Linearized Longitudinal Equations of Motion
Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2016

Learning Objectives

- 6th-order -> 4th-order -> hybrid equations
- Dynamic stability derivatives
- Long-period (phugoid) mode
- Short-period mode

Reading:
Flight Dynamics
452-464, 482-486

Review Questions

- How do numerical integration algorithms work?
- What’s the difference between a “nominal” and an “actual” trajectory?
- What assumptions must be made to derive a set of linear differential equations from a set of nonlinear equations?
- How is a nonlinear spring modeled?
- What is a “Jacobian matrix“?
- Does the point of linearization have to be fixed?
- How are the equations of motion divided into longitudinal and lateral-directional sets?
- Why is it useful to do this?
6th-Order Longitudinal Equations of Motion

Symmetric aircraft
Motions in the vertical plane
Flat earth

Nonlinear Dynamic Equations

\[
\begin{align*}
\dot{u} &= X / m - g \sin \theta - qw \\
\dot{w} &= Z / m + g \cos \theta + qu \\
\dot{x}_1 &= (\cos \theta) u + (\sin \theta) w \\
\dot{x}_2 &= (-\sin \theta) u + (\cos \theta) w \\
\dot{q} &= M / I_{xy} \\
\dot{\theta} &= q
\end{align*}
\]

State Vector, 6 components

\[
\begin{bmatrix}
u \\ w \\ x \\ z \\ q \\ \theta
\end{bmatrix} = \begin{bmatrix}
Axial Velocity \\
Vertical Velocity \\
Range \\
Altitude(–) \\
Pitch Rate \\
Pitch Angle
\end{bmatrix}
\]

\[
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6
\]

Range has no dynamic effect
Altitude effect is minimal (air density variation)

4th-Order Longitudinal Equations of Motion

Nonlinear Dynamic Equations, neglecting range and altitude

\[
\begin{align*}
\dot{u} &= f_1 = X / m - g \sin \theta - qw \\
\dot{w} &= f_2 = Z / m + g \cos \theta + qu \\
\dot{q} &= f_3 = M / I_{xy} \\
\dot{\theta} &= f_4 = q
\end{align*}
\]

State Vector, 4 components

\[
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix} = \begin{bmatrix}
Axial Velocity, m/s \\
Vertical Velocity, m/s \\
Pitch Rate, rad/s \\
Pitch Angle, rad
\end{bmatrix}
\]
Fourth-Order Hybrid Equations of Motion

Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components

\[
\begin{align*}
\dot{u} &= f_1 = \frac{X}{m} - g \sin \theta - qw \\
\dot{w} &= f_2 = \frac{Z}{m} + g \cos \theta + qu \\
\dot{q} &= f_3 = \frac{M}{I_{yy}} \\
\dot{\theta} &= f_4 = q
\end{align*}
\]

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    u \\
    w \\
    q \\
    \theta
\end{bmatrix}
\]

Axial Velocity
Vertical Velocity
Pitch Rate
Pitch Angle
Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components

$$\dot{V} = f_1 = \left[ T \cos(\alpha + i) - D - mg \sin \gamma \right]/m$$
$$\dot{\gamma} = f_2 = \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right]/mV$$
$$\dot{q} = f_3 = M / I_{yy}$$
$$\dot{\theta} = f_4 = q$$

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<thead>
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Velocity
Flight Path Angle
Pitch Rate
Pitch Angle

Hybrid Longitudinal Equations of Motion

• Replace pitch angle by angle of attack

$$\alpha = \theta - \gamma$$

$$\dot{V} = f_1 = \left[ T \cos(\alpha + i) - D - mg \sin \gamma \right]/m$$
$$\dot{\gamma} = f_2 = \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right]/mV$$
$$\dot{q} = f_3 = M / I_{yy}$$
$$\dot{\theta} = f_4 = q$$

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Velocity
Flight Path Angle
Pitch Rate
Pitch Angle
Hybrid Longitudinal Equations of Motion

- Replace pitch angle by angle of attack \( \alpha = \theta - \gamma \)

\[
\begin{align*}
\dot{V} &= f_1 = \frac{[T \cos(\alpha + i) - D - mg \sin \gamma]}{m} \\
\dot{\gamma} &= f_2 = \frac{[T \sin(\alpha + i) + L - mg \cos \gamma]}{mV} \\
\dot{q} &= f_3 = \frac{M}{I_{yy}} \\
\dot{\alpha} &= \dot{\theta} - \dot{\gamma} = q - f_2 = q - \frac{1}{mV} [T \sin(\alpha + i) + L - mg \cos \gamma]
\end{align*}
\]

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
\end{bmatrix} =
\begin{bmatrix}
    V \\
    \gamma \\
    q \\
    \alpha \\
\end{bmatrix} =
\begin{bmatrix}
    \text{Velocity} \\
    \text{Flight Path Angle} \\
    \text{Pitch Rate} \\
    \text{Angle of Attack} \\
\end{bmatrix}
\]

\( \theta = \alpha + \gamma \)

Why Transform Equations and State Vector?

Velocity and flight path angle typically have slow variations

Pitch rate and angle of attack typically have quicker variations

Coupling typically small
Small Perturbations from Steady Path

\[
\dot{x}(t) = \dot{x}_N(t) + \Delta \dot{x}(t) \\
\approx f[x_N(t), u_N(t), w_N(t), t] + F(t)\Delta x(t) + G(t)\Delta u(t) + L(t)\Delta w(t)
\]

Steady, Level Flight

\[
\dot{x}_N(t) \equiv 0 \approx f[x_N(t), u_N(t), w_N(t), t] \\
\Delta \dot{x}(t) \approx F\Delta x(t) + G\Delta u(t) + L\Delta w(t)
\]

Rates of change are “small”

Nominal Equations of Motion in Equilibrium (Trimmed Condition)

\[
\dot{x}_N(t) = 0 = f[x_N(t), u_N(t), w_N(t), t]
\]

\[
x^T_N = \begin{bmatrix} V_N & \gamma_N & 0 & \alpha_N \end{bmatrix}^T = \text{constant}
\]

\(T, D, L,\) and \(M\) contain state, control, and disturbance effects

\[
\dot{V}_N = 0 = f_1 = \left[ T \cos(\alpha_N + i) - D - mg \sin \gamma_N \right]/m \\
\dot{\gamma}_N = 0 = f_2 = \left[ T \sin(\alpha_N + i) + L - mg \cos \gamma_N \right]/mV_N \\
\dot{\alpha}_N = 0 = f_3 = M/I_{yy} \\
\dot{\alpha}_N = 0 = f_4 = (0) - \frac{1}{mV_N} \left[ T \sin(\alpha_N + i) + L - mg \cos \gamma_N \right]
\]

(See Supplemental Material for trimmed solution)
Small Perturbations from Steady Path Approximated by Linear Equations

Linearized Equations of Motion

\[
\Delta \dot{x}_{Lon} = \begin{bmatrix}
\Delta \dot{V} \\
\Delta \dot{\gamma} \\
\Delta \dot{q} \\
\Delta \dot{\alpha}
\end{bmatrix} = F_{Lon} \begin{bmatrix}
\Delta V \\
\Delta \gamma \\
\Delta q \\
\Delta \alpha
\end{bmatrix} + G_{Lon} \begin{bmatrix}
\Delta \delta T \\
\Delta \delta E \\
...\end{bmatrix} + \cdots
\]

Linearized Equations of Motion

Phugoid (Long-Period) Motion

Short-Period Motion
Approximate Decoupling of Fast and Slow Modes of Motion

Hybrid linearized equations allow the two modes to be examined separately.

\[ F_{Lon} = \begin{bmatrix} F_{Ph} & F_{Ph}^P \n \hline \hline \\ n & F_{SP} \n \hline \hline \\ F_{Ph} \n & F_{SP} \end{bmatrix} \]

Effects of **phugoid** perturbations on **phugoid** motion

Effects of **short-period** perturbations on **phugoid** motion

Effects of **phugoid** perturbations on **short-period** motion

Effects of **short-period** perturbations on **short-period** motion

\[ \begin{bmatrix} F_{Ph} \\ \hline \hline \\ \text{small} \n \hline \hline \\ F_{SP} \end{bmatrix} \begin{bmatrix} \text{small} \n \hline \hline \\ \text{small} \n \hline \hline \\ F_{SP} \end{bmatrix} \approx \begin{bmatrix} F_{Ph} \\ \hline \hline \\ 0 \n \hline \hline \\ F_{SP} \end{bmatrix} \]

Sensitivity Matrices for Longitudinal LTI Model

\[ \Delta x_{Lon}(t) = F_{Lon} \Delta x_{Lon}(t) + G_{Lon} \Delta u_{Lon}(t) + L_{Lon} \Delta w_{Lon}(t) \]

\[ F_{Lon} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \]

\[ G_{Lon} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \\ G_{41} & G_{42} & G_{43} \end{bmatrix} \]

\[ L_{Lon} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix} \]
Velocity Dynamics

First row of nonlinear equation

\[ \dot{V} = f_1 = \frac{1}{m} [T \cos \alpha - D - mg \sin \gamma] \]

\[ = \frac{1}{m} \left[ C_T \cos \alpha \frac{\rho V^2}{2} S - C_D \frac{\rho V^2}{2} S - mg \sin \gamma \right] \]

First row of linearized equation

\[ \Delta \dot{V}(t) = \left[ F_{11} \Delta V(t) + F_{12} \Delta \gamma(t) + F_{13} \Delta q(t) + F_{14} \Delta \alpha(t) \right] \]

\[ \quad + \left[ G_{11} \Delta \delta E(t) + G_{12} \Delta \delta T(t) + G_{13} \Delta \delta F(t) \right] \]

\[ \quad + \left[ L_{11} \Delta V_{\text{wind}} + L_{12} \Delta \alpha_{\text{wind}} \right] \]

Sensitivity of Velocity Dynamics to State Perturbations

\[ \dot{V} = \left[ (C_T \cos \alpha - C_D) \frac{\rho V^2}{2} S - mg \sin \gamma \right] / m \]

Coefficients in first row of \( F \)

\[ F_{11} = \frac{1}{m} \left[ (C_T \cos \alpha_N - C_D) \frac{\rho_N V_N^2}{2} S \right] \]

\[ + \left( C_T \sin \alpha_N + C_D \right) \frac{\rho_N V_N^2}{2} S \]

\[ F_{12} = -g \cos \gamma_N \]

\[ F_{13} = \frac{-1}{m} \left[ C_D \frac{\rho_N V_N^2}{2} S \right] \]

\[ F_{14} = \frac{-1}{m} \left[ (C_T \sin \alpha_N + C_D) \frac{\rho_N V_N^2}{2} S \right] \]
Sensitivity of Velocity Dynamics to Control and Disturbance Perturbations

Coefficients in first rows of $G$ and $L$

\[
G_{11} = \frac{-1}{m} \left[ C_{D_{\delta E}} \frac{\rho N V_N^2}{2} S \right]
\]

\[
G_{12} = \frac{1}{m} \left[ C_{T_{\delta T}} \cos \alpha_N \frac{\rho N V_N^2}{2} S \right]
\]

\[
G_{13} = \frac{-1}{m} \left[ C_{D_{\delta F}} \frac{\rho N V_N^2}{2} S \right]
\]

\[
L_{11} = - \frac{\partial f_1}{\partial V}
\]

\[
L_{12} = - \frac{\partial f_1}{\partial \alpha}
\]

\[
\frac{\partial C_{T_{\delta T}}}{\partial \delta T}
\]

\[
\frac{\partial C_{D_{\delta E}}}{\partial \delta E}
\]

\[
\frac{\partial C_{D_{\delta F}}}{\partial \delta F}
\]

Flight Path Angle Dynamics

Second row of nonlinear equation

\[
\dot{\gamma} = f_2 = \frac{1}{mV} \left[ T \sin \alpha + L - mg \cos \gamma \right]
\]

\[
= \frac{1}{mV} \left[ C_T \sin \alpha \frac{\rho V^2}{2} S + C_L \frac{\rho V^2}{2} S - mg \cos \gamma \right]
\]

Second row of linearized equation

\[
\Delta \dot{\gamma}(t) = \left[ F_{21} \Delta V(t) + F_{22} \Delta \gamma(t) + F_{23} \Delta q(t) + F_{24} \Delta \alpha(t) \right]
\]

\[
+ \left[ G_{21} \Delta \delta E(t) + G_{22} \Delta \delta T(t) + G_{23} \Delta \delta F(t) \right]
\]

\[
+ \left[ L_{21} \Delta V_{\text{wind}} + L_{22} \Delta \alpha_{\text{wind}} \right]
\]

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Sensitivity of Flight Path Angle Dynamics to State Perturbations

\[
\dot{\gamma} = \left( C_T \sin \alpha + C_L \right) \frac{\rho V^2}{2} S - mg \cos \gamma \right) / mV
\]

**Coefficients in second row of F**

\[
F_{21} = \frac{1}{mV_N} \left[ \left( C_T \sin \alpha_N + C_L \right) \frac{\rho N V^2}{2} S + \left( C_T \sin \alpha_N + C_L \right) \rho N V_N S \right]
\]

\[
- \frac{1}{mV_N^2} \left[ \left( C_T \sin \alpha_N + C_L \right) \frac{\rho N V^2}{2} S - mg \cos \gamma_N \right]
\]

\[
F_{22} = g \sin \gamma_N / V_N
\]

\[
F_{23} = \frac{1}{mV_N} \left[ C_{Lv} \frac{\rho N V^2}{2} S \right]
\]

\[
F_{24} = \frac{1}{mV_N} \left[ \left( C_{T_N} \cos \alpha_N + C_{L_N} \right) \frac{\rho N V^2}{2} S \right]
\]

**Pitch Rate Dynamics**

Third row of nonlinear equation

\[
\dot{q} = f_3 = \frac{M}{I_{yy}} = \frac{C_m \left( \rho V^2 / 2 \right) S \bar{c}}{I_{yy}}
\]

Third row of linearized equation

\[
\Delta \dot{q}(t) = \left[ F_{31} \Delta V(t) + F_{32} \Delta \gamma(t) + F_{33} \Delta q(t) + F_{34} \Delta \alpha(t) \right] + \left[ G_{31} \Delta \delta E(t) + G_{32} \Delta \delta T(t) + G_{33} \Delta \delta F(t) \right] + \left[ L_{31} \Delta V_{wind} + L_{32} \Delta \alpha_{wind} \right]
\]
Sensitivity of Pitch Rate Dynamics to State Perturbations

\[ \dot{q} = C_m \left( \frac{\rho V^2}{2} \right) \frac{S \bar{c}}{I_{yy}} \]

Coefficients in third row of \( F \)

\[ F_{31} = \frac{1}{I_{yy}} \left[ C_{m_v} \frac{\rho N V_n^2}{2} \bar{c} + C_{m_q} \rho N S \bar{c} \right] \]
\[ F_{32} = 0 \]
\[ F_{33} = \frac{1}{I_{yy}} \left[ C_{m_q} \frac{\rho N V_n^2}{2} \bar{c} \right] \]
\[ F_{34} = \frac{1}{I_{yy}} \left[ C_{m_u} \frac{\rho N V_n^2}{2} S \bar{c} \right] \]

Angle of Attack Dynamics

Fourth row of nonlinear equation

\[ \dot{\alpha} = f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV} \left[ T \sin \alpha + L - mg \cos \gamma \right] \]

Fourth row of linearized equation

\[ \Delta \dot{\alpha}(t) = \left[ F_{41} \Delta V(t) + F_{42} \Delta \gamma(t) + F_{43} \Delta q(t) + F_{44} \Delta \alpha(t) \right] \]
\[ + \left[ G_{41} \Delta \delta E(t) + G_{42} \Delta \delta T(t) + G_{43} \Delta \delta F(t) \right] \]
\[ + \left[ L_{41} \Delta V_{\text{wind}} + L_{42} \Delta \alpha_{\text{wind}} \right] \]
Sensitivity of Angle of Attack Dynamics to State Perturbations

\[ \dot{\alpha} = \dot{\theta} - \dot{\gamma} = q - \dot{\gamma} \]

Coefficients in fourth row of \( F \)

\[
\begin{align*}
F_{41} &= -F_{21} \quad & F_{43} &= 1 - F_{23} \\
F_{42} &= -F_{22} \quad & F_{44} &= -F_{24}
\end{align*}
\]

Alternative Approach:
Numerical Calculation of the Sensitivity Matrices ("1st Differences")

\[
\begin{align*}
\frac{\partial f_1}{\partial V}(t) &= \frac{1}{2\Delta V} \begin{bmatrix} (V + \Delta V) & (V - \Delta V) \end{bmatrix} - f_1 \begin{bmatrix} \gamma & \gamma \\
q & q \\
\alpha & \alpha \end{bmatrix} \\
\frac{\partial f_1}{\partial \gamma}(t) &= \frac{1}{2\Delta \gamma} \begin{bmatrix} V \\
(\gamma + \Delta \gamma) \\
q & q \\
\alpha & \alpha \end{bmatrix} - f_1 \begin{bmatrix} V \\
(\gamma - \Delta \gamma) \\
q & q \\
\alpha & \alpha \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f_2}{\partial V}(t) &= \frac{1}{2\Delta V} \begin{bmatrix} (V + \Delta V) & (V - \Delta V) \end{bmatrix} - f_2 \begin{bmatrix} \gamma & \gamma \\
q & q \\
\alpha & \alpha \end{bmatrix} \\
\frac{\partial f_2}{\partial \gamma}(t) &= \frac{1}{2\Delta \gamma} \begin{bmatrix} V \\
(\gamma + \Delta \gamma) \\
q & q \\
\alpha & \alpha \end{bmatrix} - f_2 \begin{bmatrix} V \\
(\gamma - \Delta \gamma) \\
q & q \\
\alpha & \alpha \end{bmatrix}
\end{align*}
\]

Remaining elements of \( F(t), G(t), \) and \( L(t) \) calculated accordingly
**Dimensional Stability and Control Derivatives**

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**Dimensional Stability-Derivative Notation**

- **Redefine force and moment symbols as acceleration symbols**
- **Dimensional stability derivatives portray acceleration sensitivities to state perturbations**

\[
\begin{align*}
\frac{\text{Drag}}{\text{mass} (m)} & \Rightarrow D \propto \ddot{V} \\
\frac{\text{Lift}}{\text{mass}} & \Rightarrow L \propto V\dot{\gamma} \\
\text{Moment} & \Rightarrow M \propto \dot{q}
\end{align*}
\]
Dimensional Stability-Derivative Notation

\[
F_{11} \equiv -D_V \triangleq \frac{1}{m} \left[ \left( C_{T_v} \cos \alpha_N - C_{D_v} \right) \rho_N V_N^2 \frac{S}{2} + \left( C_{T_N} \cos \alpha_N - C_{D_N} \right) \rho_N V_N S \right]
\]

Thrust and drag effects are combined and represented by one symbol

\[
F_{24} \equiv \frac{L_{\alpha}}{V_N} \triangleq \frac{1}{m V_N} \left[ \left( C_{T_N} \cos \alpha_N + C_{L_{\alpha}} \right) \rho_N V_N^2 \frac{S}{2} \right]
\]

Thrust and lift effects are combined and represented by one symbol

\[
F_{34} \equiv M_{\alpha} \triangleq \frac{1}{I_{yy}} \left[ C_{m_{\alpha}} \rho_N V_N^2 \frac{S c}{2} \right]
\]

Longitudinal Stability Matrix

\[
F_{Lon} = \begin{bmatrix}
F_{Ph}^P & F_{Ph}^{Sp} \\
F_{SP}^P & F_{SP} \end{bmatrix} = \begin{bmatrix}
-D_V & -g \cos \gamma_N & -D_q & -D_{\alpha} \\
\frac{L_v}{V_N} & \frac{g}{V_N} \sin \gamma_N & \frac{L_q}{V_N} & \frac{L_{\alpha}}{V_N} \\
M_v & 0 & M_q & M_{\alpha} \\
-\frac{L_v}{V_N} & -\frac{g}{V_N} \sin \gamma_N & \left( 1 - \frac{L_q}{V_N} \right) & -\frac{L_{\alpha}}{V_N} \\
\end{bmatrix}
\]

Effects of phugoid perturbations on phugoid motion

Effects of short-period perturbations on phugoid motion

Effects of phugoid perturbations on short-period motion

Effects of short-period perturbations on short-period motion
**Historical Factoids**

**Flying Cars-1**

- Curtiss Autocar, 1917
- Waterman Arrowbile, 1935
- Stout Skycar, 1931
- Consolidated/Vultee 111, 1940s
- ConvAIRCAR 116 (w/ Crosley auto), 1940s
- Taylor AirCar, 1950s
- Hallock Road Wing, 1957

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**Flying Cars-2**
“Mitzar” SkyMaster Pinto, 1970s

Lotus Elise Aerocar, concept, 2002

Haynes Skyblazer, concept, 2004

Aeromobil, 2014

Pinto separated from the airframe. Two killed.

Proposed, not built.

Aircraft entered a spin, ballistic parachute was deployed. No fatality.

Tecnam Astore

Jaguar F Type

PLUS

... or, for the same price
Comparison of 2\textsuperscript{nd}- and 4\textsuperscript{th}-
Order Model Response

4\textsuperscript{th}-Order Initial-Condition
Responses of Business Jet
at Two Time Scales

Plotted over different periods of time
4 initial conditions \([V(0), \gamma(0), q(0), \alpha(0)]\)
2nd-Order Models of Longitudinal Motion

Assume off-diagonal blocks of (4 x 4) stability matrix are negligible

\[ \mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \sim 0 \\ \sim 0 & \mathbf{F}_{SP} \end{bmatrix} \]

Approximate Phugoid Equation

\[ \Delta \mathbf{x}_{ph} = \begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -D_V & -g \cos \gamma_N \\ L_V/V_N & g/ \sin \gamma_N \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} T_{\delta t} \\ L_{\delta t}/V_N \end{bmatrix} \Delta \delta T + \begin{bmatrix} -D_V \\ L_V/V_N \end{bmatrix} \Delta V_{wind} \]

Approximate Short-Period Equation

\[ \Delta \mathbf{x}_{SP} = \begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha \\ (1 - L_q/V_N) & -L_\alpha/V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} M_\delta E/V_N \\ -L_\delta E/V_N \end{bmatrix} \Delta \delta E + \begin{bmatrix} M_\alpha/V_N \\ -L_\alpha/V_N \end{bmatrix} \Delta \alpha_{wind} \]
Comparison of Bizjet 4th- and 2nd-Order Model Responses

Phugoid Time Scale, ~100 s

4th Order, 4 initial conditions \([V(0), \alpha(0), q(0), \beta(0)]\)

2nd Order, 2 initial conditions \([V(0), \beta(0)]\)

Short-Period Time Scale, ~10 s

4th Order, 4 initial conditions \([V(0), \alpha(0), q(0), \alpha(0)]\)

2nd Order, 2 initial conditions \([q(0), \alpha(0)]\)
Approximate Phugoid Response to a 10% Thrust Increase

What is the steady-state response?

Approximate Short-Period Response to a 0.1-Rad Pitch Control Step Input

What is the steady-state response?
Normal Load Factor Response to a 0.1-Rad Pitch Control Step Input

• Normal load factor at the center of mass

\[ \Delta n_z = \frac{V_N}{g} (\Delta \dot{\alpha} - \Delta q) = \frac{V_N}{g} \left( \frac{L_{\alpha}}{V_N} \Delta \alpha + \frac{L_{\delta E}}{V_N} \Delta \delta E \right) \]

• Pilot focuses on normal load factor during rapid maneuvering

Next Time:
Lateral-Directional Dynamics

Learning Objectives

• 6\textsuperscript{th}-order → 4\textsuperscript{th}-order → hybrid equations
• Dynamic stability derivatives
• Dutch roll mode
• Roll and spiral modes
Supplemental Material

Trimmed Solution of the Equations of Motion
Flight Conditions for Steady, Level Flight

Nonlinear longitudinal model

\[ \dot{V} = f_1 = \frac{1}{m} \left[ T \cos(\alpha + i) - D - mg \sin \gamma \right] \]
\[ \dot{\gamma} = f_2 = \frac{1}{mV} \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right] \]
\[ \dot{\theta} = f_3 = \frac{M}{I_{yy}} \]
\[ \dot{\phi} = f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV} \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right] \]

Nonlinear longitudinal model in equilibrium

\[ 0 = f_1 = \frac{1}{m} \left[ T \cos(\alpha + i) - D - mg \sin \gamma \right] \]
\[ 0 = f_2 = \frac{1}{mV} \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right] \]
\[ 0 = f_3 = \frac{M}{I_{yy}} \]
\[ 0 = f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV} \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right] \]

Numerical Solution for Level Flight Trimmed Condition

- Specify desired altitude and airspeed, \( h_N \) and \( V_N \)
- Guess starting values for the trim parameters, \( \delta T_0, \delta E_0, \) and \( \theta_0 \)
- Calculate starting values of \( f_1, f_2, \) and \( f_3 \)

\[ f_1 = \dot{V} = \frac{1}{m} \left[ T (\delta T, \delta E, \theta, h, V) \cos(\alpha + i) - D (\delta T, \delta E, \theta, h, V) \right] \]
\[ f_2 = \dot{\gamma} = \frac{1}{mV_N} \left[ T (\delta T, \delta E, \theta, h, V) \sin(\alpha + i) + L (\delta T, \delta E, \theta, h, V) - mg \right] \]
\[ f_3 = \dot{\theta} = \frac{M(\delta T, \delta E, \theta, h, V)}{I_{yy}} \]

- \( f_1, f_2, \) and \( f_3 = 0 \) in equilibrium, but not for arbitrary \( \delta T_0, \delta E_0, \) and \( \theta_0 \)
- Define a scalar, positive-definite trim error cost function, e.g.,

\[ J(\delta T, \delta E, \theta) = a(f_1^2) + b(f_2^2) + c(f_3^2) \]
Minimize the Cost Function with Respect to the Trim Parameters

Error cost is “bowl-shaped”

\[ J(\delta T, \delta E, \theta) = a(f_1^2) + b(f_2^2) + c(f_3^2) \]

Cost is minimized at bottom of bowl, i.e., when

\[ \begin{pmatrix} \frac{\partial J}{\partial \delta T} & \frac{\partial J}{\partial \delta E} & \frac{\partial J}{\partial \theta} \end{pmatrix} = 0 \]

Search to find the minimum value of \( J \)

Example of Search for Trimmed Condition (Fig. 3.6-9, *Flight Dynamics*)

In MATLAB, use `fminsearch` to find trim settings

\[ (\delta T^*, \delta E^*, \theta^*) = \text{fminsearch}[J, (\delta T, \delta E, \theta)] \]
Elements of the Stability Matrix

Stability derivatives portray acceleration sensitivities to state perturbations

\[
\frac{\partial f_1}{\partial V} = -D_V; \quad \frac{\partial f_1}{\partial \gamma} = -g \cos \gamma_N; \quad \frac{\partial f_1}{\partial q} = -D_q; \quad \frac{\partial f_1}{\partial \alpha} = -D_\alpha
\]

\[
\frac{\partial f_2}{\partial V} = \frac{L_e}{V_N}; \quad \frac{\partial f_2}{\partial \gamma} = g \frac{1}{V_N} \sin \gamma_N; \quad \frac{\partial f_2}{\partial q} = \frac{L_q}{V_N}; \quad \frac{\partial f_2}{\partial \alpha} = \frac{L_\alpha}{V_N}
\]

\[
\frac{\partial f_3}{\partial V} = M_V; \quad \frac{\partial f_3}{\partial \gamma} = 0; \quad \frac{\partial f_3}{\partial q} = M_q; \quad \frac{\partial f_3}{\partial \alpha} = M_\alpha
\]

\[
\frac{\partial f_4}{\partial V} = -\frac{L_q}{V_N}; \quad \frac{\partial f_4}{\partial \gamma} = -\frac{g}{V_N} \sin \gamma_N; \quad \frac{\partial f_4}{\partial q} = 1 - \frac{L_q}{V_N}; \quad \frac{\partial f_4}{\partial \alpha} = -\frac{L_\alpha}{V_N}
\]

Control and Disturbance Sensitivities in Flight Path Angle, Pitch Rate, and Angle-of-Attack Dynamics

\[
\frac{\partial f_2}{\partial \delta E} = \frac{1}{mV_N} \left[ C_{l_{ae}} \frac{\rho V_N^2}{2} S \right]
\]

\[
\frac{\partial f_2}{\partial \delta T} = \frac{1}{mV_N} \left[ C_{l_{sr}} \cos \alpha_N \frac{\rho V_N^2}{2} S \right]
\]

\[
\frac{\partial f_2}{\partial \delta F} = \frac{1}{mV_N} \left[ C_{l_{sf}} \frac{\rho V_N^2}{2} S \right]
\]

\[
\frac{\partial f_3}{\partial \delta E} = \frac{1}{I_{yy}} \left[ C_{m_{ae}} \frac{\rho V_N^2}{2} S \right]
\]

\[
\frac{\partial f_3}{\partial \delta T} = \frac{1}{I_{yy}} \left[ C_{m_{sr}} \frac{\rho V_N^2}{2} S \right]
\]

\[
\frac{\partial f_3}{\partial \delta F} = \frac{1}{I_{yy}} \left[ C_{m_{sf}} \frac{\rho V_N^2}{2} S \right]
\]

\[
\frac{\partial f_4}{\partial \delta E} = \frac{\partial f_2}{\partial \delta E}
\]

\[
\frac{\partial f_4}{\partial \delta T} = \frac{\partial f_2}{\partial \delta T}
\]

\[
\frac{\partial f_4}{\partial \delta F} = \frac{\partial f_2}{\partial \delta F}
\]

\[
\frac{\partial f_2}{\partial \delta V} = \frac{\partial f_2}{\partial \delta V}
\]

\[
\frac{\partial f_2}{\partial \delta \alpha} = \frac{\partial f_2}{\partial \delta \alpha}
\]
Air compressibility effects are a principal source of velocity dependence.

\[
C_{DM} = \frac{\partial C_D}{\partial M} = \frac{\partial C_D}{\partial (V/a)} = a \frac{\partial C_D}{\partial V}
\]

\[a = \text{Speed of Sound}\]
\[M = \text{Mach number} = V/a\]

\[
C_{Dm} = \frac{\partial C_D}{\partial V} = \left(\frac{1}{a}\right) C_{Dm}
\]
\[
C_{Lm} = \frac{\partial C_L}{\partial V} = \left(\frac{1}{a}\right) C_{Lm}
\]
\[
C_{m_m} = \frac{\partial C_m}{\partial V} = \left(\frac{1}{a}\right) C_{m_m}
\]

**Wing Lift and Moment Coefficient Sensitivity to Pitch Rate**

**Straight-wing incompressible flow estimate (Etkin)**

\[
C_{L_{q_{wing}}} = -2C_{L_{q_{wing}}} \left( h_{cm} - 0.75 \right)
\]
\[
C_{m_{q_{wing}}} = -2C_{L_{q_{wing}}} \left( h_{cm} - 0.5 \right)^2
\]

**Straight-wing supersonic flow estimate (Etkin)**

\[
C_{L_{q_{wing}}} = -2C_{L_{q_{wing}}} \left( h_{cm} - 0.5 \right)
\]
\[
C_{m_{q_{wing}}} = -\frac{2}{3\sqrt{M^2-1}} - 2C_{L_{q_{wing}}} \left( h_{cm} - 0.5 \right)^2
\]

**Triangular-wing estimate (Bryson, Nielsen)**

\[
C_{L_{q_{wing}}} = \frac{2\pi}{3} C_{L_{q_{wing}}}
\]
\[
C_{m_{q_{wing}}} = -\frac{\pi}{3AR}
\]
Control- and Disturbance-Effect Matrices

- Control-effect derivatives portray acceleration sensitivities to control input perturbations

\[
G_{Lon} = \begin{bmatrix}
-D_{\delta E} & T_{ST} & -D_{\delta F} \\
L_{\delta E}/V_N & L_{\delta T}/V_N & L_{\delta F}/V_N \\
M_{\delta E} & M_{\delta T} & M_{\delta F} \\
-L_{\delta E}/V_N & -L_{\delta T}/V_N & -L_{\delta F}/V_N
\end{bmatrix}
\]

- Disturbance-effect derivatives portray acceleration sensitivities to disturbance input perturbations

\[
L_{Lon} = \begin{bmatrix}
-D_{\alpha\text{wind}} & -D_{\alpha\text{wind}} \\
L_{\alpha\text{wind}}/V_N & L_{\alpha\text{wind}}/V_N \\
M_{\alpha\text{wind}} & M_{\alpha\text{wind}} \\
-L_{\alpha\text{wind}}/V_N & -L_{\alpha\text{wind}}/V_N
\end{bmatrix}
\]

Primary Longitudinal Stability Derivatives

\[
D_v \triangleq -\frac{1}{m} \left[ (C_{T_v} - C_{D_v}) \frac{\rho V_N^2}{2} S + (C_{T_s} - C_{D_s}) \rho V_N S \right]
\]

\[
L_N/V_N = \frac{1}{mV_N} \left[ C_{L_v} \frac{\rho V_N^2}{2} S + C_{L_s} \rho V_N S \right] - \frac{1}{mV_N^2} \left[ C_{L_N} \frac{\rho V_N^2}{2} S - mg \right]
\]

\[
M_y = \frac{1}{I_{yy}} \left[ C_{m_y} \frac{\rho V_N^2}{2} S c \right] \quad \text{and} \quad M_{\alpha} = \frac{1}{I_{yy}} \left[ C_{m_{\alpha}} \frac{\rho V_N^2}{2} S c \right]
\]

\[
L_{\alpha}/V_N = \frac{1}{mV_N} \left[ (C_{T_N} + C_{L_s}) \frac{\rho V_N^2}{2} S \right]
\]

Small angle assumptions
Primary Phugoid Control Derivatives

\[ D_{\delta T} \approx -\frac{1}{m} \left[ C_{T_{\delta T}} \frac{\rho V_N^2}{2} S \right] \]
\[ L_{\delta F} / V_N \approx \frac{1}{m V_N} \left[ C_{L_{\delta F}} \frac{\rho V_N^2}{2} S \right] \]

Primary Short-Period Control Derivatives

\[ M_{\delta E} = C_{m_{\delta E}} \left( \frac{\rho_N V_N^2}{2 I_{yy}} \right) S\bar{c} \]
\[ L_{\delta E} / V = C_{L_{\delta E}} \left( \frac{\rho_N V_N^2}{2 m} \right) S \]
Flight Motions

Dornier Do-128 Short-Period Demonstration
http://www.youtube.com/watch?v=3hdLXE0rc9Q

Dornier Do-128 Phugoid Demonstration
http://www.youtube.com/watch?v=jzxtpQ30nLg&feature=related