust implementation in the direct mechanism.

- no conditions on how $f$ varies with type profile
  - in particular, $f$ does not have to respond to $\theta$ similar to the response of any of the aggregators $h_i$
  - no condition on the number of agents, such as $I > 2$
interaction of single crossing and contraction property

fix deception $\beta$

contraction property: there exists $i$ and $\theta'_i \in \beta_i (\theta_i)$ with (wlog) $\theta_i > \theta'_i$ such that $h_i (\theta_i, \theta_{-i}) > h_i (\theta'_i, \theta'_{-i})$ for all $\theta'_{-i} \in \beta_i (\theta_{-i})$

now fix true type profile: $\theta$, reported type profile: $\theta' \in \beta (\theta)$

choose $\theta''_i > \theta'_i$ with $h_i (\theta_i, \theta_{-i}) > h_i (\theta''_i, \theta'_{-i}) > h_i (\theta'_i, \theta'_{-i})$
\[ h_i(\theta_i, \theta_{-i}) > h_i(\theta'_i, \theta_{-i}) > h_i(\theta'_i, \theta_{-i}) \]
\[ \theta'_i \in \beta_i(\theta_i) \]
so \( h_i (\theta_i, \theta_{-i}) > h_i (\theta''_i, \theta'_{-i}) > h_i (\theta'_i, \theta'_{-i}) \)

\[ \theta'_i \in \beta_i (\theta_i) \]

\[ \text{now} \]

\[ v_i (f (\theta'), h_i (\theta''_i, \theta'_{-i})) < v_i (f (\theta''_i, \theta'_{-i}), h_i (\theta''_i, \theta'_{-i})) \quad \text{EPIC} \quad h_i (\theta) \]

\[ v_i (f (\theta'), h_i (\theta')) > v_i (f (\theta''_i, \theta'_{-i}), h_i (\theta')) \quad \text{EPIC} \quad h_i (\theta) \]
so \( h_i(\theta_i, \theta_{-i}) > h_i(\theta''_i, \theta'_{-i}) > h_i(\theta'_i, \theta'_{-i}) \)

- \( \theta'_i \in \beta_i(\theta_i) \)
- now

\[
\begin{align*}
  v_i(\mathbf{f}(\theta'), h_i(\theta)) &< v_i(\mathbf{f}(\theta''_i, \theta'_{-i}), h_i(\theta)) & \text{AGG, SC} & h_i(\theta'_i, \theta'_{-i}) \\
v_i(\mathbf{f}(\theta'), h_i(\theta')) &< v_i(\mathbf{f}(\theta''_i, \theta'_{-i}), h_i(\theta'') h_i(\theta'_i, \theta'_{-i})) & \text{EPIC} & h_i(\theta''_i, \theta'_{-i}) \\
v_i(\mathbf{f}(\theta'), h_i(\theta')) &> v_i(\mathbf{f}(\theta''_i, \theta'_{-i}), h_i(\theta')) & \text{EPIC} & h_i(\theta''_i, \theta'_{-i}) \\
\end{align*}
\]
preference aggregator \( h_i(\theta) \) is linear for each \( i \):

\[
  h_i(\theta) = \theta_i + \sum_{j \neq i} \gamma_{ij} \theta_j.
\]

\( \gamma_{ij} \) represent influence of signal of \( j \) on the value of \( i \)

interdependence matrix \( \Gamma \):

\[
  \Gamma \triangleq \begin{bmatrix}
    0 & |\gamma_{12}| & \cdots & |\gamma_{1i}| \\
    |\gamma_{21}| & 0 & \cdots & |\gamma_{2i}| \\
    \vdots & \ddots & \ddots & \ddots \\
    |\gamma_{i1}| & \cdots & \cdots & 0
  \end{bmatrix}
\]
consider the class of deceptions of the form:

$$\beta_i(\theta_i) = [\theta_i - \epsilon c_i, \theta_i + \epsilon c_i] \cap \Theta_i$$

Lemma (Linear Aggregator)

Linear aggregator functions $h$ satisfy the strict contraction property if and only if, for all $c \in \mathbb{R}^I_+$ with $c \neq 0$, there exists $i$ such that

$$c_i > \sum_{j \neq i} |\gamma_{ij}| c_j.$$

if the set of deceptions by $i$ is too large, then there is an “overhang” which can be “nipped and tucked”
Lemma (Duality)

The following two properties of $\Gamma$ are equivalent:

1. for all $c \in \mathbb{R}^l_+$ with $c \neq 0$, there exists $i$ such that:
   
   $$c_i > \sum_{j \neq i} |\gamma_{ij}| c_j;$$

   (P)

2. there exists $d \in \mathbb{R}^l_+$ such that:

   $$d_i > \sum_{j \neq i} |\gamma_{ji}| d_j,$$

   (D)

   for all $i$. 
Theorem (Contraction Property via Eigenvalue)

The interdependence matrix $\Gamma$ has the contraction property if and only if its eigenvalue $\lambda < 1$.

- contraction property if and only if we can find a vector $d$, the eigenvector of the matrix $\Gamma$ and eigenvalue $\lambda < 1$ with

$$\lambda \Gamma^T d = d$$

- for nonnegative matrices, left and right hand side eigenvalues coincide, hence also for some $c$

$$\lambda \Gamma c = c$$

- $\lambda$ rate at which the iterative procedure contracts
symmetric preferences: $\gamma_{ij} = \begin{cases} 1, & \text{if } j = i, \\ \gamma, & \text{if } j \neq i. \end{cases}$ and hence

$$\gamma < \frac{1}{l-1}$$

moderate interdependence

cyclic preferences: $\gamma_{ij} = \gamma(i-j)_{\text{mod } l}$ and

$$\sum_{j \neq i} \gamma(i-j) < 1$$
two bidders:

\[ \Gamma = \begin{bmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix} \]

sufficient condition is:

\[ \gamma_{12} \gamma_{21} < 1. \]
 ► public good example with general $\Gamma$:

$$1 + \sum_{j\neq i} \gamma_{ji} \geq 0$$

 ► single private good example

$$\gamma_{ij} \leq 1$$
Definition (Contraction Property)

The aggregator functions $h$ satisfy the contraction property if, for all $\beta \neq \beta^*$, there exists $i$ and $\theta'_i \in \beta_i (\theta_i)$ with $\theta'_i \neq \theta_i$, such that either

$$h_i (\theta_i, \theta_{-i}) = h_i (\theta'_i, \theta'_{-i}) ,$$

or

$$\text{sign} (\theta_i - \theta'_i) = \text{sign} (h_i (\theta_i, \theta_{-i}) - h_i (\theta'_i, \theta'_{-i})) ,$$

for all $\theta_{-i}$ and $\theta'_{-i} \in \beta_{-i} (\theta_{-i})$.

▸ "nongeneric" weakening by allowing equality:

$$h_i (\theta) = h_i (\theta'_i, \theta'_{-i})$$
Definition (Strict Ex Post Incentive Compatibility*)

\( scf \ f \) satisfies strict ex post incentive compatibility (EPIC) if

\[
    u_i \left( f \left( \theta_i, \theta_{-i} \right), (\theta_i, \theta_{-i}) \right) > u_i \left( f \left( \theta'_i, \theta_{-i} \right), (\theta_i, \theta_{-i}) \right)
\]

for all \( i, \theta \) and \( \theta'_i \) with \( f \left( \theta_i, \theta'_{-i} \right) \neq f \left( \theta'_i, \theta'_{-i} \right) \) for some \( \theta'_{-i} \).
Definition

If \( f \) is robustly implementable, then \( f \) satisfies strict EPIC* and the contraction property.

- Necessity holds for all mechanisms, direct or augmented
- Implementation in direct mechanism cannot be improved by augmented mechanism

The social choice function satisfies robust monotonicity if, for all \( \beta \neq \beta^* \), there exists \( i, \theta_i \) and \( \theta'_i \in \beta_i (\theta_i) \) such that, for all \( \theta'_{-i} \) and \( \psi_i \in \Delta (\beta^{-1}_i (\theta'_{-i})) \), there exists \( y \) such that

\[
\sum_{\theta'_{-i} \in \Theta_{-i}} \psi_i (\theta'_{-i}) u_i (y, (\theta_i, \theta'_{-i})) > \sum_{\theta'_{-i} \in \Theta_{-i}} \psi_i (\theta'_{-i}) u_i (f (\theta'_i, \theta'_{-i}), (\theta_i, \theta'_{-i}))
\]

while

\[
u_i (f (\theta''_i, \theta'_{-i}), (\theta_i, \theta'_{-i})) > u_i (y, (\theta''_i, \theta'_{-i})) \]

for all \( \theta''_i \in \Theta_i \).