Negative Assortative Matching of Risk-Averse Agents With Transferable Expected Utility

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Abstract

I give necessary and sufficient conditions for expected utility to be transferable among agents with different state-independent risk preferences who share a risk they cannot influence. In an assignment game where participants choose partners based on risk aversion, the conditions permit a simple proof that matching is negative assortative.

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1 Introduction

Suppose some agents with heterogeneous risk preferences can form partnerships to share an otherwise uninsured risk. Intuitively, it seems likely that opposites will attract: Very risk-averse people will prefer the least risk-averse partners, who provide good insurance, while less risk-averse people will prefer very risk-averse partners, who pay a high risk premium. But the equilibrium is difficult to characterize formally. If agents are risk averse, utility is not transferable in any single state of the world, and the standard results on matching with transferable utility (TU) appear useless.

Legros and Newman (2004) observe that some models with apparently non-transferable utility admit a TU representation. For example, in the risk-sharing problem, expected utility can be transferable even if utility in each state of the world is not. Legros and Newman (2004) show in particular that if people have utility functions of the form $u(c) = \ln(c - \theta)$ or $u(c) = \exp(-\theta c)$, where θ varies across agents, the risk-sharing problem has a TU representation and matching is negative assortative. Legros and Newman (2004) and Chiappori and Reny (2004) also show that matching is negative assortative whenever agents' risk preferences can be ranked in the Arrow-Pratt sense. However, allowing arbitrary preferences precludes an appeal to the simple methods available under transferable utility.

In Section 2, I delimit the preferences for which the risk-sharing problem has a TU representation: When two agents with state-independent preferences share a random income whose distribution they cannot influence, expected utility is transferable if and only if preferences are in the harmonic absolute risk aversion class with identical shape parameters (the ISHARA class). The class includes the logarithmic and exponential examples of Legros and Newman (2004). TU implies negative assortative matching whenever partners' types are substitutes in producing the output of the match. In Section 3, I use this fact to give a simple proof that matching on risk aversion is negative assortative for all ISHARA preferences.

2 Risk sharing with transferable expected utility

Consider an economy consisting of two mutually exclusive sets of agents who each face uninsured risk. Each agent can form a partnership with at most one agent from the other set; within such a partnership, there are no barriers to efficient risk-sharing. To fix ideas, I will describe this as a marriage market. The sets of agents are men and women, and the partnerships they form are marriages. People are expected utility maximizers. They live for two periods, $t = 0, 1$. In period 0, each person can marry at most one person of the opposite set. In period 1, each person receives a risky income and consumes a private good.

Let the type of a man m be given by some parameter θ_m, and the type of a woman f by θ_f. Let the von Neumann-Morgenstern utility function of person $i \in \{m, f\}$ be $u_i(c) = u(c; \theta_i)$. For now, focus on one man θ_m and one woman θ_f who have already decided to marry. Suppose that the man's income is $y(\omega)$ and the woman's is $z(\omega)$, where $\omega \in \Omega$ is the state of the world. Define $q(\omega) \equiv y(\omega) + z(\omega)$. I restrict attention to income processes such that expected utility is well defined.

I assume that, at $t = 0$, the couple can commit to a rule $\bar{c}\colon \Omega \to \mathbb{R}^2$ for sharing their total income $q(\omega)$ in each state of the world. Let $\bar{c}_i(\omega)$ be the consumption of person i in state ω under sharing rule $\bar{c}$. The expected utility of person i under sharing rule $\bar{c}$ is $\tilde{U}(\bar{c}; \theta_i) \equiv \mathrm{E}_i\big[u\big(\bar{c}_i(\omega); \theta_i\big)\big]$, where $\mathrm{E}_i[\cdot]$ is an expectation taken with respect to i's beliefs.

A sharing rule is feasible if it requires no more income than the couple receives. It is efficient if it is feasible and if no other feasible sharing rule would make one person strictly better off without making the other worse off. Let $C^*(q)$ be the set of efficient sharing rules. The expected utilities achieved at $\bar{c} \in C^*(q)$ generate the expected utility possibility frontier. A monotonic transformation of expected utility does not change preferences; expected utility is transferable if the frontier is a line with constant slope -1 under some such transformation.

Definition 1. *Expected utility is transferable if there exist functions $g(\tilde{U}, \theta)$ and $h(\theta_m, \theta_f; q)$*

such that 1) g is strictly increasing in $\tilde{U}$ for all θ, and 2) $\sum_i g\big(\tilde{U}(\bar{c};\theta_i),\theta_i\big) = h(\theta_m,\theta_f;q)$ for all $\bar{c} \in C^*(q)$ and all income lotteries q.

It is well known that, in a group of agents with preferences defined over commodity bundles, utility is transferable if and only if the agents' aggregate demand for each commodity does not depend on the distribution of income among agents. Bergstrom and Varian (1985) prove this theorem for finite-dimensional commodity spaces. I extend their result by considering an infinite-dimensional space in which each commodity is a contingent claim.

Mazzocco (2004), extending Pollak (1971), shows that the preferences of expected utility maximizers are exactly aggregable if and only if they are in the following class:

Definition 2. *An ISHARA household is one in which the partners have identical beliefs about the distribution of ω and in which the utility functions satisfy $\frac{\partial}{\partial c}u(c;\theta) = (c-\theta)^{-\sigma}$, where $\sigma \geq 0$ is common to all agents but θ potentially varies across individuals.*[1]

For each efficient sharing rule $\bar{c} \in C^*(q)$, there exist non-negative Pareto weights λ_i such that $\bar{c}$ solves the Pareto problem:

$$\max_{\bar{c}} \left\{ \sum_i \lambda_i \tilde{U}(\bar{c};\theta_i) \right\} \quad \text{s.t.} \quad \sum_i \bar{c}_i(\omega) \leq q(\omega) \quad \forall_\omega. \tag{1}$$

If the couple could choose a lottery q, they would seek to maximize the value of this problem. The solution to problem (1) thus characterizes both the Pareto set $C^*(q)$ and the relationship between individual and aggregate preferences over q.

It is natural to assume that both partners' preferences matter in a voluntary marriage; otherwise, one would prefer to be single. I also assume that agents do not care directly about the realized state, and I rule out moral hazard problems by assuming that the agents cannot influence which state is realized. Under these assumptions, ISHARA preferences

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[1] Constant absolute risk aversion is a special case in the limit as σ goes to infinity with θ/σ fixed. Constant relative risk aversion is a special case if there is no heterogeneity: $\theta = 0$ for everyone.

are necessary and sufficient for transferable expected utility, which in turn is necessary and sufficient for aggregate preferences to be independent of the Pareto weights. Formally:

Assumption 1. $\lambda_i > 0$ *for all* i.

Assumption 2. *The agents' utility functions* $u(c;\theta)$ *do not depend on* ω*, and the agents cannot influence the distribution of* ω.

Proposition 1. *Under Assumptions 1 and 2, if the utility functions* $u(c;\theta)$ *are continuous in* c*, the following three statements are equivalent:*

1. *Preferences are in the ISHARA class.*
2. *For any non-negative prices* $p(\omega)$ *and any wealth* $W > 0$*, the optimal choice of* q *in the problem*

$$\max_{q,\bar{c}} \left\{ \sum_i \lambda_i \tilde{U}(\bar{c};\theta_i) \right\} \quad \text{s.t.} \quad \sum_i \bar{c}_i(\omega) \leq q(\omega)\ \forall_\omega \quad \text{and} \quad \int_\Omega p(\omega)q(\omega)d\omega \leq W \qquad (2)$$

   does not depend on the Pareto weights.

3. *Expected utility is transferable.*

Proof. Theorem 1 of Mazzocco (2004) says that, under Assumption 1, statements 1 and 2 are equivalent. To show that statement 1 implies statement 3, one can construct the Pareto sets $C^*(q)$, the necessary transformations of expected utility $g(\tilde{U},\theta)$ and the aggregate utility functions $h(\theta_m,\theta_f;q)$ for all preferences in the ISHARA class. These objects are shown in Table 1. Details of the construction are available from the author on request.[2]

To complete the proof, it is sufficient to show that statement 3 implies statement 2. Assume statement 3. Construct the appropriate transformation $g(\tilde{U},\theta)$ and aggregate utility function $h(\theta_m,\theta_f;q)$ to make expected utility transferable. Consider q^1 and q^2 such

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[2] For constant absolute risk aversion, $\sigma = \infty$, it must be assumed that each person has strictly positive consumption in all states.

that $h(\theta_m, \theta_f; q^1) > h(\theta_m, \theta_f; q^2)$. Because $u(c; \theta)$ is continuous and state-independent, the Pareto frontiers are continuous and any transformed utilities $\left[g(\tilde{U}_m, \theta_m), g(\tilde{U}_f, \theta_f)\right]$ that can be achieved under q^2 can also be achieved under q^1. Hence under the transformed representation of preferences, q^1 permits a Pareto improvement over any allocation feasible under q^2. But a Pareto improvement under one cardinal representation of preferences is a Pareto improvement under all cardinal representations. The objective function in (2), which uses the original representation of preferences, is therefore larger under q^1 than under q^2 regardless of the Pareto weights. Thus the optimal q maximizes $h(\theta_m, \theta_f; q)$ and does not depend on the weights, which proves statement 2. $\square$

3 Equilibrium matching

I now consider which men will marry which women. I assume:

Assumption 3. *The distributions of* $y(\omega)$ *and* $z(\omega)$ *do not depend on* θ_i.

Assumption 4. *The variance of* $q(\omega)$ *is strictly positive.*

Assumption 3 allows me to focus solely on risk aversion. It says that all men draw income from the same distribution and all women from the same distribution, so there is no reason to choose a spouse on the basis of his or her likely income, only on the basis of his or her risk preferences. Assumption 4 guarantees that the risk-sharing problem is nontrivial.

An equilibrium in the marriage market is an assignment of men to women and vice versa, and a sharing rule for each marriage, that meet two requirements. First, no married person can prefer to be single. Second, if some man and woman are not married to each other, they cannot both want to leave their assigned mates and marry each other. The first requirement motivates Assumption 1: Anyone who had a zero Pareto weight in a marriage would prefer to be single. The second requirement determines who marries whom.

An equilibrium can be described in terms of each agent's expected utility. Let $\tilde{U}_M(\theta_m)$ (respectively, $\tilde{U}_F(\theta_f)$) be the expected utility attained under the equilibrium assignment by a married man θ_m (respectively, a married woman θ_f). Let $\hat{U}_M(\theta_m; \theta_f, \hat{V})$ be the expected utility attained by a man θ_m if he marries a woman θ_f whose expected utility is $\hat{V}$. A necessary condition for equilibrium is

$$\tilde{U}_M(\theta_m) = \max_{\theta_f} \left\{ \hat{U}_M\big(\theta_m; \theta_f, \tilde{U}_F(\theta_f)\big) \right\}. \tag{3}$$

This equation says no man would prefer to marry a woman other than the one whom he is assigned to marry and who is willing to marry him.

There is nothing special about the cardinal representation of expected utility in equation (3). In particular, with ISHARA preferences, the transformed expected utilities $U(\theta_i)$ that satisfy $U(\theta_m) + U(\theta_f) = h(\theta_m, \theta_f; q)$ can be used. The necessary condition then becomes

$$U_M(\theta_m) = \max_{\theta_f} \left\{ h(\theta_m, \theta_f; q) - U_F(\theta_f) \right\}. \tag{4}$$

This is the matching problem with transferable utility analyzed by Becker (1973) and others. As is well known, equation (4) implies that there will be negative assortative matching – high θ_m matched with low θ_f – if h is submodular. If h is twice differentiable, it is submodular if $\partial^2 h / \partial\theta_m \partial\theta_f < 0$. The following assumption helps to compute this derivative.

Assumption 5. *There exists* $\underline{q} > 0$ *such that* $q(\omega) - \theta_m - \theta_f > \underline{q}$ *for all* ω, θ_m *and* θ_f.

Suppose that in equilibrium, man θ_m marries woman θ_f, and man θ'_m marries woman θ'_f. I restrict attention to the case $\sigma > 0$, since if $\sigma = 0$, agents are risk neutral.

Proposition 2. *If preferences are of the ISHARA class and if Assumptions 3, 4 and 5 hold, then the equilibrium matching is negative assortative:* $\theta'_m \leq \theta_m$ *if and only if* $\theta'_f \geq \theta_f$.

Proof. In equilibrium, no married person would prefer to be single. Thus $\lambda_m, \lambda_f > 0$ in all marriages, Proposition 1 applies, and expected utility is transferable. Under Assumption 3, the distribution of q is the same in all marriages, so it is sufficient to show $\partial^2 h/\partial\theta_m\partial\theta_f < 0$, where h is given in Table 1. Define $r(\omega) = [q(\omega) - \theta_m - \theta_f]^{-\sigma}$. By Assumption 5, derivatives of $r(\omega)$ with respect to θ_m and θ_f are bounded, and the order of differentiation and expectation can be exchanged. Then for $\sigma \in (0, \infty)$,

$$\frac{\partial^2 h}{\partial\theta_m\partial\theta_f} = -\sigma[h(\theta_m, \theta_f; q)]^{2\sigma-1}\Big(\mathrm{E}\left[r(\omega)^{(\sigma+1)/\sigma}\right]\mathrm{E}\left[r(\omega)^{(\sigma-1)/\sigma}\right] - \mathrm{E}\left[r(\omega)\right]^2\Big). \tag{5}$$

By Assumption 4 and Jensen's inequality, this quantity is strictly negative. The proof for constant absolute risk aversion is virtually identical. □

4 Conclusion

When two agents with continuous, state-independent preferences share an uninsured risk whose distribution they cannot influence, expected utility is transferable if and only if the agents' preferences are in the ISHARA class. Matching of ISHARA agents on the basis of risk aversion is negative assortative. Different conditions for transferable expected utility and different results could obtain in models with state-dependent preferences or moral hazard, such as some examples considered by Legros and Newman (2004).

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Table 1: Transformations for transferable expected utility with ISHARA preferences.

σ	$g(\tilde{U},\theta)$	$h(\theta_m,\theta_f;q)$	efficient $\bar{c}_i(\omega)$
0	$\tilde{U}$	$\mathrm{E}[q(\omega)]$	$\rho_i q(\omega)$
1	$\exp(\tilde{U})$	$\exp\big(\mathrm{E}\big[\ln[q(\omega)-\theta_m-\theta_f]\big]\big)$	$\theta_i+\rho_i[q(\omega)-\theta_m-\theta_f]$
$(0,\infty)\backslash\{1\}$	$[(1-\sigma)\tilde{U}]^{1/(1-\sigma)}$	$\mathrm{E}\big[[q(\omega)-\theta_m-\theta_f]^{1-\sigma}\big]^{1/(1-\sigma)}$	$\theta_i+\rho_i[q(\omega)-\theta_m-\theta_f]$
∞	$-\frac{1}{\theta}\ln(-\tilde{U})$	$-\left(\frac{\theta_m+\theta_f}{\theta_m\theta_f}\right)\ln\left(\mathrm{E}\left[\exp\left(-\frac{\theta_m\theta_f q(\omega)}{\theta_m+\theta_f}\right)\right]\right)$	$K_i+\frac{\theta_i}{\theta_m+\theta_f}q(\omega)$

$\mathrm{E}[\cdot]$ is an expectation taken with respect to the agents' common beliefs. ρ_m and ρ_f are any real numbers such that $\rho_i \in (0,1)$ and $\rho_f = 1-\rho_m$. K_m and K_f are any real numbers such that $K_m = -K_f$ and $\bar{c}_i(\omega) > 0$ for all i and all ω.

Proof that statement 1 implies statement 2 in Proposition 1.

There are three cases: $\sigma = 0$, $\sigma \in (0, \infty)$ and $\sigma = \infty$.

Case 1: $\sigma = 0$. Any sharing rule that does not waste income is efficient, so $\tilde{U}(\bar{c}, \theta_i) = \rho_i \mathrm{E}[q(\omega)]$, where $\mathrm{E}[\cdot]$ is an expectation taken with respect to the agents' common beliefs and where $\rho_i \in (0, 1)$ and $\sum_i \rho_i = 1$. (Agent i's consumption share ρ_i cannot be zero when $\lambda_i > 0$.) Since $\sum_i \tilde{U}(\bar{c}, \theta_i) = \mathrm{E}[q(\omega)]$, the definition of transferable expected utility is satisfied with $g(\tilde{U}, \theta) = \tilde{U}$, and $h(\theta_m, \theta_f; q) = \mathrm{E}[q(\omega)]$.

Case 2: $\sigma \in (0, \infty)$. Let $\lambda = \lambda_m/(\lambda_m + \lambda_f)$. The solution to the Pareto problem is interior because $\lambda \in (0, 1)$ and $\lim_{c \to 0} u'(c; \theta) = \infty$. The first-order necessary and sufficient condition for the solution is

$$\forall_i \forall_\omega \quad \bar{c}_i(\omega) - \theta_i = \rho_i(\lambda, \sigma) \left[q(\omega) - \theta_m - \theta_f\right], \tag{A.1}$$

where the functions $\rho_i(\lambda, \sigma)$ satisfy $\rho_m(\lambda, \sigma) + \rho_f(\lambda, \sigma) = 1$ and $0 < \rho_i(\lambda, \sigma) < 1$ for all $\lambda \in (0, 1)$ and all σ. Thus for all efficient sharing rules $\bar{c}$, the expected utility of person i is

$$\tilde{U}_i(\bar{c}; \theta_i) = \begin{cases} \frac{[\rho_i(\lambda,\sigma)]^{1-\sigma}}{1-\sigma} \mathrm{E}\big[[q(\omega) - \theta_m - \theta_f]^{1-\sigma}\big] & \sigma \neq 1 \\ \ln\left[\rho_i(\lambda, 1)\right] + \mathrm{E}\big[\ln\left[q(\omega) - \theta_m - \theta_f\right]\big] & \sigma = 1. \end{cases} \tag{A.2}$$

A positive monotonic transformation of expected utility does not change preferences. Let the transformation be $g(\tilde{U}) = [(1-\sigma)\tilde{U}]^{1/(1-\sigma)}$ if $\sigma \neq 1$, and $g(\tilde{U}) = \exp(\tilde{U})$ if $\sigma = 1$. Define

$$h(\theta_m, \theta_f; q) = \begin{cases} \mathrm{E}\big[[q(\omega) - \theta_m - \theta_f]^{1-\sigma}\big]^{1/(1-\sigma)} & \sigma \neq 1 \\ \exp\big(\mathrm{E}\big[\ln\left[q(\omega) - \theta_m - \theta_f\right]\big]\big) & \sigma = 1. \end{cases} \tag{A.3}$$

Then i's preferences over efficient sharing rules can be represented by $U_i(\bar{c}; \theta_i) = g\big(\tilde{U}_i(\bar{c}; \theta_i)\big) = \rho_i(\lambda, \sigma) h(\theta_m, \theta_f; q)$. Because $\rho_m(\lambda, \sigma) + \rho_f(\lambda, \sigma) = 1$, the sum of the transformed expected utilities U_i is $h(\theta_m, \theta_f; q)$ for all efficient $\bar{c}$, which proves that expected utility is transferable for $\sigma \in (0, \infty)$.

Case 3: $\sigma = \infty$. It must be assumed that each person has strictly positive consumption in all states. Then the first-order necessary and sufficient condition for an interior solution to the Pareto problem is

$$\forall_i \forall_\omega \quad \bar{c}_i(\omega) = K_i(\lambda, \theta_m, \theta_f) + \frac{\theta_i}{\theta_m + \theta_f} q(\omega), \tag{A.4}$$

where $\sum_i K_i(\lambda, \theta_m, \theta_f) = 0$. Thus for all efficient sharing rules $\bar{c}$, the expected utility of person i is

$$\tilde{U}_i(\bar{c}; \theta_i) = -e^{-\theta_i K_i(\lambda, \theta_m, \theta_f)} \mathrm{E}\left[\exp\left(-\frac{\theta_m \theta_f}{\theta_m + \theta_f} q(\omega)\right)\right]. \tag{A.5}$$

A positive monotonic transformation of expected utility does not change preferences. Let the transformation be $g(\tilde{U}, \theta) = -(1/\theta)\ln(-\tilde{U})$. Define

$$h(\theta_m, \theta_f; q) = -\left(\frac{1}{\theta_m} + \frac{1}{\theta_f}\right) \ln\left(\mathrm{E}\left[\exp\left(-\frac{\theta_m \theta_f}{\theta_m + \theta_f} q(\omega)\right)\right]\right). \tag{A.6}$$

Then i's preferences over efficient sharing rules can be represented by $U_i(\bar{c}; \theta_i) = g\big(\tilde{U}_i(\bar{c}; \theta_i), \theta_i\big) = K_i(\lambda, \theta_m, \theta_f) + h(\theta_m, \theta_f; q)$. Because $K_m(\lambda, \theta_m, \theta_f) + K_f(\lambda, \theta_m, \theta_f) = 0$, the sum of the transformed expected utilities U_i is $h(\theta_m, \theta_f; q)$ for all efficient $\bar{c}$, which proves that expected utility is transferable for $\sigma = \infty$. $\square$