Equilibrium Response of Flight Control Systems*  

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Key Words—Aerospace control; closed-loop systems; control system synthesis; control theory; dynamic response; multivariable control systems; state-space methods.  

Abstract—The steady-state response of a controlled system is of particular importance in non-zero set point regulation, and nowhere is this of more concern than in the piloting of aircraft. Possible equilibrium states are, to a large extent, independent of the aircraft’s stability and transient behavior; hence, stability and command objectives can be specified separately. Singular as well as non-singular steady-state response may be desirable from the pilot’s viewpoint, i.e. a constant command vector can imply varying state and control vectors. The flight control structure can be designed to account for this characteristic, while allowing full state or state-estimate feedback if desired. Dissimilar command vectors lead to markedly different responses in systems whose eigenvalues and eigenvectors are identical. A methodology for designing to achieve proper command response is presented, and numerical and graphical illustrations are provided for a generic lightweight fighter aircraft example.  

Introduction  

Satisfactory control of an aircraft requires not only an adequate margin of stability and tolerable transient response but suitable equilibrium response to the pilot’s command inputs. From recent literature, one might get the impression that solution of the eigenvalue/eigenvector placement problem is the key to satisfactory flying qualities; however, these parameters are independent of the aircraft’s control sensitivity, which has major effect on the command response. Furthermore, flying qualities criteria generally indicate that pilots are tolerant of wide variations in eigenvalue placement, as long as command response lies within satisfactory boundaries (U.S.A.F., 1980). All of this suggests that, while stability and ‘reasonable’ transient response should be maintained, commensurate attention should be given to the aircraft’s response to commands, particularly as it is shaped by the flight control system.  

This paper explores the effects of selecting alternate controller command vectors, i.e. sets of variables whose desired levels are to be produced by the controlled system. Three areas are covered: singular and non-singular equilibrium, control design for systems in which constant command implies varying state, and the significantly different responses of systems with identical eigenvalues and eigenvectors but dissimilar command vectors.  

Equilibrium control response  

Non-singular equilibrium. The perturbation motions of an aircraft can be modeled by the ordinary differential equation  

$$\Delta x = F \Delta x + G \Delta \delta + L \Delta e$$  

(1)  

where $\Delta x$ is an $n$-dimensional state vector, $\Delta \delta$ is an $m$-dimensional disturbance vector, $F$ is the $(n \times n)$ fundamental matrix, $G$ is the $(n \times m)$ control effect matrix, and $L$ is the $(n \times p)$ disturbance effect matrix. At equilibrium, the time-derivative of the state, $\Delta x$, is zero. If $F$ is non-singular (i.e. if its inverse exists), the equilibrium state, $\Delta x^*$, can be expressed as a unique combination of constant inputs, $\Delta \delta^*$ and $\Delta e^*$.  

$$\Delta x^* = -F^{-1}(G \Delta \delta^* + L \Delta e^*).$$  

(2)  

The pilot’s cockpit command inputs to the control stick, thumb switches, throttle quadrant, foot pedals, and so on are expressed as the $l$-dimensional command vector, $\Delta y$. It is assumed that the pilot’s commands can be interpreted further as some linear combination of the state and control  

$$\Delta y = H_0 \Delta x + H_2 \Delta \delta$$  

(3)  

where $H_0$ and $H_2$ are the $(l \times n)$ and $(l \times m)$ matrices which establish the relationship. In the unaugmented case with no control crossfeeds or interconnects, $l = p$, $H_0 = 0$, and $H_2$ is a diagonal matrix of control gearings or sensitivities. With control interconnects, $H_2$ is no longer diagonal. For a state command controller, $H_2$ contains an $(l \times l)$ identity matrix plus zeros in the appropriate locations, and $H_2 = 0$. $H_2$ and $H_2$ take on more general structures for direct command of state rates, $\Delta \delta$, purposely coupled variables such as ‘C’ (Tobie, Eliott and Malcom, 1966), or coordinate transformations.  

When state equilibrium is meant to imply command equilibrium as well, (2) must be satisfied subject to (3), i.e.  

$$\Delta y^* = H_0 \Delta x^* + H_2 \Delta \delta^*.$$  

(4)  

Since (2) and (4) must be satisfied simultaneously, they can be expressed as the single equation  

$$[F \quad G] \begin{bmatrix} \Delta x^* \\ \Delta \delta^* \end{bmatrix} = A \begin{bmatrix} \Delta x^* \\ \Delta \delta^* \end{bmatrix} + \begin{bmatrix} -L \Delta e^* \\ S \Delta \delta^* \end{bmatrix}.$$  

(5)  

If $A$ is square (requiring $l = m$) and non-singular, the equilibrium state and control can be expressed as direct functions of the constant disturbance and command  

$$\begin{bmatrix} \Delta x^* \\ \Delta \delta^* \end{bmatrix} = A^{-1} \begin{bmatrix} -L \Delta e^* \\ S \Delta \delta^* \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{12} \end{bmatrix} \Delta \delta^*.$$  

(6)  

where $S$ is partitioned in blocks that are conformable with the associated vectors  

$$S_{11} = \begin{bmatrix} S_{11} \\ S_{12} \end{bmatrix}.$$  

(7)  

Hence  

$$\Delta x^* = -S_{11} L \Delta e^* + S_{12} \Delta y^*.$$  

(8)  

$$\Delta \delta^* = -S_{21} L \Delta e^* + S_{22} \Delta y^*.$$  

(9)
As in Sandell (1971) the elements of $S$ can be found by elimination in (5):

$$S_{11} = F^{-1}(-GS_{n1} + D)$$  \hspace{1cm} (10)
$$S_{12} = -F^{-1}GS_{n2}$$  \hspace{1cm} (11)
$$S_{13} = -S_{32}H_{F}F^{-1}$$  \hspace{1cm} (12)
$$S_{22} = (-H_{F}F^{-1}G + H_{F})^{-1}$$  \hspace{1cm} (13)

If there are more controls than commands ($m > n$), $A$ is nonsquare, and the inverse must be replaced by the pseudoinverse (Davison, 1976; Ben-Israel and Greville, 1974) in (6) and (13). The opposite case ($n > m$) is well-posed, and the remainder of this paper uses the assumption that $l = m$.

Table 1 provides some examples of command-state-control equilibria for a longitudinal model of a generic lightweight fighter aircraft. The state vector contains body-axis axial and normal velocities, pitch rate, and pitch angle

$$\Delta \mathbf{x}^T = [\Delta u \Delta w \Delta q \Delta \theta]$$  \hspace{1cm} (14)

while the control vector includes elevator angle and throttle setting

$$\Delta \delta^T = [\Delta \delta E \Delta \delta T]$$  \hspace{1cm} (15)

The corresponding $F$ and $G$ represent level flight at an altitude of 6096 m (20000 ft) with a true airspeed of 213 m/s (700 fps) and a statically stable center-of-gravity (c.g.) location. In stability-and-control-derivative notation, with unsteady aerodynamic effects neglected

$$F = \begin{bmatrix} X_u & X_w & (X_u - w_0) & -g \cos \theta_0 \\ M_u & M_w & M_{u,w} & 0 \\ Z_u & Z_w & (Z_u - w_0) & -g \sin \theta_0 \\ -0.036 & 0.104 & 54.3 & -32.1 \\ -0.024 & -0.873 & 698.0 & -2.62 \\ 0 & 0.003 & -0.737 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$  \hspace{1cm} (16)

$$G = \begin{bmatrix} X_E & X_T & (X_E - w_0) & -g \cos \theta_0 \\ Z_E & Z_T & (Z_E - w_0) & -g \sin \theta_0 \\ M_E & M_T & M_{E,T} & 0 \\ -7.22 & 0 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (17)

The numerical values assume that velocities and angles are measured in fps and radians, respectively, while the throttle ranges from zero to one.

Various definitions are given to the two-element command vector, and the values of $\Delta \theta^*$ and $\Delta \delta^*$ corresponding to unit command inputs, with $\Delta E^* = 0$, are shown in the table. Each command vector implies a redefinition of $H_u$ and $H_w$. For example, $H_u = 0$ for the $(\Delta \delta E, \Delta \delta T)$ mode, and

$$H_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$  \hspace{1cm} (18)

For the $(\Delta \delta \theta, \Delta u)$ mode, $H_u = 0$, and

$$H_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (19)

while for the flight path angle-velocity $(\Delta \gamma, \Delta V)$ mode

$$H_u = \begin{bmatrix} \frac{w_0 V_0^2}{g} - u_0 V_0 \\ w_0 V_0 \end{bmatrix}$$

The mixed $(\Delta \theta, \Delta \delta T)$ mode assigns

$$H_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (21)

$$H_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (22)

In every case, $A$ is nonsingular, and equilibrium state and control variables corresponding to constant command inputs are easily found.

Not all of the command vectors shown in Table 1 are practical, but they illustrate several factors which must be considered in command selection. With a given command vector, equilibrium responses for the two elements are uncoupled. This is true whether the command vector contains states, controls, or a combination of the two. For example, the $\Delta \theta^*$ equilibrium response of the $(\Delta \theta, \Delta \delta T)$ mode is obtained without throttle perturbation, while the $\Delta \delta T^*$ equilibrium response occurs with no change in the equilibrium pitch angle. As a consequence, the definition of the second command variable has a significant effect on the equilibrium response to the first command variable, and vice versa. To see this, compare $\Delta \theta^*$ and $\Delta \delta^*$ for pitch angle commands in the three modes containing $\Delta \theta$. In the first case, $\Delta \theta$ is held constant, whereas $\Delta \gamma$ and $\Delta \delta T$ are held constant in the second and third cases; however, $\Delta \delta^*$ is constant in all cases.

The equilibrium pitch rate, $\Delta \dot{\theta}^*$, is zero in all of the examples, because a constant $\Delta \theta^*$ can be obtained in no other way. This leads to the question: how can steady pitch rate be commanded? The answer is found below. An interesting result is obtained when a 1-fps normal velocity is commanded with the $(\Delta \theta, \Delta u)$ mode. This amounts to a miniscule change in the angle of attack, $\Delta \theta^* = \Delta \dot{\theta}^*/V^*$, yet it can only be obtained in a steep dive with a sharp reduction in throttle setting. This occurs because an angle-of-attack perturbation normally implies a pitch rate perturbation, yet the latter is constrained to zero. The constraint can be satisfied, together with no change in the second command variable, $\Delta u^*$, if the lift, drag, and pitch-moment balances are altered substantially. A more "conventional" equilibrium response is linked to steady pitch rate response, for which $A$ becomes singular.

<table>
<thead>
<tr>
<th>Command</th>
<th>$\Delta \theta^*$</th>
<th>$\Delta u^*$</th>
<th>$\Delta \theta^*$</th>
<th>$\Delta u^*$</th>
<th>$\Delta \theta^*$</th>
<th>$\Delta u^*$</th>
<th>$\Delta \theta^*$</th>
<th>$\Delta u^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \theta$, $\Delta u$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Delta \theta$, $\Delta \delta T$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Delta \theta$, $\Delta \delta E$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Delta \delta T$, $\Delta \delta E$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Delta \delta T$, $\Delta \delta T$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Delta \delta T$, $\Delta \delta T$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
<td>-0.05</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 1. Equilibrium response for nonsingular command modes (stable aircraft)
Although the aircraft is nominally stable in this example, the definition of equilibrium is made without regard to stability, as represented by the eigenvalues of $\mu I - F$. If the modelled aircraft is shifted aft, the numerical value of $F$ changes, and the aircraft becomes statically unstable. Nevertheless, the equilibrium is expressed by (6). Forces and moments are balanced when this equation is satisfied, but the state will diverge from equilibrium unless the system's stability is augmented by the control system. Table 2 provides an abbreviated comparison with Table 1, assuming that the c.g. shift amounts to 20% of the aircraft's mean aerodynamic chord.

The differences in Table 1 and 2 are due principally to the elevator deflection that is required to trim against the aircraft's $\Delta w$-sensitive pitch moment, represented in Table 1 by $M_{e}$. The aft c.g. shift causes $M_{e}$ to change sign; therefore, $\Delta w$ changes sign in response to $\Delta x_{1}$, and $\Delta \delta_{e}$ changes sign in the remaining cases. Equilibrium throttle setting is unaffected by the shift, as might be expected.

Singular equilibrium. In some cases of interest, $F^{-1}$ exists but $A^{-1}$ does not exist. This implies that $S_{22}$ does not exist [equation (13)]; hence, the relationship between constant commands, $\Delta y$, and $\Delta \delta^{*}$ is undefined [equation (9)].

Consider the command vector, $\Delta y = (\Delta q, \Delta \alpha)$, for which

$$H_{y} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $H_{y} = 0$ in this case

$$S_{22} = (-H_{y}F^{-1}G)^{-1}.$$  \hfill (24)

Inverting (16) leads to a matrix of the form

$$F^{-1} \Delta H = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$ \hfill (25)

which, when combined with (17) and (23) in (24), produces a noninvertible matrix.

The singular $A$ is the result of specifying a command variable whose integral is an element of the state. Steady $\Delta q^{*}$ implies steadily growing $\Delta \delta^{*}$, and a solution to (6) cannot be found. Potential solutions to the problem can be identified by partitioning the dynamic equation (1) and the command vector equation (3). The term $\Delta x_{1}$ is defined to contain those state components which are constant at command equilibrium and $\Delta x_{2}$ as those components which are pure integrals of $\Delta x_{1}$ and $\Delta \delta$.

$$\Delta x_{1} = \begin{bmatrix} \Delta q^{*} \\ \Delta \delta^{*} \end{bmatrix} = \begin{bmatrix} F_{1} \\ F_{2} \end{bmatrix} \begin{bmatrix} \Delta x_{1}^{*} \\ \Delta x_{2}^{*} \end{bmatrix} = \begin{bmatrix} G_{1} \\ G_{2} \end{bmatrix} \Delta \delta$$ \hfill (26)

$$\Delta y = H_{y} \Delta x_{1} + H_{y} \Delta \delta.$$ \hfill (27)

Disturbance inputs are not considered, and $\Delta y$ does not contain $\Delta x_{1}$.

If $F_{2} = 0$, the problem is easily solved by neglecting $\Delta x_{2}$ to form a reduced-order equilibrium. From (6)

$$\Delta x_{2}^{*} = \begin{bmatrix} x_{2}^{*} \\ x_{3}^{*} \end{bmatrix} = \begin{bmatrix} F_{1} \\ F_{2} \end{bmatrix} \begin{bmatrix} x_{1}^{*} \\ x_{2}^{*} \end{bmatrix} = \begin{bmatrix} G_{1} \\ G_{2} \end{bmatrix} \Delta \delta$$ \hfill (28)

$\Delta x_{2}^{*}$ has no effect on the equilibrium values of $\Delta x_{1}^{*}$ and $\Delta \delta^{*}$, but it could be computed from (26) and (28) as

$$\Delta x_{1}^{*}(t) = \Delta x_{1}^{*}(0) + \int_{0}^{t} \left( F_{1} \Delta x_{1}^{*} + G_{1} \Delta \delta^{*} \right) \, dt$$

$$= \Delta x_{2}^{*}(0) + \int_{0}^{t} \left( F_{1} \Delta x_{2}^{*} + G_{1} \Delta \delta^{*} \right) \, dt$$ \hfill (29)

where the partitions of $\Delta \delta$ are defined as in (7). If $F_{1} = 0$, $\Delta x_{2}^{*}$ modifies $\Delta x_{1}^{*}$ and $\Delta \delta^{*}$; a steady equilibrium is no longer possible, but a quasi-steady equilibrium, in which forces and moments are balanced, can be obtained. Substituting in (26)

$$\Delta x_{2}^{*}(t) = \Delta x_{1}^{*}(0) - \int_{0}^{t} \left( F_{1} \Delta x_{1}^{*} + F_{2} \Delta x_{2}^{*} \right) \, dt$$

subject to (29). Differentiating (30) to obtain $\Delta x_{1}^{*}$, under the assumption that $\Delta y^{*}$ is constant, yields

$$\Delta x_{1}^{*}(t) = \Delta x_{1}^{*}(0) - \int_{0}^{t} \left( F_{1} \Delta x_{1}^{*} + F_{2} \Delta x_{2}^{*} \right) \, dt$$ \hfill (31)

$$\Delta x_{2}^{*}(t) = \Delta x_{2}^{*}(0) + \int_{0}^{t} \left( F_{1} \Delta x_{1}^{*} + F_{2} \Delta x_{2}^{*} \right) \, dt$$ \hfill (32)

For a wide class of problems, $\Delta x_{1}^{*}(t) = \Delta x_{2}^{*}(t)$ is constant when $\Delta y^{*}$ is constant, so the second- and higher-order derivatives of $\Delta x_{1}^{*}(t)$ are zero, and $\Delta x_{1}^{*} = \Delta x_{1}^{*}$ constant. (This requires $G_{2} = 0$ and $F_{1} = \Delta x_{1}^{*}$ constant.) When $H_{y}$ also equals 0, (10), (12), and (13) indicate that $\Delta x_{1}^{*} = 0$; the quasi-steady solution then is simply

$$\Delta x_{1}^{*}(t) = 0$$

$$\Delta x_{2}^{*}(t) = 0$$

with $\Delta x_{2}^{*}(t)$ defined by (29).

Examples of quasi-steady equilibrium responses are presented in Table 3 for $t = 0$, using (33) and assuming that $\Delta \delta(t) = 0$. Normal velocity $\Delta w$, pitch rate, or normal acceleration $\Delta \alpha$, in $g$ units, is the first command variable, and either axial velocity or throttle setting is the second. (Note that (33) is approximate when $\Delta \delta(t)$ is a command variable, as $H_{y} = 0$.) Unlike the nonsingular $(\Delta w, \Delta \alpha)$ mode, the singular $(\Delta w, \Delta \alpha)$ mode assumes that angle-of-attack variations $(\Delta \alpha W)$ are accompanied by pitch rate variations. Similarly, normal acceleration produces pitch rate in this model, so the $(\Delta \alpha, \Delta \alpha)$ mode is singular. The trends found in Table 1 are apparent in Table 3, where it also can be noted that the $\Delta w$, $\Delta q$, and $\Delta \alpha_{q}$ quasi-steady responses differ only by a scale factor.

The time evolution of quasi-steady equilibrium responses predicted by (33) for $(\Delta q, \Delta \alpha)$ and $(\Delta q, \Delta \delta T)$ modes is described by Table 4. For the first mode, a constant pitch-rate command of $1 \, \text{deg/sec}$ yields constant $\Delta w^{*}$, $\Delta \alpha^{*}$, and $\Delta \delta^{*}$.
**Table 3. Initial Quasi-Steady Equilibrium Response for Singular Command Modes (Stable Aircraft)**

<table>
<thead>
<tr>
<th>Command</th>
<th>$\Delta y_1^*$</th>
<th>$\Delta y_2^*$</th>
<th>$\Delta u^*$,</th>
<th>$\Delta a^*$,</th>
<th>$\Delta e^*$,</th>
<th>$\Delta \delta^*$,</th>
<th>$\Delta t^*$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variables</td>
<td>fps</td>
<td>fps</td>
<td>deg/sec</td>
<td>deg</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta_1$, $\delta_2$</td>
<td>1.</td>
<td>0.</td>
<td>15.95</td>
<td>1.</td>
<td>$-37$</td>
<td>$-3$</td>
<td></td>
</tr>
<tr>
<td>$\delta_1$, $\delta T$</td>
<td>1.</td>
<td>0.</td>
<td>12.9</td>
<td>13.6</td>
<td>1.</td>
<td>$-35$</td>
<td>0</td>
</tr>
<tr>
<td>$\delta_1$, $\delta u$</td>
<td>0.</td>
<td>1.</td>
<td>0.</td>
<td>1.</td>
<td>$-07$</td>
<td>$-01$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\delta_2$, $\delta e$</td>
<td>0.</td>
<td>0.</td>
<td>-16.9</td>
<td>-2.6</td>
<td>0.</td>
<td>$-39$</td>
<td>0</td>
</tr>
<tr>
<td>$\delta_3$, $\delta x$</td>
<td>0.</td>
<td>1.</td>
<td>1.</td>
<td>$-03$</td>
<td>0.</td>
<td>$-002$</td>
<td>$-01$</td>
</tr>
<tr>
<td>$\delta_4$, $\delta u$</td>
<td>0.</td>
<td>1.</td>
<td>1.</td>
<td>0.</td>
<td>$-002$</td>
<td>$-001$</td>
<td>$-002$</td>
</tr>
<tr>
<td>$\delta_5$, $\delta e$</td>
<td>0.</td>
<td>1.</td>
<td>1.</td>
<td>0.</td>
<td>$-002$</td>
<td>$-001$</td>
<td>$-002$</td>
</tr>
</tbody>
</table>

$\Delta E^*$, as well as steadily increasing $\Delta e^*$ and $\Delta T^*$. The throttle increase counters the effect of gravity on $\Delta u^*$ as the aircraft pitches up; for the second mode, $\Delta u^*$ decreases as a consequence of the fixed throttle, and $\Delta e^*$ (or $\Delta a^*$) increases through the action of $\Delta \delta^*$ to maintain the desired pitch rate.

**Closed-loop control laws**

*Control structure.* The previous section dealt with open-loop equilibrium response; this section treats the design and effects of closed-loop controllers for various nonsingular and singular command vectors. A linear nonzero-set-point regulator can be formulated as

$$\Delta \delta = -CA\delta$$

(34)

where $'$ denotes perturbation from the set point and

$$\Delta \delta \triangleq \Delta \delta - \Delta \delta^*$$

(35)

$$\Delta \delta \triangleq \Delta \delta - \Delta \delta^*$$

(36)

The feedback gain matrix, $C$, could be found using a variety of methods; in the present case, the controller is assumed to be a linear-quadratic regulator (e.g., Kwakernaak and Sivan, 1972), i.e., a controller which minimizes the quadratic cost function

$$J = \frac{1}{2}\int_0^\tau (\Delta \delta ^T Q \Delta \delta + 2 \Delta \delta ^T M \Delta \dot{\delta} + \Delta \delta ^T R \Delta \ddot{\delta}) \, dt$$

(37)

subject to the linear constraint

$$\Delta \delta = F \Delta \delta + G \Delta \delta$$

(38)

Substituting (35) and (36) into (34)

$$\Delta \delta = \Delta \delta^* - C(A\delta - \Delta \delta^*)$$

(39)

and the control goes to its equilibrium value as the state assumes its equilibrium value.

In the nonsingular case, $\Delta \delta^*$ and $\Delta \delta^*$ are defined by (8) and (9). Assuming that disturbances are not measured for control, the nonsingular command augmentation control law can be written as

$$\Delta \delta = S_{23}\Delta \delta^* - C(A\delta - S_{23}\Delta \delta^*)$$

(40)

with

$$\Delta \delta = C_P S_{23}\Delta \delta^* + C_B \Delta \delta$$

(41)

$$C_B = -C$$

(42)

The aircraft's closed-loop response to command is described by

$$\Delta \delta = (F + GC_B)\Delta \delta + GC_D\Delta \delta^*$$

(43)

The eigenvalues and eigenvectors of (43) are associated only with $(F + GC_B)$, thus the choice of command vector has no effect on these parameters.

If the command vector is singular, $\Delta \delta^*$ and $\Delta \delta^*$ must be defined by (30) and the integral of (32). The singular command augmentation control law is described by the integral of (32) plus the following:

$$\Delta \delta = (F + C_C)\Delta \delta^* + C_C \Delta \delta^* - C_D \Delta \delta$$

(44)

The partitions of $C$ ($C_1$ and $C_2$) are conformable with $\Delta \delta_1$ and $\Delta \delta_2$. With the assumptions preceding (33) and $\Delta \delta^*_2(0) = 0$, this reduces to a control law of the form

$$\Delta \delta(t) = C_F \Delta \delta^* + C_F \int_0^t \Delta \delta^* \, dt + C_B \Delta \delta$$

(45)

Numerical examples using this controller are presented in the following section.

**Numerical examples.** A series of command response time histories was computed with the feedback gain matrix

$$C_B = \begin{bmatrix}
-0.001 & -0.047 & -0.35 & -0.094 \\
3.3 \times 10^{-5} & -1.3 \times 10^{-5} & 2.9 \times 10^{-5} & 2.3 \times 10^{-5}
\end{bmatrix}$$

(46)

The following closed-loop short period short period (sp) and phugoid (p) natural frequencies ($\omega_n$, damping ratios ($\zeta$), and eigenvector magnitudes ($\eta$) resulted from this control

$$\omega_n = 3.83 \text{ rad s}^{-1}, \quad \zeta_n = 0.67$$

$$\omega_n = 0.056 \text{ rad s}^{-1}, \quad \zeta_n = 0.89$$

$$\eta_n = 1.1 \text{ fps}, \quad 14.6 \text{ fps}, \quad 3 \text{ deg s}^{-1}, \quad 1^\circ$$

$$\eta_n = 21.4 \text{ fps}, \quad 1.1 \text{ fps}, \quad 0.1 \text{ deg s}^{-1}, \quad 1^\circ.$$
The gains, which were found by minimizing a quadratic cost function [equation (37)], provide minimal $\Delta u$ feedback and $\Delta \theta T$ activity, their principal purpose being to increase the short-period natural frequency and both damping ratios above unaugmented values. In virtually every case, $C_a$ could be adjusted to improve response for the given command vector; however, $C_a$ is held constant to facilitate comparisons of command vector effects.

The pitch angle responses of nonsingular command modes are shown in Fig. 1. In each case, a $1^\circ$ $\Delta \theta$ has been commanded through one of three feed-forward gain matrices

$$(\Delta \theta, \Delta u) \text{ mode } C_F = \begin{bmatrix} -0.09 & 0.002 \\ 0.04 & 0.03 \end{bmatrix},$$

$$(\Delta \theta, \Delta \gamma) \text{ mode } C_F = \begin{bmatrix} -0.32 & 0.23 \\ -0.28 & 0.32 \end{bmatrix},$$

$$(\Delta \theta, \Delta \theta T) \text{ mode } C_F = \begin{bmatrix} -0.12 & 0.72 \\ -0 \ & 1.01 \end{bmatrix}.$$

It can be observed that the pitch angle response occurs on the phugoid time scale and that elevator motions are not large. The $(\Delta \theta, \Delta \gamma)$ mode requires the equilibrium pitch change to result without flight path angle change. Consequently, angle of attack increases through retrimming of the elevator, and velocity decreases to prevent the climb. Unlike the remaining cases, $\Delta \theta$ overshoots its final value during the short interval simulated. Faster pitch response could be achieved by altering the design cost function, which would change $C_a$ and alter closed-loop eigenvalues.

Singular pitch rate responses for a $1 \text{ deg s}^{-1}$ command are shown in Fig. 2. Here the commanded response occurs on the short-period time scale, with the following proportional and integral feedforward gains

$$(\Delta q, \Delta u) \text{ mode } C_F = \begin{bmatrix} -1.39 & 0.002 \\ -0.03 & 0.003 \end{bmatrix}, C_I = \begin{bmatrix} -0.09 & 0 \end{bmatrix}.$$

Note that the second column of $C_a$ are identical to those of the comparable pitch angle modes and that the integral gains are identical to the first columns of the pitch angle $C_a$. The singular modes effectively add a new variable $(\Delta u^*)$ to the nonsingular command vector $(\Delta \theta \, \Delta \gamma \, \Delta \theta T)$). Furthermore, the pitch rate response is 'Type 1' due to the inclusion of pitch angle in the command and the feedback.

The transient responses of both $\Delta q$ modes are virtually identical in the first two seconds, but the difference in throttle command has increasing effect as time passes. Although the $(\Delta q, \Delta u)$ mode's throttle effect is allowed to grow without limit in this simulation, the 700-fps/20,000 ft nominal flight condition requires a trim throttle setting of 85%. Therefore, the perturbation throttle command actually would saturate at 15% (about 5 s into the run), and velocity would begin to decay as in the $(\Delta q, \Delta \theta T)$ case. Because no lift due to elevator has been modeled [equation (37)], $\Delta w$ and $\Delta \eta$, responses differ only by a scale factor, $(\Delta \theta, \Delta w \, \Delta u)$ and $(\Delta \theta, \Delta \eta, \Delta u)$ mode command responses also are virtually identical to $(\Delta q, \Delta u)$ response. The $\Delta w$ and $\Delta \eta$, responses are well damped, but there is significant overshoot in $\Delta q$. Although not shown here, this overshoot can most effectively be reduced by incorporating $\Delta \theta$ weighting in the cost function [equation (37)].

Figure 3 provides a brief look at closed-loop $\Delta q$ response of the statically unstable mode discussed earlier. Cost function weights are identical to those used for the stable
command vector has a dominant effect on aircraft response. For a fixed degree of stability, different command vectors cause markedly different response. If the state vector contains a pure integral of a command vector element, then constant command implies varying state, and the command response is singular. Designing command augmentation systems which account for this singularity leads to control structures with proportional-integral pre-filtering and conventional (e.g., linear-quadratic) feedback. The singular-command control theory is readily applied to unstable as well as stable aircraft. It is independent of the method for computing feedback gains; hence, it can be used in both 'classical' and 'modern' control system design.

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References