Topologies on Types

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Motivation

- Harsanyi (1967/68), Mertens and Zamir (1985): single "universal type space" incorporates all incomplete information

- Economic researchers work with small type spaces: leap of faith that these capture flavor of universal type space

"Finite types" dense in universal type space *under product topology,* but

- But strategic outcomes not continuous with respect to product topology
- Finite common prior types are dense in product topology (Lipman 2003)
- Types with unique rationalizable outcomes are open and dense in the product topology (Yildiz 2006)

Defining a Strategic Topology

TWO TYPES ARE CLOSE IF THEIR STRATEGIC BEHAVIOR IS SIMILAR IN ALL STRATEGIC SITUATIONS

1. "Strategic situations": Fix space of uncertainty Θ and all possible beliefs and higher order beliefs about Θ. Vary finite action sets and payoff functions depending on actions and Θ.

2. "Strategic behavior": interim (correlated) rationalizable actions, i.e., those actions consistent with common knowledge of rationality.

3. "Similar" strategic behavior: Allow ε-rationality

Results

- MAIN RESULT: Finite types are dense in the strategic topology
  - finite common prior types are NOT dense in the strategic topology
  - open sets of types have multiple rationalizable outcomes
- Product topology implies upper hemicontinuity of rationalizable outcomes
- Lower strategic convergence implies upper strategic convergence and is strictly stronger than the product topology
- Finite types are nonetheless "non-generic" (category 1) in both product and strategic topology

Electronic Mail Game

- two players 1 and 2
- two actions N and I
- two payoff states 0 and 1
- payoffs:

$\theta = 0$	N	I
N	$0, 0$	$0, -2$
I	$-2, 0$	$-2, 2$

$\theta = 1$	N	I
N	$0, 0$	$0, -2$
I	$-2, 0$	$1, 1$

- with probability δ, it is common knowledge that $\theta = 1$

- otherwise

  - $\theta = 1$ with probability α
  - player 1 only informed if θ is 0 or 1
  - if $\theta = 1$, player 1 sends a message to player 2, message lost with probability α
  - if player 2 receives a confirmation, he sends a reply - lost with probability α....

- write $t_{1\infty}$ for the type of player i for whom θ is common knowledge

- write t_{ik} for the type of player i for whom θ is not common knowledge but he has sent $k-1$ messages

  - for type t_{1k}, 1 knows that 2 knows that $(k-2$ times$)$...$\theta = 1$

– for type t_{2k}, 2 knows that $[1$ knows that 2 knows that $(k-2$ times)...$\theta = 1]$

- Thus $T_i = \{t_{i1}, t_{i2}, ...\} \cup \{t_{i\infty}\}$.

- Common Prior

$\theta = 0$	t_{21}	t_{22}	t_{23}	t_{24}	$\cdots$	$t_{2\infty}$
t_{11}	$(1-\delta)\,\alpha$	0	0	0	$\cdots$	0
t_{12}	0	0	0	0	$\cdots$	0
t_{13}	0	0	0	0	$\cdots$	0
t_{14}	0	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$t_{1\infty}$	0	0	0	0	$\cdots$	0

$\theta = 1$	t_{21}	t_{22}	$\cdots$	$\cdots$	$\cdots$	$t_{2\infty}$
t_{11}	0	0	$\cdots$	$\cdots$	$\cdots$	0
t_{12}	$(1-\delta)\,\alpha\,(1-\alpha)$	$(1-\delta)\,\alpha\,(1-\alpha)^2$	$\cdots$	$\cdots$	$\cdots$	0
t_{13}	0	$(1-\delta)\,\alpha\,(1-\alpha)^3$		$\cdots$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$t_{1\infty}$	0	0	0	0	$\cdots$	δ

- $t_{1k} \rightarrow t_{1\infty}$ in product topology

- Iterative argument establishes that N is the only ε-rationalizable action for type t_{ik} for any $\varepsilon < \frac{1+\alpha}{2-\alpha}$ and $k < \infty$
  - N is only rationalizable action for type t_{11}
  - type t_{21} assigns probability $\frac{1+\alpha}{2-\alpha}$ to type t_{11}; thus payoff to invest is at most
$$\frac{1-\alpha}{2-\alpha}(1) + \frac{1}{2-\alpha}(-2) = -\frac{1+\alpha}{2-\alpha} < -\frac{1}{2}$$
  - type t_{12} asigns probability $\frac{1+\alpha}{2-\alpha}$ to type t_{21}....

- Both N and I are rationalizable for type $t_{i\infty}$.

- Thus example shows failure of lower hemicontinuity w.r.t. product topology

- No failure of upper hemicontinuity w.r.t. product topology

- However, $t_{i\infty}$ is itself a finite type, so this example does not show a failure of denseness

Types

- two players, 1 and 2

- finite payoff states Θ

- Type $t = (\delta_1, \delta_2, ...) \in \times_{k=0}^{\infty} \Delta (X_k)$ where

$$X_0 = \Theta$$

$$X_1 = X_0 \times \Delta (X_0)$$
$$X_k = X_{k-1} \times \Delta (X_{k-1})$$

- T^* is set of "coherent" types

- There exists homeomorphism $\pi^* : T^* \rightarrow \Delta\left(T^* \times \Theta\right)$

- we ignore "redundant types" - do not matter for our soln concept

- "finite types": belong to a finite belief closed subset of the universal type space

- "product topology": $t_i^n \rightarrow t_i$ if $\pi_i^k\left(t_i^n\right) \rightarrow \pi_i^k\left(t_i\right)$ as $n \rightarrow \infty$ for all k
  - where $\pi_i^k\left(t_i\right) \in \Delta\left(X_{k-1}\right)$ is type t_i's kth level beliefs
  - often studied, but we do not expect continuous behavior

Games

- A (two player) game $G \in \mathcal{G}$ consists of
  - finite action sets A_i
  - payoff functions $g_i : A \times \Theta \to [-M, M]$

Interim Rationalizability

- $R_i^0(t_i, G, \varepsilon) = A_i$

- $$R_i^{k+1}(t_i, G, \varepsilon) = \left\{ a_i \in A_i \;\middle|\; \begin{array}{l} \exists \nu \in \Delta\left(T_j^* \times \Theta \times A_j\right) \text{ such that} \\ \text{(i) } \nu\left[\left\{(t_j, \theta, a_j) : a_j \in R_j^k(t_j, G, \varepsilon)\right\}\right] = 1 \\ \text{(ii) } \operatorname{marg}_{T^* \times \Theta} \nu = \pi_i^*(t_i) \\ \text{(iii) } \int\limits_{(t_j, \theta, a_j)} \left[\begin{array}{c} g_i(a_i, a_j, \theta) \\ -g_i(a_i', a_j, \theta) \end{array} \right] d\nu \geq -\varepsilon \\ \qquad \text{for all } a_i' \in A_i \end{array} \right\}$$

- $R_i(t_i, G, \varepsilon) = \bigcap_{k \geq 1} R_i^k(t_i, G, \varepsilon)$

- Properties (see DFM "Interim Correlated Rationalizability"

  `http://www.princeton.edu/~smorris/pdfs/interimrationalizability.`

  – Doesn't depend on redundant types
  – $R_i(t_i, G, \varepsilon)$ is the set of actions that might be played in any ε-equilibrium on any type space by a type whose higher order beliefs are given by t_i
  – Thus $R_i(t_i, G, \varepsilon)$ is the set of actions consistent with common knowledge of ε-rationality for a type whose higher order beliefs are given by t_i

Convergence Properties

- Let $h_i\left(t_i|a_i, G\right)$ be the "rationalizability" of action a_i, i.e., the smallest ε such that a_i is ε-rationalizable in game G:

$$h_i\left(t_i|a_i, G\right) = \min\{\varepsilon : a_i \in R_i\left(t_i, G, \varepsilon\right)\}$$

- Let $d\left(t_i, t_i'|a_i, G\right)$ be the difference in the rationalizability of a_i at t_i and t_i':

$$d\left(t_i, t_i'|a_i, G\right) = \left|h_i\left(t_i|a_i, G\right) - h_i\left(t_i'|a_i, G\right)\right|$$

DEFINITION: $\left(\left(t_i^n\right)_{n=1}^{\infty}, t_i\right)$ satisfies the upper strategic convergence property (written $t_i^n \rightarrow_U t_i$) if for every game G and action a_i, $h_i\left(t_i^n|a_i, G\right)$ is upper hemicontinuous in n.

DEFINITION: $\left((t_i^n)_{n=1}^{\infty}, t_i\right)$ satisfies the lower strategic convergence property (written $t_i^n \to_L t_i$) if for every game G and action a_i, $h_i(t_i^n|a_i, G)$ is lower hemicontinuous in n.

Notice:

1. USC implies that if $\varepsilon^n \to 0$ and $a_i \in R_i(t_i^n, G, \varepsilon^n)$ for each n, then $a_i \in R_i(t_i, G, 0)$.

2. convergence is not uniform over all games

A metric

$$d\left(t_i, t_i'\right) = \sum_m \beta^m \sup_{a_i, G \in \mathcal{G}^m} d\left(t_i, t_i' | a_i, G\right)$$

for some $0 < \beta < 1$.

- The distance d is a metric
- $d\left(t_i^n, t_i\right) \to 0$ if and only if $t_i^n \to_U t_i$ and $t_i^n \to_L t_i$

DEFINITION: The strategic topology is the topology generated by metric d.

- d is the coarsest metric topology with upper and lower strategic convergence satisfied.

Results

- Lower strategic convergence stronger requirement than product convergence
  - Email example
- Upper strategic convergence if and only if product convergence
  - apply a grid argument to the "report your higher order beliefs" game
- Lower strategic convergence if and only if strategic convergence

The Denseness Result

THEOREM: Finite types are dense under d.

The proof constructs a sequence of finite types lower converging to it.

1. Fix a finite grid G^* of m action games and a finite grid Z of values of ε.
   (a) Construct an exact finite type space for that grid
   (b) Let $T_i^* = \{h : G^* \times A_i \to Z\}$
   (c) Let $f_i(t_i)$ be the smallest $\varepsilon \in Z$ such that a_i is ε-rationalizable in G
   (d) Choose beliefs of each t_i to "preserve rationalizability", i.e., such that $a_i \in R_i(t_i, G, \varepsilon) \Rightarrow a_i \in R_i(f_i(t_i), G, \varepsilon)$ for all $G \in G^*$ and $\varepsilon \in Z$.

2. Fix m and $\delta > 0$. Construct finite type space such that $a_i \in$

$R_i(t_i, G, \varepsilon) \Rightarrow a_i \in R_i(f_i(t_i), G, \varepsilon + \delta)$ for all m action games G and $\varepsilon > 0$.

3. Now for any integer m, there exists a finite type $\widehat{t}_i^m$ such that

$$h_i\left(\widehat{t}_i^m | a_i, G\right) \leq h_i(t_i | a_i, G) + \frac{1}{m}$$

for all a_i and m action game G. Now fix any m and let

$$\varepsilon^n = \begin{cases} 2M, \text{ if } n \leq m \\ \frac{1}{n}, \text{ if } n > m \end{cases}.$$

Observe that $\varepsilon^n \to 0$,

$$h_i\left(\widehat{t}_i^n | a_i, G\right) \leq h_i(t_i | a_i, G) + 2M = h_i(t_i | a_i, G) + \varepsilon^n$$

for all a_i and m action game G if $n \leq m$ and

$$h_i\left(\widehat{t_i^n}|a_i, G\right) \leq h_i\left(t_i|a_i, G\right) + \frac{1}{n} = h_i\left(t_i|a_i, G\right) + \varepsilon^n$$

for all a_i and m action game G if $n > m$.

Discussion: the size of the set of finite types

- The set of finite types is dense in both product and strategic topologies
- The set of infinite types is dense in both product and strategic topologies
- The set of finite types is category 1 (the countable union of closed sets with empty interior)

Discussion: a topology that is uniform in games

$$d^*(t_i, t_i') = \sup_{a_i, G} d(t_i, t_i' | a_i, G).$$

- finer than d
- equivalent to

$$d^{**}(t_i, t_i') = \sup_{k} \sup_{f \in F_k} |E(f|\pi^*(t_i)) - E(f|\pi^*(t_i'))|,$$

where F_k are bounded functions on $T^* \times \Theta$ that are measurable with respect to kth level beliefs.

1. if $d^{**}(t_i, t_i') \leq \varepsilon$ and $a_i \in R_i(t_i, G, \delta)$, then $a_i \in R_i(t_i, G, \delta + 4\varepsilon M)$
2. thus $d^{*}(t_i, t_i') \leq \delta + 4\varepsilon M$.
3. if $d^{**}(t_i, t_i') \geq \varepsilon$, we can construct a game G and action a_i such that $d(t_i, t_i' | a_i, G) \geq \frac{\varepsilon}{2}$.

- An argument of Morris (2002) implies that finite types are not dense under d^{**}

Discussion: unbounded payoffs

- with arbitrary payoffs, the "strategic" topology would be the discrete topology
- mechanism design literature (e.g., Cremer and McLean (1985)) exploits unbounded payoffs....

Discussion: interpretating the denseness result

The approximation result shows that finite types could conceivably capture the richness of the universal type space.

Does not (of course) establish that particular finite type spaces are generic nor does it support traditional "genericity" arguments for finite type spaces