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Brief Paper

Robust control of nonlinear systems with parametric uncertainty ☆

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Abstract

Probabilistic robustness analysis and synthesis for nonlinear systems with uncertain parameters are presented. Monte Carlo simulation is used to estimate the likelihood of system instability and violation of performance requirements subject to variations of the probabilistic system parameters. Stochastic robust control synthesis searches the controller design parameter space to minimize a cost that is a function of the probabilities that design criteria will not be satisfied. The robust control design approach is illustrated by a simple nonlinear example. A modified feedback linearization control is chosen as controller structure, and the design parameters are searched by a genetic algorithm to achieve the tradeoff between stability and performance robustness. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Stochastic robustness; Monte Carlo simulation; Input-to-state stability; Genetic algorithm

1. Introduction

The problem of designing a robust state feedback control that provides global stability for an uncertain nonlinear system has been the subject of considerable research over the last decade. For nonlinear systems with unknown static nonlinearities satisfying known functional bounds, the current literature focuses on deterministic worst-case robust analysis and synthesis. Early nonlinear design methods (Gutman, 1979; Corless & Leitmann, 1981; Barmish, Corless, & Leitmann, 1983) were based on the "matching condition" assumption that the uncertainty enters into the state equation at the same point as the control input. For nonlinear systems containing unmatched uncertainty, recent results (Marino & Tomei, 1993; Slotine & Hedrick, 1993; Freeman & Kokotovic, 1993a; and references therein) are based on the technique of integrator backstepping (Kanellakopoulous, Kokotovic, & Morse, 1991).

For parametric uncertainty, guaranteed stability-bound estimates often are unduly conservative, and the resulting controller usually needs very high control effort (Freeman & Kokotovic, 1993b). Even for linear systems, many worst-case deterministic robust control problems are non-polynomial (NP) hard (Braatz, Young, Doyle, & Morari, 1994). If instead of worst-case guaranteed conclusions, probabilistic conclusions are acceptable, computation complexity can be reduced significantly. Probabilistic control design uses randomized algorithms with polynomial complexity to characterize system robustness (Ray & Stengel, 1993; Marrison, 1995; Tempo, Bai, & Dabbene, 1997; Khargonekar & Tikku, 1996; and references therein) and to identify satisfactory controllers. Furthermore, probabilistic robustness analysis and control design directly attack the real engineering problem of interest by evaluating the likelihood that the design requirements would fail subject to the uncertainties, while the deterministic robust control theories usually need to transform the engineering design criteria to fit their own frameworks.

Since many problems in robustness analysis and synthesis can be formulated as the minimization of an objective function with respect to the controller parameters, we believe that creative combinations of a variety of pre-existing control methodologies and the probabilistic approach will result in powerful tools that can address real engineering control problems. The probabilistic approach has been applied to a linear benchmark robust control problem (Wie & Bernstein, 1990) for which linear–quadratic–Gaussian regulators (Marrison & Stengel, 1995) and transfer function sweep designs (Wang & Stengel, 2001) were given to minimize a probabilistic robustness design cost.

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The probabilistic approach is readily applied to nonlinear designs as well as to linear designs. In this paper, we present a framework for nonlinear robust control by merging the stochastic point of view with feedback linearization and backstepping. In Section 2, we propose the general approach for probabilistic robust control of nonlinear systems. In Section 3, for a nonlinear model that is an adaptation of a linear benchmark problem, state feedback controller structure and robustness cost function are formulated. In Section 4, a genetic algorithm searches the design parameters of the probabilistic robust feedback control law. Simulation results are presented for stability and performance robustness of the closed-loop system.

2. Stochastic robust control design for nonlinear systems

2.1. Stochastic robustness

Denote a plant structure as $H(\boldsymbol{q})$, where $\boldsymbol{q}$ is a vector of varying plant parameters selected randomly throughout the parameter space, $\boldsymbol{Q}$, reflecting the expected distribution, $pr(\boldsymbol{q})$. Stochastic robustness characterizes a control law, $C(\boldsymbol{d})$, with $\boldsymbol{d}$ as the design parameter, in terms of the probabilities that the closed-loop system will have unacceptable stability and performance when subjected to parametric uncertainties. For each design requirement on stability or performance, define the corresponding binary indicator function $I[\cdot]$ as one if $H(\boldsymbol{q})$ and $C(\boldsymbol{d})$ form an unacceptable system and as zero otherwise. For example, if the closed-loop system is unstable, its indicator function equals one; otherwise it equals zero. If a step response lies outside a desired performance envelope, its indicator function is one; otherwise it is zero. Then for each design requirement, the probability of design requirement violation, P, can be calculated as the integral of the corresponding indicator function over the expected system parameter space:

$$P(\boldsymbol{d}) = \int_{\boldsymbol{Q}} I[H(\boldsymbol{q}), C(\boldsymbol{d})]\, pr(\boldsymbol{q})\, \mathrm{d}\boldsymbol{q}. \tag{1}$$

This probability can be viewed as the expected value of the indicator function.

The stochastic robustness cost function, $J(\boldsymbol{d})$, is formalized by combining the probabilities for different kinds of design requirements with certain tradeoffs:

$$J(\boldsymbol{d}) = fcn[P_1(\boldsymbol{d}), P_2(\boldsymbol{d}), \ldots]. \tag{2}$$

The goal is to find the optimal controller parameter $\boldsymbol{d}^*$ that minimizes the cost function J. For example, the design cost may tradeoff the probability of system instability, of unsatisfactory performance, and of excess use of control.

In most cases, Eq. (1) cannot be integrated analytically. Monte Carlo evaluation (MCE) is a practical and flexible alternative to estimate the probabilities (Kalos & Whitlock, 1986). The estimates of the probability and cost based on N samples are

$$\hat{P}(\boldsymbol{d}) = \frac{1}{N} \sum_{k=1}^{N} I[H(\boldsymbol{q}_k), C(\boldsymbol{d})], \tag{3}$$

$$\hat{J}(\boldsymbol{d}) = fcn[\hat{P}_1(\boldsymbol{d}), \hat{P}_2(\boldsymbol{d}), \ldots]. \tag{4}$$

The estimated cost $\hat{J}$ approaches the actual cost J in the limit as $N \to \infty$. For a finite N, because the probabilities defined above are all binary variables, with the outcome of a trial taking one of two possible values (acceptable or unacceptable) for each MCE, the binomial test can be applied to determine exact confidence intervals for the estimates of the probabilities. Stengel and Ray (1991) provide the formula to compute the lower and upper bounds of the confidence interval for the MCE corresponding to any specified confidence coefficients.

2.2. Stochastic robust control of nonlinear systems

In this section, we formulate the general nonlinear design problem for deterministic systems, introduce parametric uncertainty, and reformulate the problem in a probabilistic format. As a practical matter, we make certain simplifying assumptions to illustrate the design approach. Define R^+ as the interval $[0, \infty)$. A continuous function $\alpha : R^+ \to R^+$ is of class K when it is strictly increasing and $\alpha(0) = 0$. It is of class K_∞ when, in addition, $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : R^+ \times R^+ \to R^+$ is of class KL when for each fixed t, $\beta(\cdot, t)$ is of class K and for each fixed r, $\beta(r, \cdot)$ is decreasing in t with $\beta(r, \cdot) \to 0$ as $t \to \infty$.

Consider a nonlinear system with the same input and output dimensions in the presence of an uncertain parameter $\boldsymbol{q} \in \boldsymbol{Q}$:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{q}) + \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{q})\boldsymbol{u} + \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{q})\boldsymbol{w}, \\ \boldsymbol{G} &= [\boldsymbol{g}_1 \quad \boldsymbol{g}_2 \quad \cdots \quad \boldsymbol{g}_m], \\ \boldsymbol{P} &= [\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \cdots \quad \boldsymbol{p}_l], \end{aligned} \tag{5}$$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{q}), \tag{6}$$

where $\boldsymbol{G} = [\boldsymbol{g}^1 \boldsymbol{g}^2 \cdots \boldsymbol{g}^m]$, $\boldsymbol{P} = [\boldsymbol{P}^1 \boldsymbol{P}^2 \cdots \boldsymbol{P}^l]$, $\boldsymbol{f}, \boldsymbol{g}_j$ $(j = 1, 2, \ldots, m)$ and $\boldsymbol{p}_k$ $(k = 1, 2, \ldots, l)$ are smooth vector fields on R^n, $\boldsymbol{h}$ is a smooth function mapping $R^n \to R^m$, and $\boldsymbol{w}$ is an essentially bounded unknown disturbance vector.

With state feedback control $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}(t))$, the input-to-state stability (Sontag, 1989) of the closed-loop system with respect to the disturbance $\boldsymbol{w}$ is defined as follows:

Definition. The system is robust *input-to-state stable* with respect to the disturbance $\boldsymbol{w}$ if $\forall \boldsymbol{q} \in \boldsymbol{Q}$, there exist a class K function α and class KL functions β and η, such that, for each $x_0 \in R^n$, and any continuous bounded $\boldsymbol{w}$, the solution exists and satisfies

$$|\boldsymbol{x}(t)| \leqslant \beta(|\boldsymbol{x}_0|, t) + \eta(\|\boldsymbol{w}_{[0,t^*]}\|, t - t^*) + \alpha(\|\boldsymbol{w}_{(t^*,t]}\|) \tag{7}$$

for all t^* and t such that $0 \leqslant t^* \leqslant t$. $|\cdot|$ denotes the Euclidean norm, and $\|\cdot\|$ denotes the sup norm for which $\|\boldsymbol{w}\| := \operatorname{ess\,sup}\{|\boldsymbol{w}(t)|,\ t \geqslant 0\}$.

2.2.1. Nominal control system design

The following theorem describes the necessary conditions under which our design method can be applied.

Theorem (proof is given in the appendix). *In the nominal system of equations* (5)–(6), *if we assume that* $(\boldsymbol{f}, \boldsymbol{G})$ *is exact feedback linearizable, and* $(\boldsymbol{f}, \boldsymbol{P})$ *takes a strict feedback form* (*i.e., lower triangular form*), *then there exists a nonlinear coordinate transformation* $\boldsymbol{\xi} = T(\boldsymbol{x})$ *and a feedback control* $\boldsymbol{u} = -[\boldsymbol{G}^*(\boldsymbol{x})]^{-1}\boldsymbol{f}^*(\boldsymbol{x}) + [\boldsymbol{G}^*(\boldsymbol{x})]^{-1}\boldsymbol{v}$ (*the definitions of* $T(\boldsymbol{x})$, $\boldsymbol{f}^*(\boldsymbol{x})$, *and* $\boldsymbol{G}^*(x)$ *are given in the appendix*) *such that the system takes the following form*:

$$\dot{\xi}_1^i = \xi_2^i + [\boldsymbol{\varphi}_1^i(\xi_1^i)]^{\mathrm{T}}\boldsymbol{w},$$

$$\vdots$$

$$\dot{\xi}_{\gamma_i-1}^i = \xi_{\gamma_i}^i + [\boldsymbol{\varphi}_{\gamma_i-1}^i(\xi_1^i, \ldots, \xi_{\gamma_i-1}^i)]^{\mathrm{T}}\boldsymbol{w},$$

$$\dot{\xi}_{\gamma_i}^i = v_i + [\boldsymbol{\varphi}_{\gamma_i}^i(\xi_1^i, \ldots, \xi_{\gamma_i}^i)]^{\mathrm{T}}\boldsymbol{w},$$

$$i = 1, 2, \ldots, m. \tag{8}$$

The closed-loop system is input-to-state stable with respect to the disturbance $\boldsymbol{w}$ if the control is designed as

$$\boldsymbol{u} = -[\boldsymbol{G}^*(\boldsymbol{x})]^{-1}\boldsymbol{f}^*(\boldsymbol{x}) + [\boldsymbol{G}^*(\boldsymbol{x})]^{-1}[\alpha_{\gamma_1}^1 \quad \cdots \quad \alpha_{\gamma_m}^m]^{\mathrm{T}} \tag{9}$$

with intermediate controls α_j^i and error variables z_j^i calculated in terms of the backstepping process (Kristic, Kanellakopoulos, & Kokotovic, 1995)

$$z_j^i = \xi_j^i - \alpha_{j-1}^i(\xi_1^i, \ldots, \xi_{j-1}^i), \tag{10}$$

$$\alpha_j^i(\xi_1^i, \ldots, \xi_j^i) = -\delta_j^i z_j^i - z_{j-1}^i + \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}^i}{\partial \xi_k^i}\,\xi_{k+1}^i - e_j^i z_j^i \left| \boldsymbol{\varphi}_j^i - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}^i}{\partial \xi_k^i}\,\boldsymbol{\varphi}_k^i \right|^2, \tag{11}$$

where $z_0^i \equiv \alpha_0^i \equiv 0$, and δ_j^i, e_j^i, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, \gamma_i$ are positive design parameters.

2.2.2. Robust control design

Now we consider the parametric uncertainty $\boldsymbol{q} \in \boldsymbol{Q}$ in Eqs. (5) and (6). We adopt controller (9) as our stochastic robust control law structure, which is parameterized by the design parameter vector $\boldsymbol{d} = \{\delta_j^i, e_j^i,\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, \gamma_i\}$. According to system design requirements, we define corresponding robustness metrics and formulate a stochastic robustness cost function as Eq. (2). For each candidate vector of the design parameter, Monte Carlo simulations are used to evaluate the probability of metric violation. Optimization algorithms are applied to search the design parameter space to minimize the stochastic robustness cost function.

2.2.3. Simplification of control design for special cases

If in Eq. (8), the disturbance enters in a simpler way, specifically, if only the disturbance contained in $\xi_{\gamma_i}^i$ has a nonlinear coefficient, then the control law can be generated in a more conventional fashion.

Corollary. *If Eq.* (8) *takes the form*

$$\dot{\boldsymbol{\xi}}^i = \boldsymbol{A}_i\boldsymbol{\xi}^i + \boldsymbol{b}_i v_i + \boldsymbol{C}_i\boldsymbol{w} + \boldsymbol{b}_i[\boldsymbol{\psi}_i(\boldsymbol{\xi}^i)]^{\mathrm{T}}\boldsymbol{w}, \tag{12}$$

where

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\gamma_i \times \gamma_i}, \quad \boldsymbol{b}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\gamma_i \times 1}, \tag{13}$$

$\boldsymbol{C}_i$ *is a* $(\gamma_i \times l)$ *constant matrix and* $\boldsymbol{\psi}_i$ *is an* $(l \times 1)$ *nonlinear function vector, then the closed-loop system is input-to-state stable with respect to the disturbance* $\boldsymbol{w}$ *with the control law* v_i:

$$v_i = -r_i^{-1}\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{\xi}^i - \delta_i|\boldsymbol{\psi}_i|^2\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{P}_i\boldsymbol{\xi}^i, \quad r_i, \delta_i > 0, \tag{14}$$

where $\boldsymbol{P}_i$ *is the positive-definite solution to the Riccati equation with design parameters* $\boldsymbol{Q}_i$ *and* r_I:

$$\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_i - r_i^{-1}\boldsymbol{P}_i\boldsymbol{b}_i\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{P}_i + \boldsymbol{Q}_i = 0, \quad \boldsymbol{Q}_i, r_i > 0. \tag{15}$$

Considering system robustness to parameter uncertainty, we optimize the design parameter vector $\boldsymbol{d} = \{\boldsymbol{Q}_i, r_i, \delta_i;\ i = 1, 2, \ldots, m\}$ of control law (14) to minimize the stochastic robustness cost function.

In the following sections, we present an example to illustrate the design procedure. First, we derive the parameterized control law for the nominal system and formulate the stochastic robustness cost function. Then, we apply an optimization algorithm to search the controller parameter space so that the Monte Carlo estimation of the probabilistic robustness cost function is minimized.

3. Stochastic robust control design for a nonlinear spring–mass system

As a simple example to demonstrate the design method proposed in Section 2, a nonlinear spring–mass system is considered (Fig. 1). This is an adaptation of the linear benchmark problem posed by Wie and Bernstein (1990). The linear spring of the earlier problem is replaced by a linear-cubic spring therefore, the state-space model

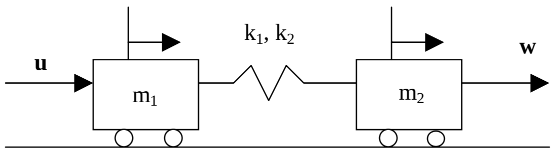

Fig. 1. A nonlinear system consists of two masses connected by a linear-cubic spring.

becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_3\\ x_4\\ -\dfrac{k_1}{m_1}(x_1-x_2) - \dfrac{k_2}{m_1}(x_1-x_2)^3\\ \dfrac{k_1}{m_2}(x_1-x_2) + \dfrac{k_2}{m_2}(x_1-x_2)^3\end{bmatrix} + \begin{bmatrix}0\\0\\ \dfrac{1}{m_1}\\ 0\end{bmatrix} u + \begin{bmatrix}0\\0\\0\\ \dfrac{1}{m_2}\end{bmatrix} w$$

$$= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{q}) + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{q})u + \boldsymbol{p}(\boldsymbol{x},\boldsymbol{q})w, \quad (16)$$

$$y = x_2 = h(\boldsymbol{x}). \quad (17)$$

We denote x_1 and x_2 as the positions of the masses, and x_3 and x_4 as their velocities. The plant is disturbed by w on the second mass, and the control u is applied to the first mass. The vector of system parameters is denoted by $\boldsymbol{q}=[k_1\ \ k_2\ \ m_1\ \ \ m_2]^{\mathrm{T}}$, with nominal values as $k_1^0=m_1^0=m_2^0=1$, and $k_2^0=-0.1$. The limits of mass and spring-constant variations are

$$0.5 < k_1 < 2, \qquad -0.5 < k_2 < 0.2,$$
$$0.5 < m_1 < 1.5, \quad 0.5 < m_2 < 1.5. \quad (18)$$

The single input/single output form of the transformation $\boldsymbol{\xi} = T(\boldsymbol{x})$ to Lie-derivative coordinates (defined in the appendix) is calculated as

$$\xi_1 = h(\boldsymbol{x}) = x_2,$$

$$\xi_2 = L_f h = x_4,$$

$$\xi_3 = L_f^2 h = \frac{k_1^0}{m_2^0}(x_1-x_2) + \frac{k_2^0}{m_2^0}(x_1-x_2)^3,$$

$$\xi_4 = L_f^3 h = (x_3-x_4)\left[\frac{k_1^0}{m_2^0} + \frac{3k_2^0}{m_2^0}(x_1-x_2)^2\right]. \quad (19)$$

By this nonlinear transformation, the nominal system of Eq. (16) becomes

$$\dot{\xi}_1 = \xi_2,$$

$$\dot{\xi}_2 = \xi_3 + \frac{1}{m_2^0}w,$$

$$\dot{\xi}_3 = \xi_4,$$

$$\dot{\xi}_4 = L_f^4 h + L_g L_f^3 h u + L_p L_f^3 h w, \quad (20)$$

where $L_f^4 h$, $L_g L_f^3 h$, and $L_p L_f^3 h$ are Lie derivatives given below:

$$L_f^4 h = \frac{6k_2^0}{m_2^0}(x_1-x_2)(x_3-x_4)^2 - \left(\frac{1}{m_1^0}+\frac{1}{m_2^0}\right)\left[\frac{k_1^0}{m_2^0} + \frac{3k_2^0}{m_2^0}(x_1-x_2)^2\right] \times[k_1^0(x_1-x_2) + k_2^0(x_1-x_2)^3], \quad (21)$$

$$L_g L_f^3 h = \frac{1}{m_1^0 m_2^0}[k_1^0 + 3k_2^0(x_1-x_2)^2], \quad (22)$$

$$L_p L_f^3 h = -\frac{1}{(m_2^0)^2}[k_1^0 + 3k_2^0(x_1-x_2)^2]. \quad (23)$$

If $L_g L_f^3 h$ is not singular, the control law u for the nominal system can be defined as

$$u = (L_g L_f^3 h)^{-1}(-L_f^4 h + v). \quad (24)$$

Then Eq. (20) takes the form of Eq. (12)

$$\dot{\boldsymbol{\xi}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{b}v + \boldsymbol{c}w - \frac{3k_2^0}{(m_2^0)^2}(x_1-x_2)^2\boldsymbol{b}w \quad (25)$$

with

$$\boldsymbol{c} = \begin{bmatrix}0\\ 1/m_2^0\\ 0\\ -k_1/(m_2^0)^2\end{bmatrix}. \quad (26)$$

By Eq. (14), we formulate the control law as

$$u = \frac{1}{L_g L_f^3 h}(-L_f^4 h - r^{-1}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\xi} - \delta(x_1-x_2)^2\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\xi}), \quad (27)$$

where $\boldsymbol{P}$ is the positive-definite solution to the Riccati equation parameterized by $\boldsymbol{Q}$ and r as Eq. (15). For simplicity, the matrix $\boldsymbol{Q}$ is chosen as a diagonal matrix, $\boldsymbol{Q} = diag\{q_1,q_2,q_3,q_4\}$; therefore, the design parameter vector is defined as

$$\boldsymbol{d} = \{q_1,q_2,q_3,q_4,r,\delta\}. \quad (28)$$

We adopt the nominal control law (27) as our stochastic robust controller structure. Three aspects of system robustness are of concern in this design: stability, settling time, and control effort. The appropriate design parameter vector $\boldsymbol{d}$ achieves tradeoffs between stability and performance robustness by minimizing a stochastic robustness cost function

$$J = w_i P_i^2 + w_{ts} P_{ts}^2 + w_u P_u^2, \quad (29)$$

where P_i, P_{ts}, and P_u represent probability of instability, of settling-time exceedance, and of excess control effort respectively. The probabilities are squared in Eq. (29) to place higher penalty on large probabilities of unsatisfactory behavior, which may be of greater concern than low probabilities of design metric violation. The choice of the weights w_i, w_{ts}, and w_u has been addressed extensively in Marrison and Stengel (1995). In this paper, we choose the cost function weights as $w_i = 1$, $w_{ts} = 0.01$ and $w_u = 0.01$, which are the same weights as those of Design 1 in Marrison and Stengel (1995).

4. Searching design parameters and simulation results

In Monte Carlo estimation of P_i, P_{ts}, and P_u, each candidate control law is evaluated subject to variations of the system parameters, variations of the initial state, and variations of the magnitude of the disturbance. We separate the design procedure into several stages. In the first stage, we fix the initial state at the origin and the disturbance as a unit impulse at the initial time, estimating only the effect of the system parameter variations on the closed-loop system. In the second stage, we change the initial condition as well as the system parameters, using a random number generator to specify the initial condition over the region of interest. In the final stage, we add the variations of the magnitude of the disturbance, using a random number generator to specify the size of disturbance within the interval of interest.

In this paper, we address just the first step, i.e., we assume that the initial state is at the origin and that the disturbance is a unit impulse at the initial time. In addition to the definition of system stability, we identify the measures of system performance for the design at this specific step as:

- *Probability of exceeding settling time* P_{ts}: $m_{ts} = 1$ if the time history of the output $|x_2(t)| > 0.1$ for any $t \geqslant 15$ s; $m_{ts} = 0$ otherwise.
- *Probability of exceeding control limit* P_u: $m_u = 1$ if the time history of the control input $|u(t)| > 1$ for any t; $m_u = 0$ otherwise.

Because Mount Carlo estimation of probability is used in the process of searching control design parameters, the discrepancy between the Monte Carlo estimate and the true value results in apparent "noise" in the evaluation of the cost function. Furthermore, the cost function is not convex; therefore, a genetic algorithm (Goldberg, 1989) is used to search for the control design parameters. The genetic algorithm is a randomized adaptive search method. It deals with the design parameter vectors as though they are the chromosomes of organisms trying to compete and survive in the environment specified by the cost J from generation to generation. Each element in the design parameter vector is represented by a binary number sequence; the elements are connected to compose a chromosome.

There are four operations in each generation of the chromosome evolution: evaluation, selection, crossover, and mutation. The initial population is formed by randomly generating a number of chromosomes. Each chromosome is evaluated by Monte Carlo simulation, and high-fitness chromosomes are selected to survive to the next generation. A chromosome with lower cost J has higher fitness. The selected chromosomes are paired randomly and subjected to crossover with a probability usually in $(0.6, 1.0)$. Crossover is carried out by swapping the tails of a pair of chromosomes at a random point along the binary sequence. After crossover, the binary number sequence in each chromosome may be mutated with a very low probability (usually < 0.1). Each mutated binary number of the chromosome is altered from 0 to 1 or from 1 to 0. For this design, the initial population contains 50 chromosomes.

We examine the effects of probability distributions of uncertainties on control law design and robustness. Stengel and Ray (1991) show that linear–quadratic regulator design is relatively insensitive to the assumed probability distribution of the parametric uncertainty. Barmish and Lagoa (1996) suggest that the uniform distribution is the worst case among bounded unimodal distributions on which robustness analysis may be based. In this paper, we design two controllers based on uniform (bounded) and Gaussian (unbounded) distributions respectively, then compare the robustness of the two controllers when the actual distributions are either uniform or Gaussian.

First, we assume that the system's uncertain parameters have uniform probability distributions over the range given in Eq. (18). During the process of optimization, each candidate control law is evaluated by 500 Monte Carlo simulations. After 20 generations the genetic algorithm produces control law (27) as

$$u = \frac{1}{L_g L_f^3 h}\left\{ -L_f^4 h - [0.2616 \quad 1.3831 \quad 2.9157 \quad 1.214] \times \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} - [0.0967 \quad 0.5115 \quad 1.078 \quad 0.449] \times \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} (x_1 - x_2)^2 \right\}. \tag{30}$$

This control law contains two terms that are linear in the Lie-derivative coordinate vector, whose gains are determined by linear–quadratic regulation of the transformed system. The second term is weighted by the square of the difference in positions of the two masses, a direct effect of the cubic nonlinearity.

The design process is repeated with the assumption that the uncertain parameters have Gaussian distributions as follows:

$$k_1 = N(1.25, 0.1875),\ k_2 = N(-0.15, 0.0408),$$
$$m_1 = N(1, 0.0833), \quad m_2 = N(1, 0.0833). \tag{31}$$

For each system parameter, the mean and variance of the Gaussian distribution are chosen to be the same as those of the uniform distribution listed in Eq. (18). The design for Gaussian distributions produces control law (27) as

$$u = \frac{1}{L_g L_f^3 h} \left\{ -L_f^4 h - [0.1403\ \ 0.7705\ \ 2.0611\ \ 0.9214] \times \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} - [0.0657\ \ 0.3297\ \ 0.8086\ \ 0.9835] \times \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} (x_1 - x_2)^2 \right\}. \tag{32}$$

Table 1 shows the comparison of the robustness profiles of the two controllers. Controller 1 (Eq. (30)) assumes uniform uncertainty and Controller 2 (Eq. (32)) assumes Gaussian uncertainty. Each evaluation data is based on 25,000 Monte Carlo simulations, against either uniform or Gaussian evaluation distributions. In each cell, the first number is the probability or cost, and the numbers in brackets denote the 95% confidence interval of the Monte Carlo estimate of the probability or cost.

As should be expected, if the evaluation distribution is uniform, the controller designed under that assumption yields a lower design cost than that of the controller designed for Gaussian uncertainty. Conversely, Controller 2 produces lower cost than Controller 1 when the uncertainty is Gaussian. For all but one case, evaluation with uniform distributions produces a lower metric than evaluation with Gaussian distributions, contradicting the notion that the assumption of uniform uncertainty is conservative. For the present case, the reason is quite clear: the tails of Gaussian distributions are unbounded, and unsatisfactory values outside the bounds of uniform distributions are likely to occur.

Controller 2 has lower probability of instability than Controller 1 for both uniform and Gaussian evaluations, and it is less likely to use too much control with Gaussian uncertainty. However, it is twice as likely to use excessive control if the uncertainty is uniform, and its probability of unsatisfactory settling time is marginally higher than that of Controller 1.

The design procedure proposed by this paper can be applied very easily to any nonlinear systems with feedback linearizable nominal vector fields. The probabilistic robust control framework can be further extended to nonlinear systems that are not feedback linearizable by utilizing approximate feedback linearization (Hauser, Sastry, & Kokotovic, 1992) and backstepping, etc. In Wang and Stengel (2000), for the longitudinal motion of a hypersonic aircraft containing 28 inertial and aerodynamic uncertain parameters and 39 design metrics, the probabilistic design approach produces an efficient flight control system that achieves better stability and performance robustness than a comparable stochastic robust linear controller (Marrison & Stengel, 1998).

5. Conclusions

A general framework for designing robust controllers for uncertain nonlinear systems is presented. The method combines probabilistic design and evaluation with feedback linearization and backstepping techniques. The design approach reveals new insights about the robustness of feedback-linearization-type controllers, and it shows the relationship between nominal system characteristics and the probability of maintaining satisfactory performance when parameters of the system are uncertain. The proposed probabilistic approach is illustrated with a nonlinear design example that shows excellent stability and performance robustness. The example illustrates the effects of uniform

Table 1
Comparison of system robustness of controllers designed with uniform and Gaussian distributions

Controller design	J_{cost}		P_i		P_{ts}		P_u	
	Uniform	Gaussian	Uniform	Gaussian	Uniform	Gaussian	Uniform	Gaussian
	Eval. Dist.	Eval. Dist.	Eval. Dist.	Eval. Dist.	Eval. Dist.	Eval. Dist.	Eval. Dist.	Eval. Dist.
Controller 1	4.15e-4 [3.54e-4, 4.81e-4]	2.5e-3 [2.29e-3, 2.79e-3]	0.016 [0.0144, 0.0176]	0.044 [0.0415, 0.0465]	0.615 [0.0585, 0.0645]	0.0952 [0.0916, 0.0988]	0.11 [0.1061, 0.1139]	0.225 [0.2198, 0.2302]
Controller 2	6.22e-4 [5.7e-4, 6.79e-4]	1.4e-3 [1.2e-3, 1.6e-3]	0.0107 [0.0094, 0.012]	0.0319 [0.0297, 0.0341]	0.0809 [0.0775, 0.0843]	0.0968 [0.0931, 0.1005]	0.2104 [0.2053, 0.2155]	0.1721 [0.1674, 0.1768]

and Gaussian uncertainties on controller design and evaluation. For example, evaluation with Gaussian uncertainty produces more conservative results, while the assumption of Gaussian uncertainty in control design leads to lower probability of instability when evaluated against actual uncertainties that are uniform or Gaussian. The method is not restricted by the number of uncertainties or the degree of nonlinearity of the system, and it provides a logical extension to prior results for robust linear control.

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Appendix.

Proof of the theorem. Define a transformation from the state to Lie-derivative coordinates $\boldsymbol{\xi} = T(\boldsymbol{x})$ as

$$\xi_1^i = h_i,$$
$$\xi_2^i = L_{\boldsymbol{f}} h_i,$$
$$\vdots$$
$$\xi_{\gamma_i}^i = L_{\boldsymbol{f}}^{\gamma_i - 1} h_i,$$
$$i = 1, 2, \ldots, m, \tag{A.1}$$

where

$$L_{\boldsymbol{f}} h_i = \frac{\partial h_i}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}), \quad L_{\boldsymbol{f}}^s h_i = L_{\boldsymbol{f}}(L_{\boldsymbol{f}}^{s-1} h_i). \tag{A.2}$$

For each h_i, γ_i is defined as the smallest integer such that at least one $j \in \{1, 2, \ldots, m\}$ satisfies $L_{\boldsymbol{g}_j}(L_{\boldsymbol{f}}^{\gamma_i - 1} h_i) \neq 0$. With the assumption that $(\boldsymbol{f}, \boldsymbol{G})$ is exact feedback linearizable (Isidori, 1989), Eq. (A.1) defines a set of coordinates that transforms the nominal system of Eq. (5) into

$$\dot{\xi}_1^i = \xi_2^i + \sum_{k=1}^{l} (L_{\boldsymbol{p}_k} h_i) w_k,$$
$$\vdots$$
$$\dot{\xi}_{\gamma_i - 1}^i = \xi_{\gamma_i}^i + \sum_{k=1}^{l} (L_{\boldsymbol{p}_k} L_{\boldsymbol{f}}^{\gamma_i - 2} h_i) w_k,$$
$$\dot{\xi}_{\gamma_i}^i = L_{\boldsymbol{f}}^{\gamma_i} h_i + \sum_{j=1}^{m} (L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^{\gamma_i - 1} h_i) u_j + \sum_{k=1}^{l} (L_{\boldsymbol{p}_k} L_{\boldsymbol{f}}^{\gamma_i - 1} h_i) w_k,$$
$$i = 1, 2, \ldots m. \tag{A.3}$$

The exact feedback linearization assumption also allows the existence of a feedback control law $\boldsymbol{u} = -[\boldsymbol{G}^*(\boldsymbol{x})]^{-1} \boldsymbol{f}^*(\boldsymbol{x}) + [\boldsymbol{G}^*(\boldsymbol{x})]^{-1} \boldsymbol{v}$ with $\boldsymbol{f}^*(\boldsymbol{x})$ and a nonsingular $\boldsymbol{G}^*(\boldsymbol{x})$ defined as

$$\boldsymbol{f}^*(\boldsymbol{x}) = \begin{bmatrix} L_{\boldsymbol{f}}^{\gamma_1} h_1 \\ L_{\boldsymbol{f}}^{\gamma_2} h_2 \\ \vdots \\ L_{\boldsymbol{f}}^{\gamma_m} h_m \end{bmatrix},$$

$$\boldsymbol{G}^*(\boldsymbol{x}) = \begin{bmatrix} L_{\boldsymbol{g}_1} L_{\boldsymbol{f}}^{\gamma_1 - 1} h_1 & L_{\boldsymbol{g}_2} L_{\boldsymbol{f}}^{\gamma_1 - 1} h_1 & \cdots & L_{\boldsymbol{g}_m} L_{\boldsymbol{f}}^{\gamma_1 - 1} h_1 \\ L_{\boldsymbol{g}_1} L_{\boldsymbol{f}}^{\gamma_2 - 1} h_2 & L_{\boldsymbol{g}_2} L_{\boldsymbol{f}}^{\gamma_2 - 1} h_2 & \cdots & L_{\boldsymbol{g}_m} L_{\boldsymbol{f}}^{\gamma_2 - 1} h_2 \\ \vdots & \vdots & \vdots & \vdots \\ L_{\boldsymbol{g}_1} L_{\boldsymbol{f}}^{\gamma_m - 1} h_m & L_{\boldsymbol{g}_2} L_{\boldsymbol{f}}^{\gamma_m - 1} h_m & \cdots & L_{\boldsymbol{g}_m} L_{\boldsymbol{f}}^{\gamma_m - 1} h_m \end{bmatrix} \tag{A.4}$$

such that Eq. (A.3) becomes

$$\dot{\xi}_1^i = \xi_2^i + \sum_{k=1}^{l} (L_{\boldsymbol{p}_k} h_i) w_k,$$
$$\vdots$$
$$\dot{\xi}_{\gamma_i - 1}^i = \xi_{\gamma_i}^i + \sum_{k=1}^{l} (L_{\boldsymbol{p}_k} L_{\boldsymbol{f}}^{\gamma_i - 2} h_i) w_k,$$
$$\dot{\xi}_{\gamma_i}^i = v_i + \sum_{k=1}^{l} (L_{\boldsymbol{p}_k} L_{\boldsymbol{f}}^{\gamma_i - 1} h_i) w_k. \tag{A.5}$$

By the assumption that $(\boldsymbol{f}, \boldsymbol{P})$ takes a strict feedback form (i.e. Eq. (A.5) in a lower-triangular form), which is a general assumption for nonlinear control system designs, there exist functions $\boldsymbol{\varphi}_1^i(\xi_1^i), \boldsymbol{\varphi}_2^i(\xi_1^i, \xi_2^i), \ldots, \boldsymbol{\varphi}_{\gamma_i}^i(\xi_1^i, \xi_2^i, \ldots, \xi_{\gamma_i}^i)$, which are $l \times 1$ vectors such that Eq. (A.5) becomes

$$\dot{\xi}_1^i = \xi_2^i + [\boldsymbol{\varphi}_1^i(\xi_1^i)]^{\mathrm{T}} \boldsymbol{w},$$
$$\vdots$$
$$\dot{\xi}_{\gamma_i - 1}^i = \xi_{\gamma_i}^i + [\boldsymbol{\varphi}_{\gamma_i - 1}^i(\xi_1^i, \ldots, \xi_{\gamma_i - 1}^i)]^{\mathrm{T}} \boldsymbol{w},$$
$$\dot{\xi}_{\gamma_i}^i = v_i + [\boldsymbol{\varphi}_{\gamma_i}^i(\xi_1^i, \ldots, \xi_{\gamma_i}^i)]^{\mathrm{T}} \boldsymbol{w}. \tag{A.6}$$

The new input v_i is designed to control a linear system subjected to disturbances with nonlinear coefficients. Following the backstepping procedure in Kristic, Kanellakopoulos, and Kokotovic (1995), error variables z_j^i and intermediate controls α_j^i are defined to aid in the expression of the control law v_i

$$z_j^i = \xi_j^i - \alpha_{j-1}^i(\xi_1^i, \ldots, \xi_{j-1}^i), \tag{A.7}$$

$$\alpha_j^i(\xi_1^i, \ldots, \xi_j^i) = -\delta_j^i z_j^i - z_{j-1}^i + \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}^i}{\partial \xi_k^i} \xi_{k+1}^i - e_j^i z_j^i \left| \boldsymbol{\varphi}_j^i - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}^i}{\partial \xi_k^i} \boldsymbol{\varphi}_k^i \right|^2, \tag{A.8}$$

where $z_0^i \equiv \alpha_0^i \equiv 0$, and $\delta_j^i, e_j^i,\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,\gamma_i$ are positive design parameters. The control law v_i is defined as $v_i = \alpha_{\gamma_i}^i(\xi_1^i,\ldots,\xi_{\gamma_i}^i)$. The derivative of the error variable is

$$\begin{aligned}
\dot{z}_j^i &= \xi_{j+1}^i + (\boldsymbol{\varphi}_{j+1}^i)^{\mathrm{T}}\boldsymbol{w} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}[\xi_{k+1}^i + (\boldsymbol{\varphi}_{k+1}^i)^{\mathrm{T}}\boldsymbol{w}] \\
&= \alpha_j^i + z_{j+1}^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\xi_{k+1}^i \\
&\quad + \left[\boldsymbol{\varphi}_{j+1}^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\boldsymbol{\varphi}_{k+1}^i\right]^{\mathrm{T}}\boldsymbol{w} \\
&= -\delta_j^i z_j^i - z_{j-1}^i + z_{j+1}^i \\
&\quad + \left[\boldsymbol{\varphi}_{j+1}^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\boldsymbol{\varphi}_{k+1}^i\right]^{\mathrm{T}}\boldsymbol{w} \\
&\quad - e_j^i z_j^i\left|\boldsymbol{\varphi}_j^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\boldsymbol{\varphi}_k^i\right|^2.
\end{aligned} \tag{A.9}$$

Define the quadratic Lyapunov function

$$V_{\gamma_i} = \frac{1}{2}\sum_{j=1}^{\gamma_i}(z_{j+1}^i)^2. \tag{A.10}$$

The derivative of (A.10) along the solutions of (A.9) is

$$\begin{aligned}
\dot{V}_{\gamma_i} &= \sum_{j=1}^{\gamma_i}\left[-\delta_j^i(z_j^i)^2 + z_j^i\left(\boldsymbol{\varphi}_{j+1}^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\boldsymbol{\varphi}_{k+1}^i\right)^{\mathrm{T}}\boldsymbol{w}\right. \\
&\quad \left. - e_j^i(z_j^i)^2\left|\boldsymbol{\varphi}_j^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\boldsymbol{\varphi}_k^i\right|^2\right] \\
&\leqslant \sum_{j=1}^{\gamma_i}\left[-\delta_j^i(z_j^i)^2 + |z_j^i|\left|\boldsymbol{\varphi}_{j+1}^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\boldsymbol{\varphi}_{k+1}^i\right|\|\boldsymbol{w}\|_\infty\right. \\
&\quad \left. - e_j^i(z_j^i)^2\left|\boldsymbol{\varphi}_j^i - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}^i}{\partial\xi_k^i}\,\boldsymbol{\varphi}_k^i\right|^2\right] \\
&\leqslant \sum_{j=1}^{\gamma_i}\left[-\delta_j^i(z_j^i)^2 + \frac{\|\boldsymbol{w}\|_\infty^2}{4e_j^i}\right].
\end{aligned} \tag{A.11}$$

This implies that $z_j^i(t)$ is globally uniformly bounded. Since α_j^i are smooth functions, by Eq. (A.7), $\boldsymbol{\xi}^i$ is globally uniformly bounded and the closed-loop system is input-to-state stable with respect to the disturbance $\boldsymbol{w}$. The controller is

$$\begin{aligned}
\boldsymbol{u} &= -[\boldsymbol{G}^*(\boldsymbol{x})]^{-1}\boldsymbol{f}^*(\boldsymbol{x}) \\
&\quad + [\boldsymbol{G}^*(\boldsymbol{x})]^{-1}[\alpha_{\gamma_1}^1 \quad \cdots \quad \alpha_{\gamma_m}^m]^{\mathrm{T}}.
\end{aligned} \tag{A.12}$$

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