A Universal Operator Theoretic Framework for Quantum Fault Tolerance

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Recent Papers

“Reliable Final Computational Results from Faulty Quantum Computation”
G Gilbert, M Hamrick, YS Weinstein (MITRE) submitted to New Journal of Physics

“A Universal Operator Theoretic Framework for Quantum Fault Tolerance”
G Gilbert, M Hamrick, YS Weinstein (MITRE) V Aggarwal, A Robert Calderbank (Princeton) submitted to Physical Review A

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Quantum Computational Path

- Initial Quantum State $\rho$
- Quantum Computation
- Final Quantum State $\rho'$
- Final Measurement
- Probabilistically Distributed Set of Possible Outcomes
  - $y_1$
  - $y_2$
  - $\ldots$
  - $y_i$

Overall Computation

Final Result $y_i$
Accuracy of Quantum Computational Results (1)

Kitaev’s Model of Ideal Quantum Computation

Kitaev (1997)

\[ \begin{array}{c}
\text{I}_m & \xrightarrow{G} & \text{O}_m \\
\text{T}(H_{\text{logical}}) & \xrightarrow{G} & \text{T}(H_{\text{logical}}) \\
X & \xrightarrow{F} & Y
\end{array} \]

Probability that the ideal quantum computation followed by measurement produces outcome \( y \):

\[ \Pr[(\text{O}_m \circ G \circ \text{I}_m)(x) = y] = \text{tr}\left(\sqrt{E_y} G(I_m(x)) \sqrt{E_y}\right) \]

Probability that the ideal quantum computation followed by measurement produces the correct result \( F(x) \):

\[ \text{tr}\left(\sqrt{E_{F(x)}} G(I_m(x)) \sqrt{E_{F(x)}}\right) > 1 - p \]

The quantity \( p \) bounds the probability that the ideal quantum computation followed by measurement fails to produce the correct result.

But real quantum computers don’t implement \( U \) exactly. What’s the effect on the failure probability bound of implementation errors?
Accuracy of Quantum Computational Results (2)

The Quantum Computer Condition (QCC)
Gilbert, Hamrick & Weinstein (2007)

Kitaev model gives failure probability for ideal computation.
QCC describes deviation of the quantum computation from the ideal.
What’s the combined effect?
Accuracy of Quantum Computational Results (3)

Kitaev model shows ideal quantum computation can produce correct output, within $p$

the QCC states that real quantum computation can realize ideal quantum computation, within $\alpha$

our result shows that real quantum computation can produce correct output, within $p + \alpha$
Achieving a Specified Probability of Success

We can now derive a bound on the implementation inaccuracy that will guarantee the correct result with a specified probability

- End user specifies maximum probability of failure
  - \( \hat{p} < \frac{1}{2} \) means majority voting can be used
- Our result bounds total probability of failure
- Bounds \( p_b \) on failure probability, \( p \), for ideal quantum computation are known for algorithms of interest:
  - e.g., in Grover search for 1 out of \( n \) items \( p_b = \frac{1}{n} \)
- We require that the implementation error tolerance be bounded by \( \hat{p} - p_b \)
- It follows that the probability that the computation fails to return the correct result is bounded, as required, by \( \hat{p} \)

We wish to achieve \( p_f \leq \hat{p} \)

\[
p_f \leq p + \alpha
\]

\[
p \leq p_b
\]

\[
\alpha \leq \hat{p} - p_b
\]

\[
p_f \leq p + \alpha \leq p_b + \alpha \leq \hat{p}
\]

But the standard theory of fault tolerance yields a bound on the probability that the quantum computation results in the correct quantum state, not the implementation inaccuracy, which is a bound on the error norm of the state.

Can we relate the two?
Fault Tolerance Theory

• Fault tolerance techniques enable correct dynamical evolution of the quantum computation in the presence of noise
  – Circuit is subject to (random) errors
  – Techniques such as concatenated quantum error correcting codes are used to correct errors
  – Fault tolerant circuit operates on encoded states
  – Not all errors are corrected, only the more likely ones
  – There is a residual error probability that the circuit fails to produce the desired quantum state as its output
    – Residual error probability can be estimated from theory
      – Doesn’t address quantum uncertainties in the final measurement
    – Need to find connection between residual error probability and probability of obtaining the correct final result

Residual Error Probability $\mathcal{E}_N$  

\[
\frac{\mathcal{E}_N}{\mathcal{E}_{th}} \approx \left( \frac{\mathcal{E}_0}{\mathcal{E}_{th}} \right)^{2^N}
\]

(for a single logical gate using simplified model for concatenated error codes)

$\mathcal{E}_0$  Elementary gate failure probability

$\mathcal{E}_{th}$  Error threshold (function of circuit structure)

$N$  Number of levels of concatenation

Preskill (1998)
Specifying Fault Tolerance to Guarantee Probability of Obtaining Correct Final Result

• Fault tolerance theory yields a residual error probability \( \approx N_g \varepsilon_N \)
  – \( N_g \) is the number of logical gates in the formal description of the algorithm

• We derive from this a bound on implementation inaccuracy given by:
  \[
  \| P^M \cdot \rho - U \rho U^\dagger \| \leq 2N_g \varepsilon_N
  \]

• Success criterion is achieved if this bound satisfies \( 2N_g \varepsilon_N \leq \hat{p} - p_b \)

• For a given circuit and a given probability of elementary gate failure, this gives the number of levels of concatenation required to achieve the success criterion for the overall result

\[
N_{\text{min}} \approx \log_2 \frac{\ln \frac{2N_g \varepsilon_{th}}{\hat{p} - p_b}}{\ln \frac{\varepsilon_{th}}{\varepsilon_0}}
\]
Example

Levels of Concatenation Required to Guarantee Specified Tolerance of 0.4 for Failure to Compute Correct Final Result

- Failure probability tolerance for final result: $\hat{p} = 0.4$
- Number of gates in circuit: $\mathcal{N}_g = 10^{12}$
- Measurement failure bound for quantum algorithm: $p_b = 0.2$
- Fault tolerant error threshold: $\epsilon_{th} = 10^{-9}$
Quantum Error Correction (1)

QEC

\[ V_{\text{dec}} R \varepsilon V_{\text{enc}} \rho = \rho \]

OQEC

\[ V_{\text{dec}} Tr_B F_{AB} R \varepsilon W_{\rho_B} V_{\text{enc}} \rho = \rho \]

EAQEC

\[ V_{\text{dec}} R \varepsilon V_{\text{enc}} \rho = \rho \]

EAOQEC

\[ V_{\text{dec}} Tr_B F_{AB} R \varepsilon W_{\rho_B} V_{\text{enc}} \rho = \rho \]

\[ V_{\text{enc}}, V_{\text{dec}} \text{ encoding, decoding} \]

\[ V_{\text{enc}}, V_{\text{dec}} \text{ encoding, decoding w/ebits} \]

\[ W_{\rho_B} : W_{\rho_B} \equiv \rho_A \otimes \rho_B \oplus 0_K \]

\[ W_{\rho_B} : W_{\rho_B} \equiv \rho_A \otimes \rho_B \oplus 0_K \text{ w/ebits} \]

\[ \varepsilon \text{ error dynamics} \]

\[ R \text{ measurement and recovery} \]

\[ R \text{ measurement and recovery w/ebits} \]

\[ F_{A,B} \text{ projects } H \text{ onto } A \otimes B \]

\[ F_{A,B} \text{ projects } H \text{ onto } A \otimes B \text{ w/ebits} \]
Quantum Error Correction (2)

\[ \lim_{c \to 0} V_{enc} = V_{dec} \]
\[ \lim_{c \to 0} F_{AB} = F_{AB} \]
\[ \lim_{c \to 0} R = R \]
\[ \lim_{\dim B \to 0} Tr_B = 1 \]
\[ \lim_{\dim B \to 0} W_{\rho_B} = 1 \]

OQEC contains QEC
EAQEC contains QEC
EAOQEC contains EAQEC and OQEC

EAOQEC: \[ V_{dec} Tr_B F_{AB} R E W_{\rho_B} V_{enc} \rho = \rho \]

\[ \Rightarrow V_{dec} Tr_B F_{AB} R E W_{\rho_B} V_{enc} \rho - \rho = 0 \quad \text{(EAOQEC)} \]

\[ \Rightarrow \left\| \tilde{\rho} \rho - U \rho U^\dagger \right\|_1 \leq \alpha \quad \text{(QCC)} \]

with \[ \tilde{\rho} = V_{dec} Tr_B F_{AB} R E W_{\rho_B} V_{enc} \], \( \alpha = 0 \) and \( U = I \)
Quantum Error Correction (3)

all forms of error correction
are special cases of the QCC
Operator Quantum Fault Tolerance (OQFT)

Would like “top-down” approach to fault tolerance based on system-level dynamical constraint

\[ \left\| \tilde{P} \rho - U \rho U^\dagger \right\|_1 \leq \alpha \]

full specification of system dynamics with success criterion

\[ \left\| \tilde{P}^{(i)} \rho - U \rho U^\dagger \right\|_1 \]

characterization system dynamics at \( i \)th level of concatenation

\[ \Rightarrow \sup_{\rho} \left\| \tilde{P}^{(i+1)} \rho - U \rho U^\dagger \right\|_1 < 1 \]

\[ \sup_{\rho} \left\| \tilde{P}^{(i)} \rho - U \rho U^\dagger \right\|_1 \]

Operator Quantum Fault Tolerance (OQFT)
Operator Quantum Fault Tolerance (2)

\[ \mathcal{M}_{\{c \rightarrow l\}} \left[ Y \left( \mathcal{M}_{\{l \rightarrow c\}} \rho \right) Y^\dagger \right] = U \rho U^\dagger \]

Y (acting on \( H_{\text{comp}} \)) faithfully implements \( U \) (acting on \( H_{\log} \))

\[ P^{(i)} \left( \mathcal{M}_{\{l \rightarrow c\}} \rho \right) = \left(1 - \varepsilon_f^{(i)} \right) Y \left( \mathcal{M}_{\{l \rightarrow c\}} \rho \right) Y^\dagger + \varepsilon_f^{(i)} Q_f^{(i)} \left( \mathcal{M}_{\{l \rightarrow c\}} \rho \right) \]

model for \( P \) that includes local stochastic noise and locally correlated stochastic noise (\textit{i.e.}, standard quantum fault tolerance)

<table>
<thead>
<tr>
<th>( \varepsilon_f^{(i+1)} )</th>
<th>( \sup_\rho | \widetilde{Q}^{(i+1)} \rho - U \rho U^\dagger |_1 )</th>
<th>( \frac{\varepsilon_f^{(i+1)}}{\varepsilon_f^{(i)}} )</th>
<th>( \sup_\rho | \widetilde{Q}^{(i)} \rho - U \rho U^\dagger |_1 )</th>
<th>( \frac{\varepsilon_f^{(i+1)}}{\varepsilon_f^{(i)}} &lt; 1 )</th>
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OQFT success criterion

standard FT success criterion

OQFT can produce more accurate (larger) error thresholds than standard FT
Conclusion

• The probability that a quantum computation produces the correct final answer can be simply expressed in terms of the intrinsic failure probability due to quantum uncertainty in measurement and the implementation inaccuracy.

• The resulting expression can be used to specify the “amount” of fault tolerance required to achieve a specified success probability that the quantum computer yields the correct final answer.

• All forms of error correction and avoidance are special cases of the QCC.

• The QCC provides a universal operator-theoretic framework for quantum fault tolerance based on a top-down criterion, leading to improved accuracy for error threshold values.