JOSEPHSON JUNCTION ARRAYS
AND
TOPOLOGICAL QUANTUM COMPUTATION

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A quantum computer has to be a very unusual physical system!

1) Start with:

\[ |\Psi_0\rangle \otimes |\chi_0\rangle \]

2) In an ideal world, we would like to get, at the end of computation:

\[ U(t)|\Psi_0\rangle \otimes |\chi_0\rangle \]

where \( U(t) \) is a unitary operation, corresponding to a certain quantum program

3) In real life, we most of the time get, instead:

\[ \sum_{j=1}^{N} p_j(t)|\Psi_j(t)\rangle \otimes |\chi_j(t)\rangle \]
As soon as $N > 1$, this is an entangled state, and the computer’s reduced density matrix becomes a statistical mixture:

$$\rho(t) = \sum_{j=1}^{N} |\psi_j(t)\rangle\langle \psi_j(t)|$$

$$p_j(t) = \langle \chi_j(t) | \chi_j(t) \rangle$$

This decoherence phenomenon occurs in general very quickly!
Example for a Brownian particle

\[ T_D = \tau \left( \frac{\lambda_T}{|x_1 - x_2|} \right)^2 \]

Order of magnitude estimates: Sphere of 1 \( \mu \text{m} \) in diameter, of density \( 1 \text{ g/cm}^{-3} \), in water at room temperature:
\( \tau \): classical friction time \( \simeq 4 \times 10^{-8} \text{ s} \)
\( \lambda_T \): thermal wave length \( \simeq 10^{-16} \text{ m} \)
\( |x_1 - x_2| \): spatial separation between two wave packets.
Kitaev’s proposal (quant-ph/9707021)

1) computer should be a large system, of which only a small number of eigenstates are used for computation.
2) Different internal states used for computation should be:
   • macroscopically different
   • perfectly degenerate, even in the presence of environmental noise
3) Physical operators coupling the computer to its environment are local
   “Topologically protected” subspace

\[ M^+ \quad \Delta \quad M \]
Constraints on system-environment coupling

Environment-induced splittings in protected subspace

\[ \Delta E \simeq M \left( \frac{M}{\Delta} \right)^L \]

Can be very small if \( L \) is large, provided \( \frac{M}{\Delta} < 1 \)

So we can't use environment to distinguish states in the protected subspace
Basics of Josephson junction arrays

$\phi_j$; local phase of Cooper pair condensate

$\hat{n}_j = \frac{\partial}{i \partial \phi_j}$: number of Cooper pairs on island $j$

$\Delta \phi_j \Delta n_j \simeq 2\pi$

$A_{ij} = \frac{2\pi}{\Phi_0} \int_i^j \vec{A}_{ij}.d\vec{r}$

$H = -E_J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j - A_{ij}) + \frac{E_C}{2} \sum_{ij} (C_{ij}^{-1}) \hat{n}_i \hat{n}_j$

$E_J$: Josephson coupling energy

$E_C$: Charging energy
A rhombus with half a flux quantum

Define $\theta_{ij} = \phi_i - \phi_j - A_{ij}$, then:

$$\theta_{12} + \theta_{23} + \theta_{34} + \theta_{41} \equiv \pi, \mod 2\pi$$

→ Get two-fold degenerate classical ground-state, with $\theta_{ij} = \pm \frac{\pi}{4}$
→ Quantum fluctuations ($E_C \neq 0$) of phases lift this degeneracy
There is an extensive residual entropy for the classical states! Remark: this classical degeneracy is NOT $2^{|\text{bonds}|}$, but $2^{|\text{sites}|}$, where $|\text{bonds}|$ and $|\text{sites}|$ refer to the underlying square lattice.
Local constraints on the classical ground-states (1)
Local constraints on the classical ground-states (2)

Define Ising plaquette variables $\tau_{ij}$, where $i$, $j$ are nodes connecting several rhombi:

The constraints are now expressed as:

$$\prod_{\square} \tau_{ij}^z = 1$$

Reminiscent of a gauge theory, (zero flux condition).
Basic tunneling process acts as: $\prod_{j}^{(i)} \tau_{ij}^{x}$

Tunnel rate: $r \simeq E_{j}^{3/4} E_{C}^{1/4} \exp(-4S_{0})$

where: $S_{0} = 1.61(E_{j}/E_{C})^{1/2}$, (Ioffe and Feigel'man, 2002)
Key properties of elementary flips

Tunnel processes

\[ \prod_j \tau_{ij}^{(i)} \prod_{ij} \tau_{ij}^x \]

Constraints

\[ \prod \tau_{ij}^z \]

Elementary flips \( \prod_j \tau_{ij}^{(i)} \tau_{ij}^x \) mutually commute, and they also commute with all local constraints \( \prod \tau_{ij}^z \).

→ Tunneling Hamiltonian is easy to solve!
Physical picture: Localization of Cooper pairs, and charge $4e$ condensate

Introduce $P_j$ such that: $P_j|\phi_j\rangle = |\phi_j + \pi\rangle$

From $|n_j\rangle = \int \frac{d\phi_j}{\sqrt{2\pi}} \exp(in_j\phi_j)|\phi_j\rangle$, $P_j|n_j\rangle = (-1)^{n_j}|n_j\rangle$

$P_j$: basic flip

The parity of $n_j$ is conserved and single Cooper pairs are localized.

\[\langle \exp(i\phi_j) \rangle = 0 \text{ but } \langle \exp(i2\phi_j) \rangle \neq 0\]
Aharonov-Bohm cages (1)

Aharonov-Bohm cages (2)
A two-fold degenerate ground-state

Total charge is conserved →
Even: (even, even) or (odd, odd)
Odd: (even, odd) or (odd, even)

Important property:
Matrix elements between two states appear only at order $L$ in perturbation theory →
Suppression factor $(w/r)^L$, where $w$ is amplitude of local flux noise.
Moving a $\pi$ vortex around the inner hole (1)
Moving a $\pi$ vortex around the inner hole (2)

Application of local flip operators in the bulk does not change final state.
Moving a $\pi$ vortex around the inner hole (3)

Final state $|\Psi_f\rangle$

$|\Psi_f\rangle = \exp(i\pi \hat{n}_{\text{hole}}) |\Psi_{\text{in}}\rangle$

Adiabatic phase analogous to Aharonov-Casher phase!
SMALL ARRAYS
AS
PARTIALLY PROTECTED
QUBITS

Non-local symmetries and protection against local noise
Assume \([H, P_i] = [H, Q_j] = 0\)

\[P_i^2 = 1, \quad [P_i, P_j] = 0\]

\[Q_i^2 = 1, \quad [Q_i, Q_j] = 0\]

\[\{P_{\text{row}}, Q_{\text{column}}\} = 0\]

Can diagonalize simultaneously:

\(P_1, P_2, \ldots, P_M, Q_1 Q_2, \ldots, Q_1 Q_N\)

Gives only two-dimensional irreducible representations!

1) Start with \(|\uparrow\rangle\), such that:

\[P_i |\uparrow\rangle = \alpha_i |\uparrow\rangle\]

\[Q_1 Q_j |\uparrow\rangle = \beta_j |\uparrow\rangle\]

2) Define \(|\downarrow\rangle\) as:

\[|\downarrow\rangle = Q_1 |\uparrow\rangle\]

3) \(|\downarrow\rangle\) satisfies:

\[P_i |\downarrow\rangle = -\alpha_i |\downarrow\rangle\]

\[Q_1 Q_j |\downarrow\rangle = \beta_j |\downarrow\rangle\]

4) Furthermore:

\[Q_j |\uparrow\rangle = \beta_j |\downarrow\rangle\]
Doublets exist as long as one can find at least ONE pair $P_i$, $Q_j$, commuting with $H$.

Example of a static disorder configuration which does NOT lift any degeneracy!

Local noise breaks degeneracies only at high orders ($M$, or $N$) in perturbation theory!
Important example

Local noise operators $\mathcal{N}_{ij}$ such that they commute with all $P_{\text{row}}$'s and $Q_{\text{column}}$'s, with the exception of:

$$\{\mathcal{N}_{ij}, Q_j\} = 0$$

$$\mathcal{N}_{ij}|{\alpha}, \{\beta\}, \tau\rangle = |{\alpha}, \beta_1, ..., -\beta_j, ..., \beta_N, \tau\rangle$$

$$\mathcal{N}_{i1}|{\alpha}, \{\beta\}, \tau\rangle = |{\alpha}, -\beta_1, ..., -\beta_j, ..., -\beta_N, \tau\rangle$$

1) This noise induces no transitions inside the doublets ($\tau$ is conserved):

   $\rightarrow$ No relaxation!

2) Degeneracies are lifted only at order $N$ or larger:

   $\rightarrow$ Dephasing rate is exponentially small!
Example: X-Z Ising model

\[ H = -J_x \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - J_z \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z \]

Douçot, Feigel’man, Ioffe, Iosele-vich, P. R. B. 71, (2005)

Conservation laws

\[
\begin{align*}
P_{\text{row}} &= \prod_{r \in \text{row}} \sigma_r^z \\
Q_{\text{column}} &= \prod_{r \in \text{column}} \sigma_r^x \\
P_i^2 &= 1, \ [P_i, P_j] = 0 \\
Q_i^2 &= 1, \ [Q_i, Q_j] = 0 \\
\{P_{\text{row}}, Q_{\text{column}}\} &= 0
\end{align*}
\]
Main question: how does energy gap depend on $J_x$, $J_z$, $M$, $N$?
Seems to close exponentially with system size
Dorier, Becca, and Mila, P. R.  \textbf{B 72}, 024448, (2005)
\rightarrow One should work with a relatively small array, that is $M, N \simeq 5$
Effect of quenched disorder

\[ H_{\text{dis}} = \sum_r h^z_r \sigma^z_r + h^x_r \sigma^x_r \]

Random field \( h^z_r \in [-0.05, 0.05] \), \( h^x_r = 0 \)

All states are doubly-degenerate, in a way largely unsensitive to noise from environment
Simplest protected physical device

Josephson element that has two (approximately) degenerate classical states with phase differences $\phi_2 - \phi_1 = 0$ or $\phi_2 - \phi_1 = \pi$ and fluctuates between them → effective ‘spin’ degree of freedom.

In order to protect against flip errors one needs large potential barrier in the potential $V(\phi)$ that separates states with $\phi=0$ and $\phi=\pi$.

Need $M\sim N$ chains to add these potentials and decrease charging energy of $\phi$. 
Single chain of rhombi: Josephson energy

\[ \Delta \phi_n = -\frac{\pi}{2} \sigma_n^z \]

\[ -\frac{\pi}{2} \sum_{n=1}^{N} \sigma_n^z \equiv \frac{\pi}{2} \prod_{n=1}^{N} \sigma_n^z - \frac{\pi}{2} (N+1) [2\pi] \]

Boundary condition:

\[ \phi_R = \frac{\pi}{2} \prod_{n=1}^{N} \sigma_n^z - \frac{\pi}{2} (N + 1) + \phi_{el} \]

Classical ground-states for rhombus:

\[ H_{eff,J} = \frac{c(N)}{2} \left( \phi_R - \frac{\pi}{2} \prod_{n=1}^{N} \sigma_n^z + \frac{\pi}{2} (N + 1) \right)^2 \]
Josephson energy in the classical limit
Effect of quantum fluctuations on low-energy spectrum

If $\frac{E_c}{E_J}$ not very small, single rhombi can flip, with tunneling amplitude:

$$b \simeq (E_J^3 E_c)^{1/4} \exp(-1.61 \sqrt{\frac{E_J}{E_c}})$$

Elementary flips:

\[ \begin{align*}
\phi_R & \rightarrow +\pi \\
\phi_R & \rightarrow -\pi
\end{align*} \]
Model Hamiltonian for low-energy spectrum 

**CONSTRAINT:** \( \phi_R = \frac{\pi}{2} \Pi_n \sigma_n^z - \frac{\pi}{2} (N + 1) + \phi_{el} \)

Introduce translation operator \( T \) such that \( T | \phi_{el} \rangle = | \phi_{el} + \pi \rangle \):

\[
H_{\text{eff}} = -a(N) \left( \sum_n \sigma_n^x \right)^2 - b(N) \left( \sum_n \sigma_n^x \right) (T + T^\dagger) + \frac{c(N)}{2} \phi_{el}^2
\]

Keeping only two lowest branches in Josephson energy

\[
H_{\text{eff}} = -a(N) \left( \sum_n \sigma_n^x \right)^2 - b(N) \sum_n \sigma_n^x + \frac{c(N)}{2} \left( \phi_R - \frac{\pi}{2} \prod_{n=1}^{N} \sigma_n^z + \frac{\pi}{2} (N + 1) \right)^2
\]

Spectrum depends only on \( \Sigma = | \sum_n \sigma_n^x | \)
Comparison between effective model and numerical diagonalizations

\[ E_J/E_C = 6 \]

\[ E_{\pm}(\Sigma) = a(N)\Sigma^2 + \frac{c(N)}{2}(\phi_R + \frac{\pi^2}{4}) \pm \left( b(N)^2\Sigma^2 + \left( \frac{\pi}{2} c(N)\phi_R \right)^2 \right)^{1/2} \]
Non-local symmetries in the low-energy sector

Introduce $\tau^z$, such that $\phi_R = \frac{\pi}{2} \tau^z - \frac{\pi}{2} (N + 1)$

(Global qubit state)

$$H_{\text{eff}} = -a(N) \left( \sum_n \sigma_n^x \right)^2 - b(N) \sum_n \sigma_n^x + \frac{\pi^2 c(N)}{4} \frac{1}{2} \left( \prod_{n=1}^N \sigma_n^z \right) \tau^z$$

$$P = \tau^z$$
$$Q_n = \sigma_n^x \tau^x$$

Effect of local noises: $H_{\mathcal{N}} = \sum_n h_n^z \sigma_n^z + h_n^x \sigma_n^x$

$h_n^x \sigma_n^x$ preserves $P$ and $Q_m \rightarrow$ No degeneracy lifting

$h_n^z \sigma_n^z$ preserves $P$ and $Q_m$, $m \neq n$, but anticommutes with $Q_n$

$\rightarrow$ Degeneracy lifting, but at order $N$ in perturbation theory!
Effect of static perturbations (1)
Offset charges and local variations of $E_J$ do not lift doublet degeneracy provided $N$ is odd

Proof: for $N$ odd, total phase variation across the chain is $\pm \frac{\pi}{2}$. So one goes from $\tau^z = 1$ to $\tau^z = -1$ by changing all local phases $\phi_r$ into $-\phi_r$.

This operation commutes with Josephson energy

$$-\sum_{\langle r, r' \rangle} E_{J,rr'} \cos(\phi_r - \phi_{r'} - A_{rr'})$$

because for half-flux quantum/rhombus, we may choose $A_{rr'} \in \{0, \pi\}$.

To preserve the charging energy (in the presence of offset charges), we need to preserve $\hat{n}_r = \frac{\partial}{i \partial \phi_r}$, so the desired operation reads:

$$\Psi_{\text{new}}(\phi_1, \ldots, \phi_{3N+1}) = \bar{\Psi}(-\phi_1, \ldots, -\phi_{3N+1})$$

Remark: for $N$ even, these perturbations lift doublet degeneracy only at order $N$. 

35
Effect of static perturbations (2)

Effect of area variations $\longrightarrow \Phi \neq \frac{\Phi}{\Phi_0}$ in a given rhombus.

$\longrightarrow$ lifts degeneracy between opposite chiralities

$$\Delta H_{\text{eff}} = \sum_n h_n \sigma_n^z$$

$$h_n \simeq \frac{\Delta \Phi_n}{\Phi_0} E_J \sim 0.01 E_J$$

By choosing $E_c$ not too small, we can get $|h_n| < \Delta$

Numerical results for $N = 3$:

$$\frac{\Delta}{E_J} \simeq 0.03 \quad (E_J = 6E_c)$$

$$\frac{\Delta}{E_J} \simeq 0.1 \quad (E_J = 4E_c)$$

Remark: here again, this perturbation lifts doublet degeneracy only at order $N$. 

36
Read-out

Requires moving the system out of topologically protected subspace

Can be either destructive (critical current measurement) or non destructive (ac impedance measurement)

Measuring $\tau^z$

Apply a small uniform magnetic field, which lifts degeneracy between $\tau^z = \pm 1$. This has to be applied fast compared to the inverse qubit splitting, and slowly compared to the inverse gap $\Delta^{-1}$.

The two states are now macroscopically distinct.
Conclusions

1) Possibility to simulate lattice-gauge theories based on finite groups $G$ using Josephson junction arrays.
2) Low energy excitations in these models are anyons, whose braiding properties are sufficient to generate non-trivial unitary operations, if $G$ is non-Abelian.
3) Ground-state degeneracies occur in the presence of large holes, and are protected against local noise terms to high order $\sim L$ in perturbation theory.
4) Small Josephson junction arrays with a large set of non-local symmetries are expected to be good candidates for protected qubits.
5) Possible implementation in ion traps.