Flight Control Design Using Non-linear Inverse Dynamics*

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Flight control systems based upon non-linear inverse dynamics offer the potential of providing improved levels of safety and performance over conventional designs developed using linearizing assumptions.

Key Words—Non-linear systems; control system design; decoupling; inverse systems; aerospace control.

Abstract—Aircraft in extreme flight conditions can encounter severe non-linear effects generated from high angles of attack and high angular rates. Flight control systems based upon non-linear inverse dynamics offer the potential for providing improved levels of safety and performance in these flight conditions over the competing designs developed using linearizing assumptions. Inverse dynamics are generated for specific command variable sets of a 12-state non-linear aircraft model to develop a control system which is valid over the entire flight envelope. Detailed descriptions of the inertial dynamic and aerodynamic models are given, and it is shown how the command variable sets are altered as a function of the system state to add stall prevention features to the system. Simulation results are presented for various mission objectives over a range of flight conditions to confirm the effectiveness of the design.

INTRODUCTION

The problem of stall/spin accidents is a particularly severe one for general aviation (GA) aircraft. In recent years more than a tenth of all single-engine light plane accidents, and nearly a third of all fatal light plane accidents have been related to stall (Stengel and Nixon, 1982). Conventional flight control designs assume the aircraft dynamics are linear and time invariant about some nominal flight condition. They feature stability and command augmentation systems to meet handling qualities criteria, with gains scheduled as functions of the nominal flight condition. In extreme flight conditions the performance of these systems starts to deteriorate due to the unmodelled effects of strong non-linearities inherent in the flight dynamics, which only become significant at high angles of attack or high angular rates.

Control laws that are based on the non-linear inverse dynamics (NID) of the aircraft offer the potential for providing improved levels of performance over conventional flight control designs in these extreme flight conditions. This is due to the NID controller's more accurate representation of the forces and moments that arise in response to large state and control perturbations. These control laws also allow specific state variables to be commanded directly. This simplifies the pilot's task of capturing desired flight trajectories, and it is useful in adding stall prevention features to the system.

The control of non-linear systems through the use of their inverse dynamics is a topic that has received a great deal of attention in recent years (Falb and Wolovich, 1967; Singh and Rugh, 1972; Freund, 1973, 1975; Asseo, 1973; Singh and Schy, 1979, 1980; Meyer and Cicolani, 1981; Meyer et al. 1984; Menon et al. 1985). Asseo (1973) showed how a simplified aircraft model could be decoupled to provide flight path angle and heading angle commands to the pilot. Singh and Schy (1979, 1980) applied this same theory to maneuvering aircraft and demonstrated how the roll coupling divergences associated with rapid open-loop maneuvers could be eliminated from the closed-loop response. In a slightly different approach, Meyer and Cicolani (1981) and Meyer et al. (1984) construct the inverse dynamics of a VSTOL aircraft by using linearizing transformations. A regulator is then designed for the transformed system, forcing it to track the output of a reference model. Menon et al. (1985) simplify the calculations of the linearizing transformations by using singular perturbation theory. Assuming that there is a sufficient time-scale separation between the fast and slow modes in the system, they have shown

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that the steady-state values of the fast modes can be used as control inputs to the slow modes.

This paper is part of a continuing effort of analytical and experimental studies at Princeton (Frutter, 1982; Sri-Jayantha, 1983a,b; Ehrenstrom, 1983; Silhouette, 1986) investigating the control of aircraft at high angles of attack. It extends the above works by developing a stall prevention flight control system that is valid over the entire flight envelope for both highly maneuverable and GA aircraft. Through the use of non-linear inverse dynamics, this controller decouples specific state variables that are of particular interest to the pilot. The command variables are organized in sets that can be varied as functions of the flight phase, to provide the pilot with a maximum of control over the aircraft with a minimum of effort. The controller incorporates the full non-linear inertial dynamics and aerodynamics into its design using an aerodynamic model based upon the Navion GA aircraft, with elevator, aileron, rudder and throttle deflections as the control inputs.

GENERAL THEORY OF NON-LINEAR INVERSE DYNAMICS

The objective of this section is to review the techniques that can be applied to develop a flight control system that is valid over the entire flight envelope. These techniques are based on the construction of inverse dynamics as presented in Singh and Rugh (1972) and Freund (1973) for systems of the form,

\[ \dot{x} = A(x) + B(x)u \]  
\[ y = C(x) \]  
(1)

where \( A(x) = (n \times 1) \) vector, \( B(x) = (n \times m) \) matrix, \( C(x) = (m \times 1) \) vector. Since the aircraft dynamics take the general non-linear form,

\[ \dot{x}' = f(x', u') \]  
\[ y = Cx' \]  
(3)

where \( x' = (n \times 1) \) state vector, \( u' = (m \times 1) \) control vector, \( y = (l \times 1) \) output vector, and \( C = (l \times n) \) constant matrix, a transformation into the linear analytic form of (1) and (2) is required. This can be accomplished by augmenting the system dynamics with derivatives of appropriate control inputs.

The inverse dynamics of (1) and (2) are constructed by differentiating the individual elements of \( y \) a sufficient number of times until a term containing a \( u \) appears. Since only \( m \) outputs can be controlled independently with \( m \) inputs, it will be assumed that \( \text{dim}(y) = \text{dim}(u) = m \). Introducing the \( k \)-th order differentiation operator, \( L^k_A(\cdot) \), such that,

\[ L^k_A(x) = \left[ \frac{\partial}{\partial x} L^{k-1}_A(x) \right] A(x) \]  
(5)

simplifies the subsequent development. Using this notation to differentiate the \( i \)-th component of \( y \) yields,

\[ \ddot{y}_i = C_i \ddot{x} = C_i L^1_A(x) \]  
\[ + C_i \left[ \frac{\partial}{\partial x} L^0_A(x) \right] A(x) \]  
(7)

\[ \dot{y}_i = C_i \dot{x} = C_i L^2_A(x) \]  
\[ + C_i \left[ \frac{\partial}{\partial x} L^1_A(x) \right] B(x)u = C_i L^3_A(x) \]  
(8)

\[ y_i^{cd} = C_i x^{cd} = C_i L^d_A(x) \]  
\[ + C_i \left[ \frac{\partial}{\partial x} L^{d-1}_A(x) \right] A(x) \]  
\[ + C_i \left[ \frac{\partial}{\partial x} L^{d-1}_A(x) \right] B(x)u \]  
(9)

where \( d_i \) is the order of the derivative of \( y_i \) necessary to ensure that,

\[ \frac{\partial}{\partial x} L^{d_i-1}_A(x) B(x) \neq 0. \]  
(10)

After differentiating the \( m \) elements of \( y \), each the appropriate number of times, the output dynamics can be represented as,

\[ y^{cd} = \begin{bmatrix} y_1^{cd} \\ y_2^{cd} \\ \vdots \\ y_m^{cd} \end{bmatrix} = \begin{bmatrix} C_1 L_{d_1}^1(x) \\ C_2 L_{d_2}^2(x) \\ \vdots \\ C_m L_{d_m}^m(x) \end{bmatrix} \]  
(11)

Using the notation of Singh and Rugh (1972) and Freund (1973) let,

\[ A^*_i(x) = C_i [L_{d_i}^i(x)] \]  
(12)

\[ B^*_i(x) = C_i \left[ \frac{\partial}{\partial x} L^{d_i-1}_A(x) \right] B(x). \]  
(13)

This allows (11) to be written in more compact notation as,

\[ y^{cd*} = A^*(x) + B^*(x)u. \]  
(14)

A sufficient condition for the existence of an inverse system model to (1) and (2) is that \( B^* \) in (14) be non-singular (Singh and Rugh, 1972;
Freund, 1973). If this is the case, then the inverse system model takes the form,
\[ \dot{x} = [A(x) - B(x)F(x)] + B(x)G(x)v \]  \hspace{1cm} (15)
\[ u = -F(x) + G(x)v \]  \hspace{1cm} (16)
where \( v = y^{cd}\) is the input to the inverse system, \( u \) is its output and
\[ G = [B^*(x)]^{-1} \]  \hspace{1cm} (17)
\[ F = [B^*(x)]^{-1}A^*(x) \]  \hspace{1cm} (18)

Applying the NID control law,
\[ u = -F(x) + G(x)v \]  \hspace{1cm} (19)
to the original system of (1) and (2) leaves it in the integrator-decoupled form,
\[ y^{cd} = v. \]  \hspace{1cm} (20)

Setting,
\[ v = -\sum_{k=0}^{d-1} P_k y^{ck} + P_0 w \]  \hspace{1cm} (21)
with \( y^{ck} \) the \( k \)th derivative of the output vector \( y \), and the \( P_k \) chosen as \( (m \times m) \) constant diagonal matrices, gives the original system the decoupled linear, time-invariant dynamics,
\[ y^{cd} + P_{d-1} y^{cd-1} + \cdots + P_0 y = P_0 w \]  \hspace{1cm} (22)
where \( w \) is the new external control input. Figure 1 illustrates the NID approach to control law development.

The maximum number of poles that can be placed with a NID control law is highly dependent upon the choice of the elements in the output vector \( y \). For cases where \( \sum_{i=1}^{m} d_i = n \), all the system poles can be placed, and closed-loop stability can be guaranteed if closed-loop observability can be proven (Freund, 1975). For cases where \( \sum_{i=1}^{m} d_i < n \), closed-loop stability can be guaranteed only locally by showing that the modes made unobservable by the NID control law have stable dynamics over the regions of interest in the state space.

**MATHEMATICAL MODEL OF THE AIRCRAFT**

Now that the methods of construction of NID control laws for general non-linear systems of the form of (1) and (2) are clear, the next step is to apply these techniques to the equations of motion of the aircraft. Choosing a hybrid coordinate system consisting of combined wind and body axes, the equations of motion of an aircraft take the form (Etkin, 1972),

\[ \dot{V} = -\frac{D}{m} - g \sin \gamma \]  \hspace{1cm} (23)
\[ \dot{\phi} = q_w \cos \varphi - r_w \sin \varphi \]  \hspace{1cm} (24)
\[ \dot{\psi} = p_w \sin \varphi + r_w \cos \varphi \tan \gamma \]  \hspace{1cm} (25)
\[ \dot{\psi} = (q_w \sin \varphi + r_w \cos \varphi) \sec \gamma \]  \hspace{1cm} (26)
\[ \dot{\alpha} = q - q_w \sec \beta - (p \cos \alpha + r \sin \alpha) \tan \beta \]  \hspace{1cm} (27)
\[ \dot{\beta} = r_w + p \sin \alpha - r \cos \alpha \]  \hspace{1cm} (28)
\[ \dot{\dot{q}} = \frac{1}{l_{yy}} [M + I_{zz}(r^2 - p^2) + (I_{zz} - I_{xx})r p] \]  \hspace{1cm} (29)
\[ \dot{\dot{\beta}} = \frac{1}{l_{xx}} [I_{xx} \dot{p} + I_{zz} \dot{q} + (I_{yy} - I_{zz})qr] \]  \hspace{1cm} (30)
\[ \dot{\dot{\alpha}} = K \]  \hspace{1cm} (31)

![Fig. 1. Non-linear inverse dynamics control system.](image-url)
where
\[
q_w = \frac{1}{mV} (L - mg \cos \gamma \cos \varphi) \\
r_w = \frac{1}{mV} (-S + mg \cos \gamma \sin \varphi) \\
p_w = p \cos \alpha \cos \beta + (q - \dot{\alpha}) \sin \beta + r \sin \alpha \cos \beta
\] (32)

and \(q_w, r_w, p_w\) are the wind-axis angular rates; \(V\) is the flight path velocity; \(\varphi, \gamma, \psi\) are the wind-axis Euler angles; \(p, q, r\) are the body-axis angular rates; \(D, S, L\) are the drag, side and lift forces; \(\mathcal{L}, \mathcal{M}, \mathcal{N}\) are the rolling, pitching and yawing moments; \(\alpha = \) angle of attack; and \(\beta = \) sideslip angle. A thrust coefficient \((C_T)\) is also used in the calculation of the aerodynamic forces \((D, S, L)\) and aerodynamic moments \((\mathcal{L}, \mathcal{M}, \mathcal{N})\). It is found from the thrust model,
\[
C_T = \frac{\eta P_{\text{max}}}{\dot{q}SV} \delta T
\] (35)

where \(\delta T\) is the throttle setting, \(\eta\) is the propeller efficiency, \(\dot{q}\) is the dynamic pressure, \(S\) is the wing planform area and \(P_{\text{max}}\) is the maximum power of the engine.

The non-dimensional aerodynamic moment coefficients \((C_\mathcal{L}, C_\mathcal{M}, C_\mathcal{N})\) used in the calculation of \(\mathcal{L}, \mathcal{M}\) and \(\mathcal{N}\) are assumed to be non-linear functions of \(\alpha, \beta\) and \(C_T\), and linear functions of the elevator (\(\delta E\)), aileron (\(\delta A\)) and rudder (\(\delta R\)) deflections. These non-dimensional coefficients take the general functional form,
\[
\mu = \mu_0(\alpha, \beta, C_T) + \mu_1(\alpha, \beta, C_T) \delta E \\
+ \mu_2(\alpha, \beta, C_T) \delta A + \mu_3(\alpha, \beta, C_T) \delta R
\] (36)

where the \(\mu_i\) \((i = 0, 1, 2, 3)\) are obtained by interpolating actual aerodynamic data using functions of the form,
\[
\mu_i(\alpha, \beta, C_T) = \mu_{i0} + \mu_{i1} \alpha + \mu_{i2} \beta + \mu_{i3} C_T + \mu_{i4} \alpha \beta \\
+ \mu_{i5} C_T + \mu_{i6} \alpha C_T + \mu_{i7} \alpha \beta C_T
\] (37)

The \(\mu_i\) \((i = 0, 1, 2, \ldots, 7)\) are unique constants associated with each subspace of aerodynamic data, and they are derived by compressing linear interpolation operations into minimal realizations. The data subspaces are cubes defined by \(\alpha, \beta\) and \(C_T\) on the intervals,
\[
\begin{bmatrix}
\alpha_i & \leq & \alpha & \leq & \alpha_{i+1} \\
\beta_i & \leq & \beta & \leq & \beta_{i+1} \\
C_{T_i} & \leq & C_T & \leq & C_{T_{i+1}}
\end{bmatrix}
\] (38)

where the subscripted values of \(\alpha, \beta\) and \(C_T\) define the cube boundaries shown in Fig. 2. The non-dimensional moment coefficients interpolated in this way are \(C_{\mathcal{L}_0}, C_{\mathcal{M}_0}, C_{\mathcal{N}_0}\) and \(C_{\mathcal{L}_{i+1}}, C_{\mathcal{M}_{i+1}}, C_{\mathcal{N}_{i+1}}\).

The non-dimensional aerodynamic force coefficients \((C_D, C_S, C_L)\) used in the calculation of \(D, S\) and \(L\) take the general functional form,
\[
\lambda = \lambda_0(\alpha, \beta) + \lambda_1(\alpha, \beta) C_T + \lambda_2(\alpha, \beta) \delta E \\
+ \lambda_3(\alpha, \beta) \delta A + \lambda_4(\alpha, \beta) \delta R + \lambda_5(\alpha, \beta) \delta EC_T \\
+ \lambda_6(\alpha, \beta) \delta AC_T + \lambda_7(\alpha, \beta) \delta RC_T
\] (39)

where the \(\lambda_i\) \((i = 0, 1, 2, \ldots, 7)\) are obtained from the compressed bi-cubic spline representations,
\[
\lambda_i(\alpha, \beta) = \lambda_{i0} + \lambda_{i1} + \lambda_{i2} \beta + \lambda_{i3} \alpha \beta + \lambda_{i4} \alpha^3 \\
+ \lambda_{i5} \beta^3 + \lambda_{i6} \alpha^2 \beta + \lambda_{i7} \alpha \beta^3
\] (40)

of appropriate aerodynamic data on each data subspace of (38). The bi-cubic form is required to ensure that the derivatives of the aerodynamic force coefficients used in the NID control laws are continuous across the subspace boundaries. The aerodynamic force coefficients interpolated in this way are: \(C_{D_0}, C_{S_0}, C_{L_0}; C_{D_1}, C_{S_1}, C_{L_1}; C_{D_2}, C_{S_2}, C_{L_2}; C_{D_3}, C_{S_3}, C_{L_3};\) and \(C_{D_{i+1}}, C_{S_{i+1}}, C_{L_{i+1}}\).

The aircraft equations of motion can be put in the triangular form,
\[
\dot{x}_1 = A_1(x_1, x_2, x_3) \\
\dot{x}_2 = A_2(x_1, x_2, x_3, x_4) + B_2(x_1, x_2, x_3, u_1) u_2 \\
\dot{x}_3 = A_3(x_1, x_2, x_3, u_1, u_2) + B_3(x_1, x_2, x_3, u_1, u_2) u_3
\] (41, 42, 43)

where
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix},
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
V \\
\psi \\
\varphi
\end{bmatrix},
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} =
\begin{bmatrix}
p \\
r \\
\delta E
\end{bmatrix}
\] (44, 45, 46)

and
\[
u_1 = [\delta T], \quad u_2 = [0], \quad u_3 = [\delta A, \delta R]
\] (47, 48, 49)

These definitions allow stable inverse dynamics...
to be constructed for the NID control laws. This triangular model of the aircraft dynamics neglects the derivatives of the aerodynamic forces (but not the aerodynamic moments) with respect to the control surface deflections ($\delta E$, $\delta A$, $\delta R$) and the body angular rates ($p$, $q$, $r$). Although the system is controllable through these force effects, they are small for most aircraft configurations and are not primary paths of aerodynamic control. The principal function of the control surface deflections is to impart aerodynamic moments about the various body axes. It can be shown that if NID control laws are derived exploiting these weak force effects, either unrealistically large control deflections will be required for small state perturbations, or the system will be destabilized by cancelling the non-linear equivalent of non-minimum phase transmission zeros with unstable poles. The neglected force effects will however be included in the numerical calculations of $D$, $S$, and $L$ used by the real-time control laws, and they can be considered as disturbances ($d_i$) in the block diagram representation of the triangular system in Fig. 3. The control input $u_2$ in the general triangular model of (41)–(43) is not applicable in this study.

Engine dynamics up to second order of the form,

$$\delta T = k_1 \delta \dot{T} + k_2 \delta T + k_3 \delta T_{\text{com}}$$

also can be accommodated in the triangular model by making $\delta T$ and its derivative ($\delta \dot{T}$) states of the system. $\delta T_{\text{com}}$ represents the new commanded throttle setting, while $k_1$–$k_3$ are engine parameters that can vary as a function of the flight condition.

APPLICATION OF NON-LINEAR INVERSE DYNAMICS TO FLIGHT CONTROL

The first step in the design of the flight control system is to decide upon the command variable sets. These sets can be selected as functions of the flight phase; examples include:

$$y_{\text{com}_1} = \begin{bmatrix} V \\ \gamma \\ q \\ \varphi \\ \beta \end{bmatrix}, \quad y_{\text{com}_2} = \begin{bmatrix} V \\ \gamma \\ p \\ \beta \end{bmatrix}$$

for aircraft with four independent control inputs. Body-axis roll rate ($p$) is also a candidate command variable for $y_{\text{com}_1}$ and $y_{\text{com}_2}$ replacing $\varphi$. For purposes of simulation, however, it was found more convenient to work with the wind-axis roll angle ($\varphi$). $y_{\text{com}_1}$ and $y_{\text{com}_2}$ are intended primarily for the take-off, landing and cruising phases of flight. $y_{\text{com}_2}$ has the option of commanding either $\psi$ or $\psi_0$, depending upon the mission objectives. The addition of $y_{\text{com}_2}$ provides the pilot with a maneuver mode. This allows direct control over attitude, simplifying the performance of tracking tasks and landing flare maneuvers. $y_{\text{com}_1}$ is activated by the stall prevention logic, and supersedes all other command variable sets in the event of stall.

The NID control laws are derived by differentiating $V$, $\gamma$ and $\psi$ three times; $\alpha$, $\beta$ and $\varphi$ twice; and $p$ and $q$ once, using the aircraft dynamics of (23)–(34) in the triangular form of (41)–(43). Using the derivatives of the various command variables in Appendix A, $A^*_x$ and $B^*$ can be assembled for each of the command variable sets. This allows the $F$ and $G$ of the NID control law of (19) to be computed. Elements of $P_x$ that give each command variable the approximate response of a single-input–single-output (SISO) linear system must also be chosen with the speed of response of the original system in mind. Arbitrary pole placement can lead to a high sensitivity to plant parameter uncertainty and an increased tendency for control saturation. Table 1 contains the diagonal elements of the $P_x$ matrices that correspond to systems with time constants of from 1–2 s. Given the dynamics of the Navion aircraft, these choices of time constants are reasonable.

Because $B^*$ must be inverted to form the NID control law (19), any flight conditions that cause $B^*$ to be singular must be avoided. $B^*$ is a function of the command set definition, so different command vectors have different points of singularity. As noted in Appendix B, the
**Table 1. Diagonal Elements of Linear Dynamics Parameters (Pₖ)**

<table>
<thead>
<tr>
<th>Command set</th>
<th>Diagonal elements of</th>
<th>Command set</th>
<th>Diagonal elements of</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P₂</td>
<td>P₁</td>
<td>P₀</td>
</tr>
<tr>
<td>y_com₁</td>
<td>-7</td>
<td>-15</td>
<td>-9</td>
</tr>
<tr>
<td></td>
<td>-7</td>
<td>-15</td>
<td>-9</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-2.5</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>-7</td>
<td>-15</td>
<td>-9</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-2.5</td>
<td>-1</td>
</tr>
<tr>
<td>y_com₂</td>
<td>0</td>
<td>-5</td>
<td>-4</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-5</td>
<td>-4</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-2.5</td>
<td>-1</td>
</tr>
</tbody>
</table>

Singularities occur outside the normal flight envelope of the aircraft studied.

Command variable independence also can be maintained in the event of control saturation. This is accomplished by dropping appropriate variables from the command set, and incorporating the control saturation effects into the calculation of the inverse dynamics of the remaining command variables. Table 2 shows the correspondence between control saturation and command variable elimination.

To prevent the pilot from commanding trim conditions beyond the edges of the normal flight envelope, these edges have been implemented as hard constraints in the various command sets. Based upon the Navion aircraft model, Fig. 4 shows the maximum flight path angle command (y_com) for a given velocity command (V_com), while Fig. 5 shows the maximum bank angle command (φ_com) given V_com and y_com. For each value of V in Fig. 5 the left φ boundary is based upon a zero throttle setting, the right φ boundary a full throttle setting, and the upper boundary is computed assuming α = α_max.

Given V_com and y_com, the maximum ψ command can be found from,

$$
\dot{\psi}_{\text{max}} = \frac{g}{V_{\text{com}}} \cos y_{\text{com}} \tan \varphi_{\text{max}}
$$

where

$$
\varphi_{\text{max}} = \tan^{-1} \left[ \sqrt{\left( \frac{q S C_{L_{\text{max}}}}{mg \cos y_{\text{com}}} \right)^2 - 1} \right]
$$

**Table 2. Correspondence Between Control Saturation and Command Variable Elimination**

<table>
<thead>
<tr>
<th>Control saturated</th>
<th>Command variable affected</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Command set 1 2 3 4</td>
</tr>
<tr>
<td>δT</td>
<td>V V V γ</td>
</tr>
<tr>
<td>δE</td>
<td>γ q γ α</td>
</tr>
<tr>
<td>δA</td>
<td>φ φ γ φ</td>
</tr>
<tr>
<td>ΔR</td>
<td>β β β β</td>
</tr>
</tbody>
</table>

Fig. 4. Maximum steady-state flight path angle versus velocity.

Fig. 5. Maximum steady-state bank angle flight envelope.

and C_{L\text{max}} is the trim value of the lift coefficient (C_l) at the maximum angle of attack (α_{max}). These constraints help to keep the pilot from stalling the plane in the steady state. y_com is used in conjunction with the activation logic of Fig. 6.
to prevent stalls during transient motions. The angle-of-attack limiter becomes active whenever the system enters the region designated as “Active Stall Prevention” in the \((\alpha, \dot{\alpha})\) phase plane. Once activated, \(\alpha\) is driven towards \(\alpha_{\text{max}}\) by \(\gamma_{\text{com}}\), until the angle-of-attack acceleration, \(\ddot{\alpha}\), associated with the pre-empted command variable set is less than or equal to the angle-of-attack acceleration commanded by the angle-of-attack limiter. At this point the stall prevention system is disengaged, allowing the aircraft to be flown with the originally selected command variable set. Defining \(\alpha_{\text{stall}}\) as the angle of attack where the lift curve slope equals zero, \(\alpha_{\text{max}}\) must be chosen just below \(\alpha_{\text{stall}}\) to ensure that if the throttle saturates, the singularity conditions of \(\gamma_{\text{com}}\), and \(\gamma_{\text{com}}\), in Appendix B will not be encountered.

Simulations were conducted to evaluate the performance of the NID system by flying the trajectory shown in Fig. 7. The maneuvers were chosen to highlight the capabilities of the command variables, to test the control saturation logic, and to probe the integrity of the stall prevention system under transient and steady-state flight conditions. The trajectories of Fig. 7 can be divided roughly into five main segments. Segment 1 consists of an ascent at the maximum rate of climb using \(\gamma_{\text{com}}\). The stall prevention system limits \(\gamma_{\text{com}}\) to be within the steady-state flight envelope during this maneuver. Segment 2 is a constant-altitude coordinated turn performed using \(\psi\) as a command variable of \(\gamma_{\text{com}}\). Segment 3 is a climbing banked turn using \(\varphi\) in \(\gamma_{\text{com}}\) as the relevant command variable. Segment 4 shows that sideslips can be executed with either constant bank or heading angles. This depends on whether \(\gamma_{\text{com}}\) or \(\gamma_{\text{com}}\) is chosen as the command variable set. In the final segment, \(\gamma_{\text{com}}\) is used to perform a descending banked turn while simultaneously reducing the velocity along the flight path. The complexity of the maneuver in Segment 5 tests both the stall prevention system and the saturation logic. Table 3 contains a detailed summary of the simulation broken down into 10-s maneuver blocks. The simulation results of this study are shown in Figs 8–12. Figure 8 shows the
### Table 3. Simulation command variable summary

<table>
<thead>
<tr>
<th>Trajectory segment</th>
<th>Maneuver block</th>
<th>Time (s)</th>
<th>Command set</th>
<th>$V$ (ft/s$^{-1}$)</th>
<th>$\gamma$ (deg)</th>
<th>$\phi$ (deg)</th>
<th>$\dot{\psi}$ (deg/s)</th>
<th>$\psi$ (deg)</th>
<th>$\beta$ (deg)</th>
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<td>0–10</td>
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<td>$110 \rightarrow 125$</td>
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<td>0</td>
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<td>2</td>
<td>2</td>
<td>10–20</td>
<td>3</td>
<td>$125\rightarrow 145$</td>
<td>10–0</td>
<td>10–0</td>
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<td>30–40</td>
<td>3</td>
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<td>3</td>
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<td>-5</td>
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### Figures

**Figure 8.** Command variables ($V$, $\gamma$, $\phi$, $\psi$, $\beta$) versus time.
Fig. 9. Angular rates \((p, q, r)\) versus time.

Fig. 10. Control deflections \((\delta T, \delta E, \delta A, \delta R)\) versus time.

Fig. 11. Angle of attack versus time.
command variable inputs and the corresponding time histories, while Figs 9 and 10 show the resulting angular rates and control deflections.

The stall prevention results can be most dramatically seen in Figs 11 and 12. Figure 11 shows angle of attack (α) being limited to \( \alpha_{\text{max}} = 17^\circ \) in maneuver 13, while Fig. 12 shows the corresponding phase plane trajectory. Immediately after issuing the \( \psi_{\text{com}} = 0 \) at the start of maneuver 14, the stall prevention system disengages and angle of attack is allowed to again follow an unconstrained course.

**CONCLUSIONS**

As can be seen from the simulated results, the flight control system designed using NID allows the aircraft to be flown by the pilot with command variable inputs that reflect the goal of the particular flight phase, and which are valid over the entire flight envelope. In addition, the command variables have low order dynamics, and time constants which can be selected as functions of the flight condition. The system maintains command variable independence even in the event of control saturation, and it prevents the pilot from entering stalls through steady-state flight envelope boundary constraints and angle-of-attack limiting.

These results indicate that the issues of handling qualities, tracking capabilities, and operational safety at high angles of attack can be addressed directly using this type of flight control system. This will lead to improved levels of performance over conventional flight controller designs developed using linearizing approximations.

Current research activities include a discrete-time formulation of the NID control laws (Lane and Stengel, 1987), their digital implementation in a multi-microprocessor environment and testing using a fixed-base flight simulator. Future work will concentrate on updating the non-linear interpolation operators defined over each aerodynamic data subspace by using parameter identification techniques. This will produce an adaptive flight control system that continually improves with time as better aerodynamic estimates are used in the construction of the aircraft’s inverse dynamics.

**REFERENCES**


\[ Q_1 = [(L_u - m g \dot{w}_u), mg \sin \gamma \cos \varphi, L_x, L_y, L_z, C_{\delta \beta}, mg \cos \gamma \sin \varphi] \]

\[ Q_2 = \begin{bmatrix} L_{uv} & 0 & L_{\alpha \alpha} & L_{\alpha \beta} & L_{uv \alpha} & 0 \\ 0 & -mg \cos \gamma \cos \varphi & 0 & 0 & 0 & -mg \sin \gamma \sin \varphi \\ L_{uv} & 0 & 0 & L_{\alpha \beta} & L_{uv \beta} & 0 \\ L_{uv} & 0 & L_{\alpha \alpha} & 0 & L_{uv \gamma} & 0 \\ L_{C_{\delta \beta}} & 0 & L_{C_{\delta \beta}} & 0 & L_{C_{\delta \beta}} & 0 \\ 0 & 0 & 0 & 0 & 0 & mg \cos \gamma \cos \varphi \end{bmatrix} \]

\[ R_1 = [(S_w + m r_w), mg \sin \gamma \sin \varphi, S_x, S_y, S_C_{\delta \beta}, mg \cos \gamma \cos \varphi] \]

\[ R_2 = \begin{bmatrix} S_{uv} & 0 & S_{u\alpha} & S_{u\beta} & S_{uv \alpha} & 0 \\ 0 & mg \cos \gamma \sin \varphi & 0 & 0 & 0 & mg \sin \gamma \cos \varphi \\ S_{uv} & 0 & 0 & S_{u\beta} & S_{uv \gamma} & 0 \\ S_{uv} & 0 & S_{u\alpha} & 0 & S_{uv \gamma} & 0 \\ S_{C_{\delta \beta}} & 0 & S_{C_{\delta \beta}} & 0 & S_{C_{\delta \beta}} & 0 \\ 0 & 0 & 0 & 0 & 0 & mg \cos \gamma \cos \varphi \end{bmatrix} \]

Non-linear inverse dynamics flight control systems

APPENDIX A. COMMAND VARIABLE DERIVATIVES

The derivatives of appropriate order of the command variables \( V, \gamma, q, \alpha, \beta, \varphi \) and \( \psi \) are presented assuming the non-dimensional aerodynamic coefficients take the form of the interpolating functions in (36) and (37) and (39) and (40). In order to simplify the expressions for the derivatives of the aerodynamic forces \( D, L \) and \( S \), the following notation is introduced. Let,

\[ z^T = [V \gamma \alpha \beta C_{\delta \beta} q] \]

\[ S_1 = [D_v, mg \cos \gamma, D_x, D_y, D_z, D_{C_{\delta \beta}}, 0] \]

\[ S_2 = \begin{bmatrix} D_{uv} & 0 & D_{u\alpha} & D_{u\beta} & D_{uv \alpha} & 0 \\ 0 & -mg \sin \gamma & 0 & 0 & 0 & 0 \\ D_{uv} & 0 & 0 & D_{u\beta} & D_{uv \beta} & 0 \\ D_{uv} & 0 & D_{u\alpha} & 0 & D_{uv \gamma} & 0 \\ D_{C_{\delta \beta}} & 0 & D_{C_{\delta \beta}} & 0 & D_{C_{\delta \beta}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ R_1 = [(S_w + m r_w), mg \sin \gamma \sin \varphi, S_x, S_y, S_C_{\delta \beta}, -mg \cos \gamma \cos \varphi] \]

\[ R_2 = \begin{bmatrix} S_{uv} & 0 & S_{u\alpha} & S_{u\beta} & S_{uv \alpha} & 0 \\ 0 & mg \cos \gamma \sin \varphi & 0 & 0 & 0 & mg \sin \gamma \cos \varphi \\ S_{uv} & 0 & 0 & S_{u\beta} & S_{uv \gamma} & 0 \\ S_{uv} & 0 & S_{u\alpha} & 0 & S_{uv \gamma} & 0 \\ S_{C_{\delta \beta}} & 0 & S_{C_{\delta \beta}} & 0 & S_{C_{\delta \beta}} & 0 \\ 0 & 0 & 0 & 0 & 0 & mg \cos \gamma \cos \varphi \end{bmatrix} \]

Derivatives up to second order of the wind-axis pitch and yaw rates \( (q_w, r_w) \) will be required in the NID control laws. These can be expressed in terms of the previously defined \( z \), \( Q_1 \), \( Q_2 \), \( R_1 \), \( R_2 \) as

\[ \dot{q}_w = \frac{1}{m V} (Q_1 \dot{z}) \]

\[ \dot{r}_w = \frac{1}{m V} (\dot{z}^T Q_2 \dot{z} + Q_1 z - 2m \dot{V} \dot{q}_w) \]

\[ \dot{r}_w = \frac{1}{m V} (\dot{z}^T R_2 \dot{z} + R_1 z + 2m \dot{V} \dot{r}_w) \]

The first derivative of the wind-axis \( (p_w) \) roll rates will also be used. This is given by,

\[ \dot{p}_w = (\dot{p} \cos \alpha + \dot{r} \sin \alpha) \sec \beta + (r \cos \alpha - p \sin \alpha) \sec \beta \dot{\alpha} + (p \cos \alpha + r \sin \alpha) \tan \beta + q \sec \beta \dot{\beta} + \dot{q}_w \tan \beta \]

\[ q \frac{1}{I_{yy}} [I_{yy} (r^2 - p^2) + (I_{xx} - I_{yy}) \dot{r}_w] \]

\[ \dot{\dot{V}} = -\frac{1}{m} (D + mg \sin \gamma) \]

\[ \dot{V} = \frac{1}{m} (S_1 \dot{z}) \]

\[ \dot{\dot{V}} = -\frac{1}{m} (\dot{z}^T S_2 \dot{z} + S_1 \dot{z}) \]

\[ \dot{\dot{g}} = q_w \cos \varphi - r_w \sin \varphi \]

\[ \ddot{\gamma} = (\dot{q}_w \cos \varphi - \dot{r}_w \sin \varphi) - (q_w \sin \varphi + r_w \cos \varphi) \psi \]

\[ \ddot{\gamma} = -(q_w \cos \varphi - r_w \sin \varphi) \psi^2 - 2(q_w \sin \varphi + r_w \cos \varphi) \dot{\psi} \]

\[ q_w \sin \varphi + r_w \cos \varphi \dot{\psi} + (q_w \cos \varphi - r_w \sin \varphi) \dot{\psi} \]

\[ \dot{q} = \frac{1}{I_{yy}} [I_{yy} (r^2 - p^2) + (I_{xx} - I_{yy}) \dot{r}_w] \]
\[
\dot{\alpha} = q - q_{\omega} \sec \beta - (p \cos \alpha + r \sin \alpha) \tan \beta \\
\dot{\alpha} = q - q_{\omega} \sec \beta + (p \sin \alpha - r \cos \alpha) \tan \beta \\
-\left(q_{\omega} \sin \beta + p \cos \alpha + r \sin \alpha\right) \sec^2 \beta \\
-\left(-\rho \cos \alpha + r \sin \alpha\right) \tan \beta \\
\quad (A20)
\]

\[
\dot{\beta} = r + p \sin \alpha - r \cos \alpha \\
\dot{\beta} = r + p \sin \alpha + r \cos \alpha + (p \cos \alpha + r \sin \alpha) \dot{\alpha} \\
\dot{\psi} = p_{\omega} + (\sin \varphi \tan \gamma) q_{\omega} + (\cos \varphi \tan \gamma) r_{\omega} \\
\dot{\psi} = \rho_{\omega} + (\cos \varphi \cos \rho - r_{\omega} \sin \varphi) \tan \gamma \\
\quad + k(\sin \varphi + r_{\omega} \cos \varphi) \sec^2 \gamma \\
\quad + (\tan \gamma \sec \gamma) \gamma \\
\quad (A25)
\]

\[
\psi = (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi) \sec \gamma \\
\psi = (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi + (q_{\omega} \cos \varphi - r_{\omega} \sin \varphi) \dot{\varphi}) \sec \gamma \\
\quad + (q_{\omega} \cos \varphi + r_{\omega} \cos \varphi) \tan \gamma \gamma \\
\quad (A26)
\]

\[
\psi = (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi + 2(q_{\omega} \cos \varphi - r_{\omega} \sin \varphi) \dot{\varphi}) \sec \gamma \\
\quad + 2(q_{\omega} \cos \varphi + r_{\omega} \cos \varphi) \tan \gamma \gamma \\
\quad (A27)
\]

\[
\psi = -\left(q_{\omega} \sin \varphi + r_{\omega} \cos \varphi\right) \dot{\varphi} \\
+ (q_{\omega} \cos \varphi - r_{\omega} \sin \varphi) \dot{\varphi} \sec \gamma \\
+ 2(q_{\omega} \cos \varphi + r_{\omega} \cos \varphi) \tan \gamma \gamma \\
+ (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi)(1 + \sin^2 \gamma)(\sec^2 \gamma) \gamma^2 \\
+ (\tan \gamma \sec \gamma) \gamma \\
\quad (A28)
\]

\[
\begin{bmatrix}
-D/m \delta T & (q_{\omega T} \cos \varphi - r_{\omega T} \sin \varphi) & 0 & 0 \\
-1/m D_{\alpha} \dot{\alpha}_{\delta E} & (q_{\omega E} \cos \varphi - r_{\omega E} \sin \varphi) & 0 & 0 \\
-1/m (D_{\alpha} \dot{\alpha}_{\delta A} + D_{\beta} \dot{\beta}_{\delta A}) & (q_{\omega A} \cos \varphi - r_{\omega A} \sin \varphi) - (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi) \dot{\varphi}_{\delta A} & \dot{\varphi}_{\delta A} & \dot{\beta}_{\delta A} \\
-1/m (D_{\alpha} \dot{\alpha}_{\delta R} + D_{\beta} \dot{\beta}_{\delta R}) & (q_{\omega A} \cos \varphi - r_{\omega A} \sin \varphi) - (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi) \dot{\varphi}_{\delta R} & \dot{\varphi}_{\delta R} & \dot{\beta}_{\delta R}
\end{bmatrix}
\quad (B9)
\]

This matrix becomes singular when,
\[
\det(B^*) = 0 = -\frac{1}{mV} \dot{q}_{\delta E} \sec \beta (\dot{\rho}_{\delta A}r_{\delta A} - \dot{\rho}_{\delta A}r_{\delta A})
\cdot \left[(D_{\alpha}L_{\omega} - D_{\beta}L_{\delta T}) \cos \varphi + (D_{\omega S_{\omega r} - D_{\delta T}S_{\delta}) \sin \varphi] \quad (B10)
\]

Since (\dot{\rho}_{\delta A}r_{\delta A} - \dot{\rho}_{\delta A}r_{\delta A}) < 0 over the entire flight envelope for the Navion aircraft, \det(B^*) = 0 when,
\[
\dot{\varphi} = \tan^{-1} \left(\frac{(D_{\alpha}L_{\omega} - D_{\beta}L_{\delta T})}{(S_{\delta T} + D_{\delta T}S_{\delta})} \right) \quad (B11)
\]

Normal flight conditions for this aircraft produce singular bank angles of approximately 90° as,
\[
\left|\frac{(D_{\delta T}L_{\omega} - D_{\delta T}L_{\omega})}{(S_{\delta T} + D_{\delta T}S_{\delta})}\right| = \left|\frac{\alpha}{\beta}\right| > 5. \quad (B12)
\]

If the throttle saturates, however, the first row and column of \(B^*\) must be truncated. In this case a singularity condition occurs when,
\[
\det(B^*) = -\frac{1}{mV} \dot{q}_{\delta E} (L_{\omega} \cos \varphi + S_{\omega} \sin \varphi)(\dot{\rho}_{\delta A}r_{\delta A} - \dot{\rho}_{\delta A}r_{\delta A}) = 0 \quad (B13)
\]

\[
\begin{bmatrix}
-D/m \delta T & \dot{q}_{\omega T} \cos \varphi - r_{\omega T} \sin \varphi & 0 & 0 \\
-1/m D_{\alpha} \dot{\alpha}_{\delta E} & \dot{q}_{\omega E} \cos \varphi - r_{\omega E} \sin \varphi & 0 & 0 \\
-1/m (D_{\alpha} \dot{\alpha}_{\delta A} + D_{\beta} \dot{\beta}_{\delta A}) & \dot{q}_{\omega A} \cos \varphi - r_{\omega A} \sin \varphi - (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi) \dot{\varphi}_{\delta A} & \dot{\varphi}_{\delta A} & \dot{\beta}_{\delta A} \\
-1/m (D_{\alpha} \dot{\alpha}_{\delta R} + D_{\beta} \dot{\beta}_{\delta R}) & \dot{q}_{\omega A} \cos \varphi - r_{\omega A} \sin \varphi - (q_{\omega} \sin \varphi + r_{\omega} \cos \varphi) \dot{\varphi}_{\delta R} & \dot{\varphi}_{\delta R} & \dot{\beta}_{\delta R}
\end{bmatrix}
\quad (B14)
\]

or when \(\dot{\varphi} = -\tan^{-1}(C_{\alpha r}/C_{\delta})\). Since \(C_{\delta} \ll C_{\alpha}\), except for angles of attack near the point where \(L_{\omega} = 0\), special condition is avoided as the stall prevention system enforces \(\alpha_{\max} < \alpha_{\text{stall}}\). For \(\dot{y}_{\text{com}} = [V \; \psi \; \beta]^{T}\),

\[
\begin{bmatrix}
-D/m \delta T & 0 & 0 & 0 \\
-1/m D_{\alpha} \dot{\alpha}_{\delta E} & \dot{q}_{\delta E} & 0 & 0 \\
-1/m (D_{\alpha} \dot{\alpha}_{\delta A} + D_{\beta} \dot{\beta}_{\delta A}) & 0 & \dot{\beta}_{\delta A} & \dot{\beta}_{\delta A} \\
-1/m (D_{\alpha} \dot{\alpha}_{\delta R} + D_{\beta} \dot{\beta}_{\delta R}) & 0 & \dot{\beta}_{\delta R} & \dot{\beta}_{\delta R}
\end{bmatrix}
\quad (B15)
\]

Since \(\det(B^*) = 0\) only when \(\alpha = 90°\), \(\dot{y}_{\text{com}}\) is valid over the full flight envelope.

For \(\dot{y}_{\text{com}} = [V \; \psi \; \beta]^{T}\)
This matrix becomes singular when,
\[
\det (B^*) = 0 = (mg \cos \gamma \cos \varphi + mVq)(D_{\phi T}L_{\alpha} - D_{\alpha}L_{\phi T}) + (mg \cos \gamma \sin \varphi - mVr_{\alpha})(D_{\phi T}S_{\alpha} - D_{\alpha}S_{\phi T}).
\] (B17)

Since
\[
L = mVq + mg \cos \gamma \cos \varphi
\] (B18)
and
\[
S = mg \cos \gamma \sin \varphi - mVr_{\alpha}
\] (B19)
the singularity condition for this $B^*$ matrix is,
\[
\left| \frac{S}{L} \right| = \left| \frac{(D_{\phi T}L_{\alpha} - D_{\alpha}L_{\phi T})}{(D_{\phi T}S_{\alpha} - D_{\alpha}S_{\phi T})} \right| = \left| \frac{c}{d} \right|.
\] (B20)

Assuming the aircraft is in a coordinated turn (zero sideslip angle) as shown in Fig. 13, where $L = n(mg)$, and $n$ is the load factor, then the singularity condition occurs when the lateral acceleration of the flight path is given by,
\[
\frac{S'}{m} = ng \left[ \sin \varphi + \frac{c}{d} \cos \varphi \right].
\] (B21)

**Fig. 13.** Forces in a coordinated turn ($\beta = 0$).

When $\beta = 0$, $|c/d| > 25$. Therefore bank angles up to $75^\circ$ would require lateral accelerations in excess of 7 gs (when $n = 1$) to encounter these singularity conditions within the normal flight envelope. In the event of throttle saturation,
\[
\det (B^*) = \left( \frac{1}{mV} \right)^2 \frac{\dot{\psi}_{\alpha T} \sec \beta (\dot{\dot{p}}_{\alpha R} \dot{\dot{\alpha}}_{\alpha} - \dot{\dot{p}}_{\phi} \dot{\dot{R}}_{\phi}) (L_{\alpha} L_{\phi} + S_{\alpha} S_{\phi}).
\] (B22)

Since $C_{\alpha} \ll C_{\phi}$ when $\alpha_{\text{max}} < \alpha_{\text{stall}}$, this singularity condition also occurs outside the normal flight envelope.

For $y_{\text{comm}} = [\gamma \varphi \beta]^T$,
\[
(B^*)^T = \begin{bmatrix}
(q_{\phi T} \cos \varphi - r_{\phi T} \sin \varphi) & 0 & 0 \\
(q_{\phi T} \cos \varphi - r_{\phi T} \sin \varphi) & \dot{\alpha}_{\phi T} & 0 \\
(q_{\alpha T} \cos \varphi - \dot{r}_{\alpha T} \sin \varphi) & -q_{\alpha T} \sin \varphi + r_{\alpha T} \cos \varphi & \dot{\alpha}_{\phi T} \dot{\dot{\alpha}}_{\alpha} - \dot{\dot{p}}_{\phi} \dot{\dot{R}}_{\phi} \\
(q_{\alpha T} \cos \varphi - \dot{r}_{\alpha T} \sin \varphi) & -q_{\alpha T} \sin \varphi + r_{\alpha T} \cos \varphi & \dot{\dot{p}}_{\phi} \dot{\dot{R}}_{\phi}
\end{bmatrix}
\] (B23)

\[
\det (B^*) = \dot{\psi}_{\alpha T} \sec \beta (L_{\alpha} L_{\phi} + S_{\alpha} S_{\phi}) \times (\dot{\dot{p}}_{\phi} \dot{\dot{R}}_{\phi} - \dot{\dot{p}}_{\phi} \dot{\dot{R}}_{\phi}).
\] (B24)

Since $y_{\text{comm}}$ is only active when $\alpha > \alpha_{\text{active}}$, and for the Navion aircraft at these values of angle of attack $C_{\alpha_{\text{l}}}, > 3 C_{\alpha_{\text{st}}}$, the singularity condition,
\[
\varphi = -\tan^{-1} \left[ \left( C_{\alpha_{\text{l}}} / C_{\alpha_{\text{st}}} \right) \right]
\] (B25)
occurs at values of $\varphi$ in excess of $70^\circ$. In the event of throttle saturation, however, $y_{\text{comm}}$ has no singularity conditions.