Linearized Longitudinal Equations of Motion
Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2014

Learning Objectives

- 6th-order -> 4th-order -> hybrid equations
- Dynamic stability derivatives
- Long-period (phugoid) mode
- Short-period mode

Reading:
Flight Dynamics
452-464, 482-486
Airplane Stability and Control
Chapter 7

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http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

The Jets at an Awkward Age

Chapter 7, Airplane Stability and Control, Abzug and Larrabee

- What are the principal subject and scope of the chapter?
- What technical ideas are needed to understand the chapter?
- During what time period did the events covered in the chapter take place?
- What are the three main "takeaway" points or conclusions from the reading?
- What are the three most surprising or remarkable facts that you found in the reading?
**Longitudinal LTI Dynamics “Wordle”**

**6th-Order Longitudinal Equations of Motion**

- Symmetric aircraft
- Motions in the vertical plane
- Flat earth

**Nonlinear Dynamic Equations**

\[
\begin{align*}
\dot{u} &= \frac{X}{m} - g \sin \theta - qw \\
\dot{w} &= \frac{Z}{m} + g \cos \theta + qu \\
\dot{x}_1 &= (\cos \theta)u + (\sin \theta)w \\
\dot{x}_2 &= (-\sin \theta)u + (\cos \theta)w \\
\dot{q} &= \frac{M}{I_{yy}} \\
\dot{\theta} &= q
\end{align*}
\]

**State Vector, 6 components**

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix}
= \begin{bmatrix} x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \end{bmatrix}_{Long}
\]

\[
\begin{bmatrix}
u \\
w \\
x \\
z \\
q \\
\theta
\end{bmatrix}
= \begin{bmatrix} Axial Velocity \\
Vertical Velocity \\
Range \\
Altitude(–) \\
Pitch Rate \\
Pitch Angle \end{bmatrix}
\]
4\textsuperscript{th}-Order Longitudinal Equations of Motion

Nonlinear Dynamic Equations, neglecting range and altitude

\[
\begin{align*}
\dot{u} &= f_1 = \frac{X}{m} - g \sin \theta - qw \\
\dot{w} &= f_2 = \frac{Z}{m} + g \cos \theta + qu \\
\dot{q} &= f_3 = \frac{M}{I_{yy}} \\
\dot{\theta} &= f_4 = q
\end{align*}
\]

State Vector, 4 components

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = x_{Lon_4}
\]

\[
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} = Axial Velocity, m/s
\]

\[
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} = Vertical Velocity, m/s
\]

\[
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} = Pitch Rate, rad/s
\]

\[
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} = Pitch Angle, rad
\]

Fourth-Order Hybrid Equations of Motion
Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components
Replace \( X \) and \( Z \) by \( T \), \( D \), and \( L \)

\[
\begin{align*}
\dot{u} &= f_1 = \frac{X}{m} - g \sin \theta - qw \\
\dot{w} &= f_2 = \frac{Z}{m} + g \cos \theta + qu \\
\dot{q} &= f_3 = \frac{M}{I_{yy}} \\
\dot{\theta} &= f_4 = q
\end{align*}
\]

\[
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix} =
\begin{bmatrix}
u \\ w \\ q \\ \theta
\end{bmatrix}
\]

Axial Velocity
Vertical Velocity
Pitch Rate
Pitch Angle

\[
\dot{V} = f_1 = \frac{[T \cos(\alpha+i) - D - mg \sin \gamma]}{m} \\
\dot{\gamma} = f_2 = \frac{[T \sin(\alpha+i) + L - mg \cos \gamma]}{mV} \\
\dot{q} = f_3 = \frac{M}{I_{yy}} \\
\dot{\theta} = f_4 = q
\]

\[
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix} =
\begin{bmatrix}
V \\ \gamma \\ q \\ \theta
\end{bmatrix}
\]

Velocity
Flight Path Angle
Pitch Rate
Pitch Angle

\( i = \text{Incidence angle of the thrust vector with respect to the centerline} \)
Hybrid Longitudinal Equations of Motion

- Replace pitch angle by angle of attack \( \alpha = \theta - \gamma \)

\[
\begin{align*}
\dot{V} &= f_1 = \frac{T \cos(\alpha + i) - D - mg \sin \gamma}{m} \\
\dot{\gamma} &= f_2 = \frac{T \sin(\alpha + i) + L - mg \cos \gamma}{mV} \\
\dot{q} &= f_3 = \frac{M}{I_{yy}} \\
\dot{\theta} &= f_4 = q
\end{align*}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
V \\
\gamma \\
q \\
\theta
\end{bmatrix} =
\begin{bmatrix}
\text{Velocity} \\
\text{Flight Path Angle} \\
\text{Pitch Rate} \\
\text{Pitch Angle}
\end{bmatrix}
\]

Hybrid Longitudinal Equations of Motion

- Replace pitch angle by angle of attack \( \alpha = \theta - \gamma \)

\[
\begin{align*}
\dot{V} &= f_1 = \frac{T \cos(\alpha + i) - D - mg \sin \gamma}{m} \\
\dot{\gamma} &= f_2 = \frac{T \sin(\alpha + i) + L - mg \cos \gamma}{mV} \\
\dot{q} &= f_3 = \frac{M}{I_{yy}} \\
\dot{\theta} &= f_4 = q - \frac{1}{mV} \left[ T \sin(\alpha + i) + L - mg \cos \gamma \right]
\end{align*}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
V \\
\gamma \\
q \\
\alpha
\end{bmatrix} =
\begin{bmatrix}
\text{Velocity} \\
\text{Flight Path Angle} \\
\text{Pitch Rate} \\
\text{Angle of Attack}
\end{bmatrix}
\]

\( \theta = \alpha + \gamma \)
Why Transform Equations and State Vector?

- Velocity and flight path angle typically have slow variations
- Pitch rate and angle of attack typically have quicker variations

Small Perturbations from Steady Path Approximated by Linear Equations

\[
\begin{align*}
\dot{x}(t) &= \dot{x}_N(t) + \Delta \dot{x}(t) \\
&\approx f[x_N(t), u_N(t), w_N(t), t] + F(t) \Delta x(t) + G(t) \Delta u(t) + L(t) \Delta w(t)
\end{align*}
\]

**Steady, Level Flight**

\[
\begin{align*}
\dot{x}(t) &= 0 + \Delta \dot{x}(t) \\
&\approx f[x_N(t), u_N(t), w_N(t), t] + F \Delta x(t) + G \Delta u(t) + L \Delta w(t)
\end{align*}
\]

Rates of change are “small”
Nominal Equations of Motion in Equilibrium (Trimmed Condition)

\[
\begin{align*}
\dot{x}_N(t) &= 0 = f[x_N(t), u_N(t), w_N(t), t] \\
x_N^T &= \begin{bmatrix} V_N & \gamma_N & 0 & \alpha_N \end{bmatrix}^T = \text{constant}
\end{align*}
\]

\(T, D, L,\) and \(M\) contain state, control, and disturbance effects

\[
\begin{align*}
V_N &= 0 = f_1 = \left[ T \cos(\alpha_N + i) - D - mg \sin \gamma_N \right]/m \\
\gamma_N &= 0 = f_2 = \left[ T \sin(\alpha_N + i) + L - mg \cos \gamma_N \right]/mV_N \\
\dot{\alpha}_N &= 0 = f_3 = M/I_{yy} \\
\alpha_N &= 0 = f_4 = (0) - \left( \frac{1}{mV_N} \right) \left[ T \sin(\alpha_N + i) + L - mg \cos \gamma_N \right]
\end{align*}
\]

(See Supplemental Material for trimmed solution)

Small Perturbations from Steady Path Approximated by Linear Equations

**Linearized Equations of Motion**

\[
\begin{align*}
\Delta \dot{x}_{Lon} &= \begin{bmatrix} \Delta \dot{V} \\
\Delta \dot{\gamma} \\
\Delta \dot{q} \\
\Delta \dot{\alpha} \end{bmatrix} = F_{Lon} \begin{bmatrix} \Delta V \\
\Delta \gamma \\
\Delta q \\
\Delta \alpha \end{bmatrix} + G_{Lon} \begin{bmatrix} \Delta \delta T \\
\Delta \delta E \\
\ldots \end{bmatrix} + \ldots
\end{align*}
\]
Approximate Decoupling of Fast and Slow Modes of Motion

Hybrid linearized equations allow the two modes to be examined separately

\[
\begin{bmatrix}
F_{Lon} \\
F_{SP}
\end{bmatrix}
\approx
\begin{bmatrix}
F_{Ph} & 0 \\
0 & F_{SP}
\end{bmatrix}
\]

Effects of phugoid perturbations on phugoid motion
Effects of short-period perturbations on phugoid motion
Effects of phugoid perturbations on short-period motion
Effects of short-period perturbations on short-period motion
Sensitivity Matrices for Longitudinal LTI Model

\[ \Delta \dot{x}_{\text{Lon}}(t) = F_{\text{Lon}} \Delta x_{\text{Lon}}(t) + G_{\text{Lon}} \Delta u_{\text{Lon}}(t) + L_{\text{Lon}} \Delta w_{\text{Lon}}(t) \]

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial \gamma} & \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial \alpha}
\end{bmatrix} = F_{\text{Lon}}
\]

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial T} & \frac{\partial f_1}{\partial \delta E}
\end{bmatrix} = G_{\text{Lon}}
\]

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial V_{\text{wind}}} & \frac{\partial f_1}{\partial \alpha_{\text{wind}}}
\end{bmatrix} = L_{\text{Lon}}
\]

Velocity Dynamics

Nonlinear equation

\[
\dot{V} = f_1 = \frac{1}{m} \left[ T \cos \alpha - D - mg \sin \gamma \right]
\]

\[
= \frac{1}{m} \left[ C_T \cos \alpha \frac{\rho V^2}{2} - C_D \frac{\rho V^2}{2} S - mg \sin \gamma \right]
\]

First row of linearized dynamic equation

\[
\Delta \dot{V}(t) = \left[ \frac{\partial f_1}{\partial V} \Delta V(t) + \frac{\partial f_1}{\partial \gamma} \Delta \gamma(t) + \frac{\partial f_1}{\partial q} \Delta q(t) + \frac{\partial f_1}{\partial \alpha} \Delta \alpha(t) \right]
\]

\[
+ \left[ \frac{\partial f_1}{\partial E} \Delta \delta E(t) + \frac{\partial f_1}{\partial T} \Delta \delta T(t) + \frac{\partial f_1}{\partial \delta F} \Delta \delta F(t) \right]
\]

\[
+ \left[ \frac{\partial f_1}{\partial V_{\text{wind}}} \Delta V_{\text{wind}} + \frac{\partial f_1}{\partial \alpha_{\text{wind}}} \Delta \alpha_{\text{wind}} \right]
\]
Sensitivity of Velocity Dynamics to State Perturbations

\[ \dot{V} = \left[ (C_T \cos \alpha - C_D) \frac{\rho V^2}{2} S - mg \sin \gamma \right] / m \]

Coefficients in first row of \( F \)

\[ \frac{\partial f_1}{\partial V} = \frac{1}{m} \left( (C_T \cos \alpha_N - C_D) \frac{\rho_N V_N^2}{2} S + (C_T \sin \alpha_N + C_D) \frac{\rho_N V_N^2}{2} \right) \]

\[ \frac{\partial f_1}{\partial \gamma} = -\frac{1}{m} [mg \cos \gamma_N] = -g \cos \gamma_N \]

\[ \frac{\partial f_1}{\partial q} = -\frac{1}{m} \left[ C_D \frac{\rho_N V_N^2}{2} \right] \]

\[ \frac{\partial f_1}{\partial \alpha} = -\frac{1}{m} \left[ (C_T \sin \alpha_N + C_D) \frac{\rho_N V_N^2}{2} \right] \]

Sensitivity of Velocity Dynamics to Control and Disturbance Perturbations

Coefficients in first rows of \( G \) and \( L \)

\[ \frac{\partial f_1}{\partial \delta E} = -\frac{1}{m} \left[ C_D \frac{\rho_N V_N^2}{2} S \right] \]

\[ \frac{\partial f_1}{\partial \delta T} = \frac{1}{m} \left[ C_T \cos \alpha_N \frac{\rho_N V_N^2}{2} S \right] \]

\[ \frac{\partial f_1}{\partial \delta F} = -\frac{1}{m} \left[ C_D \frac{\rho_N V_N^2}{2} S \right] \]

\[ \frac{\partial f_1}{\partial V_{wind}} = -\frac{\partial f_1}{\partial V} \]

\[ \frac{\partial f_1}{\partial \alpha_{wind}} = -\frac{\partial f_1}{\partial \alpha} \]

\[ C_{Tv} = \frac{\partial C_T}{\partial V} \]

\[ C_{Tv} = \frac{\partial C_D}{\partial V} \]

\[ C_{Tv} = \frac{\partial C_D}{\partial q} \]

\[ C_{Tv} = \frac{\partial C_D}{\partial \alpha} \]
Flight Path Angle Dynamics

Nonlinear equation

\[
\dot{\gamma} = f_2 = \frac{1}{mV} [T \sin \alpha + L - mg \cos \gamma]
\]

\[
= \frac{1}{mV} \left[ C_T \sin \alpha \frac{\rho V^2}{2} S + C_L \frac{\rho V^2}{2} S - mg \cos \gamma \right]
\]

Second row of linearized equation

\[
\Delta \dot{\gamma}(t) = \left[ \frac{\partial f_2}{\partial V} \Delta V(t) + \frac{\partial f_2}{\partial \gamma} \Delta \gamma(t) + \frac{\partial f_2}{\partial q} \Delta q(t) + \frac{\partial f_2}{\partial \alpha} \Delta \alpha(t) \right]
\]

\[
+ \left[ \frac{\partial f_2}{\partial \delta E} \Delta \delta E(t) + \frac{\partial f_2}{\partial \delta T} \Delta \delta T(t) + \frac{\partial f_2}{\partial \delta F} \Delta \delta F(t) \right]
\]

\[
+ \left[ \frac{\partial f_2}{\partial \Delta V_{\text{wind}}} \Delta V_{\text{wind}} + \frac{\partial f_2}{\partial \Delta \alpha_{\text{wind}}} \Delta \alpha_{\text{wind}} \right]
\]

Sensitivity of Flight Path Angle Dynamics to State Perturbations

\[
\dot{\gamma} = \left[ (C_T \sin \alpha + C_L) \frac{\rho V^2}{2} S - mg \cos \gamma \right] / mV
\]

Coefficients in second row of \( F \)

\[
\frac{\partial f_2}{\partial V} = \frac{1}{mV} \left[ (C_T \sin \alpha_N + C_L) \frac{\rho N V^2}{2} S + (C_T \sin \alpha_N + C_L) \rho N V_N S \right]
\]

\[
- \frac{1}{mV^2} \left[ (C_T \sin \alpha_N + C_L) \frac{\rho N V^2}{2} S - mg \cos \gamma_N \right]
\]

\[
\frac{\partial f_2}{\partial \gamma} = \frac{1}{mV_N} \left[ mg \sin \gamma_N \right] = g \sin \gamma_N / V_N
\]

\[
\frac{\partial f_2}{\partial q} = \frac{1}{mV_N} \left[ C_L \frac{\rho N V^2}{2} S \right]
\]

\[
\frac{\partial f_2}{\partial \alpha} = \frac{1}{mV_N} \left[ (C_T \cos \alpha_N + C_L) \frac{\rho N V^2}{2} S \right]
\]

\[
C_T = \frac{\partial C_T}{\partial V}
\]

\[
C_L = \frac{\partial C_L}{\partial V}
\]

\[
C_T = \frac{\partial C_T}{\partial q}
\]

\[
C_L = \frac{\partial C_L}{\partial \alpha}
\]
Pitch Rate Dynamics

Nonlinear equation

\[ \dot{q} = f_3 = \frac{M}{I_{yy}} = \frac{C_m \left( \rho V^2 / 2 \right) SC}{I_{yy}} \]

Third row of linearized equation

\[
\Delta \dot{q}(t) = \left[ \frac{\partial f_3}{\partial V} \Delta V(t) + \frac{\partial f_3}{\partial \gamma} \Delta \gamma(t) + \frac{\partial f_3}{\partial q} \Delta q(t) + \frac{\partial f_3}{\partial \alpha} \Delta \alpha(t) \right] \\
+ \left[ \frac{\partial f_3}{\partial \delta E} \Delta \delta E(t) + \frac{\partial f_3}{\partial \delta T} \Delta \delta T(t) + \frac{\partial f_3}{\partial \delta F} \Delta \delta F(t) \right] \\
+ \left[ \frac{\partial f_3}{\partial V_{\text{wind}}} \Delta V_{\text{wind}} + \frac{\partial f_3}{\partial \alpha_{\text{wind}}} \Delta \alpha_{\text{wind}} \right]
\]

Sensitivity of Pitch Rate Dynamics to State Perturbations

\[ \dot{q} = C_m \left( \rho V^2 / 2 \right) SC / I_{yy} \]

Coefficients in third row of F

\[
\frac{\partial f_3}{\partial V} = 1 \left[ \frac{C_{m_v} \rho_N V_N^2}{2} SC + C_{m_q} \rho_N V_N SC \right]
\]

\[
\frac{\partial f_3}{\partial \gamma} = 0
\]

\[
\frac{\partial f_3}{\partial q} = 1 \left[ \frac{C_{m_q} \rho_N V_N^2}{2} SC \right]
\]

\[
\frac{\partial f_3}{\partial \alpha} = 1 \left[ \frac{C_{m_\alpha} \rho_N V_N^2}{2} SC \right]
\]
Angle of Attack Dynamics

Nonlinear equation

\[ \dot{\alpha} = f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV} \left[ T \sin \alpha + L - mg \cos \gamma \right] \]

Fourth row of linearized equation

\[
\Delta \dot{\alpha}(t) = \left[ \frac{\partial f_4}{\partial V} \Delta V(t) + \frac{\partial f_4}{\partial \gamma} \Delta \gamma(t) + \frac{\partial f_4}{\partial q} \Delta q(t) + \frac{\partial f_4}{\partial \alpha} \Delta \alpha(t) \right] \\
+ \left[ \frac{\partial f_4}{\partial \delta E} \Delta \delta E(t) + \frac{\partial f_4}{\partial \delta T} \Delta \delta T(t) + \frac{\partial f_4}{\partial \delta F} \Delta \delta F(t) \right] \\
+ \left[ \frac{\partial f_4}{\partial V_{\text{wind}}} \Delta V_{\text{wind}} + \frac{\partial f_4}{\partial \alpha_{\text{wind}}} \Delta \alpha_{\text{wind}} \right]
\]

Sensitivity of Angle of Attack Dynamics to State Perturbations

\[ \dot{\alpha} = \dot{\theta} - \dot{\gamma} = q - \dot{\gamma} \]

Coefficients in fourth row of F

\[ \frac{\partial f_4}{\partial V} = - \frac{\partial f_2}{\partial V} \]
\[ \frac{\partial f_4}{\partial q} = 1 - \frac{\partial f_2}{\partial q} \]
\[ \frac{\partial f_4}{\partial \gamma} = - \frac{\partial f_2}{\partial \gamma} \]
\[ \frac{\partial f_4}{\partial \alpha} = - \frac{\partial f_2}{\partial \alpha} \]
**Alternative Approach:**
Numerical Calculation of the Sensitivity Matrices ("1st Differences")

\[
\begin{align*}
\frac{\partial f_1}{\partial \gamma} (t) &= \frac{1}{2\Delta V} \begin{bmatrix} (V + \Delta V) \\ (V - \Delta V) \end{bmatrix} \frac{\partial f_1}{\partial q} (t) &= \frac{1}{2\Delta V} \begin{bmatrix} \gamma \\ q \end{bmatrix} \frac{\partial f_1}{\partial \alpha} (t) &= \frac{1}{2\Delta V} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \\
\frac{\partial f_1}{\partial \gamma} (t) &= \frac{1}{2\Delta V} \begin{bmatrix} V \\ (\gamma + \Delta \gamma) \end{bmatrix} \frac{\partial f_1}{\partial q} (t) &= \frac{1}{2\Delta V} \begin{bmatrix} q \\ \gamma \end{bmatrix} \frac{\partial f_1}{\partial \alpha} (t) &= \frac{1}{2\Delta V} \begin{bmatrix} \alpha \\ V \end{bmatrix}
\end{align*}
\]

Remaining elements of \( F(t), G(t), \) and \( L(t) \) calculated accordingly

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**Current Events**
SpaceShipTwo Accident
October 31, 2014
Flight Profile of SpaceShipOne (Precursor to SpaceShipTwo)

SpaceShipTwo Cutaway Diagram
Probable Cause, as of 11/3/14
Premature Feathering at M = 1

Chris Hart, ‘68, ‘70
Acting Chairman, NTSB

SpaceShipTwo Accident
October 31, 2014

Probable Cause, as of 11/3/14
Premature Feathering at M = 1

Chris Hart, ‘68, ‘70
Acting Chairman, NTSB

SpaceShipOne
Ansari X Prize, December 17, 2003

Brian Binnie, ‘78
Pilot, Astronaut
Interest in space tourism is growing, as several companies compete to offer suborbital rides into space for paying customers. The door was opened on October 4, 2004 when Mojave Aerospace Ventures SpaceShipOne (Fig. 1) flew higher than 100 km for the second time in less than three weeks, winning the Ansari X-Prize (http://en.wikipedia.org/wiki/SpaceShipOne). Princeton alumnus, Brian Binnie, was at the controls for the award-winning flight.

Figure 1. SpaceShipOne, in flight, and with astronaut/test pilot Brian Binnie, MAE 478.

This week's assignment is to simulate SpaceShipOne's flight. There are two major parts to the assignment. First, you will develop a longitudinal aerodynamic, inertial, and thrust model for the aircraft. Then, you will calculate the flight trajectory (Fig. 2) using point-mass longitudinal equations of motion. The dynamic equations are similar to those of Assignment #2, but are modified to include thrust, to portray the vehicle in conventional and "feathered" re-entry configuration over a range of angles of attack and Mach numbers, and to account for altitude-dependent variations in gravitational acceleration and atmospheric properties.
**SpaceShipOne State Histories**

![Graphs showing Velocity, Flight Path Angle, Altitude, and Range histories for different angles and times.]

**SpaceShipOne Dynamic Pressure and Mach Number Histories**

![Graphs showing Dynamic Pressure and Mach Number histories over time.]
Dimensional Stability-Derivative Notation

- **Redefine force and moment symbols as acceleration symbols**
- **Dimensional stability derivatives portray acceleration sensitivities to state perturbations**

\[
\begin{align*}
\text{Drag} & \quad \Rightarrow \quad D \propto \dot{V} \\
\text{mass (m)} & \\
\text{Lift} & \quad \Rightarrow \quad L \propto V\dot{\gamma} \\
\text{mass} & \\
\text{Moment} & \quad \Rightarrow \quad M \propto \dot{q} \\
\text{moment of inertia (}I_{yy}\text{)} & 
\end{align*}
\]
Dimensional Stability-Derivative Notation

\[
\frac{\partial f_1}{\partial V} = -D_v \triangleq \frac{1}{m} \left[ \left( C_{T_v} \cos \alpha_N - C_{D_v} \right) \frac{\rho_N V_N^2}{2} - S + \left( C_{T_N} \cos \alpha_N - C_{D_N} \right) \rho_N V_N S \right]
\]

Thrust and drag effects are combined and represented by one symbol.

\[
\frac{\partial f_2}{\partial \alpha} = \frac{L_\alpha / V_N \triangleq \frac{1}{mV_N} \left[ \left( C_{T_N} \cos \alpha_N + C_{L_\alpha} \right) \frac{\rho_N V_N^2}{2} S \right]}
\]

Thrust and lift effects are combined and represented by one symbol.

\[
\frac{\partial f_3}{\partial \alpha} \triangleq M_\alpha \triangleq \frac{1}{I_{yy}} \left[ C_{m_\alpha} \frac{\rho_N V_N^2}{2} S \right]
\]

Longitudinal Stability Matrix

\[
F_{Lon} = \begin{bmatrix}
F_{Ph} & F_{SP}^{Ph} \\
F_{SP}^{Ph} & F_{SP}
\end{bmatrix}
\]

Effects of phugoid perturbations on phugoid motion

Effects of short-period perturbations on short-period motion
Comparison of Fourth- and Second-Order Dynamic Models

4th-Order Initial-Condition Responses of Business Jet at Two Time Scales

Plotted over different periods of time
4 initial conditions \([V(0), \gamma(0), q(0), \alpha(0)]\)
Approximate Phugoid Equation

\[ \Delta \mathbf{x}_{ph} = \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} = \begin{bmatrix} -D_V & -g \cos \gamma_N \\ L_v / V_N & g / V_N \sin \gamma_N \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} T_{\delta T} \\ L_{\delta T} / V_N \end{bmatrix} \Delta \delta T + \begin{bmatrix} -D_V \\ L_v / V_N \end{bmatrix} \Delta V_{wind} \]
Comparison of Bizjet 4th- and 2nd-Order Model Responses

Phugoid Time Scale, \(\sim\)100 s

4th Order, 4 initial conditions 
\([V(0), \gamma(0), q(0), \alpha(0)]\)

2nd Order, 2 initial conditions 
\([V(0), \gamma(0)]\)

Short-Period Time Scale, \(\sim\)10 s

4th Order, 4 initial conditions 
\([V(0), \gamma(0), q(0), \alpha(0)]\)

2nd Order, 2 initial conditions 
\([q(0), \alpha(0)]\)
Approximate Phugoid Response to a 10% Thrust Increase

What is the steady-state response?

Approximate Short-Period Response to a 0.1-Rad Pitch Control Step Input

What is the steady-state response?
Normal Load Factor Response to a 0.1-Rad Pitch Control Step Input

- Normal load factor at the center of mass

\[ n_z = \frac{V_N}{g} (\Delta \dot{\alpha} - \Delta q) = \frac{V_N}{g} \left( \frac{L_{\alpha}}{V_N} \Delta \alpha + \frac{L_{\delta E}}{V_N} \Delta \delta E \right) \]

- Pilot focuses on normal load factor during rapid maneuvering

Next Time:
Lateral-Directional Dynamics

Reading:
*Flight Dynamics*
574–591
Supplemental Material

Trimmed Solution of the Equations of Motion
Flight Conditions for Steady, Level Flight

Nonlinear longitudinal model

\[ \dot{V} = f_1 = \frac{1}{m}[T \cos(\alpha + i) - D - mg \sin \gamma] \]
\[ \dot{\gamma} = f_2 = \frac{1}{mV}[T \sin(\alpha + i) + L - mg \cos \gamma] \]
\[ \dot{q} = f_3 = \frac{M}{I_{yy}} \]
\[ \dot{\alpha} = f_4 = \ddot{\gamma} - q = -\frac{1}{mV}[T \sin(\alpha + i) + L - mg \cos \gamma] \]

Nonlinear longitudinal model in equilibrium

\[ 0 = f_1 = \frac{1}{m}[T \cos(\alpha + i) - D - mg \sin \gamma] \]
\[ 0 = f_2 = \frac{1}{mV}[T \sin(\alpha + i) + L - mg \cos \gamma] \]
\[ 0 = f_3 = \frac{M}{I_{yy}} \]
\[ 0 = f_4 = \ddot{\gamma} - q = -\frac{1}{mV}[T \sin(\alpha + i) + L - mg \cos \gamma] \]

Numerical Solution for Level Flight Trimmed Condition

- Specify desired altitude and airspeed, \( h_N \) and \( V_N \)
- Guess starting values for the trim parameters, \( \delta T_0, \delta E_0, \) and \( \theta_0 \)
- Calculate starting values of \( f_1, f_2, \) and \( f_3 \)

\[ f_1 = \frac{1}{m}[T(\delta T, \delta E, \theta, h, V) \cos(\alpha + i) - D(\delta T, \delta E, \theta, h, V)] \]
\[ f_2 = \frac{1}{mV_N}[T(\delta T, \delta E, \theta, h, V) \sin(\alpha + i) + L(\delta T, \delta E, \theta, h, V) - mg] \]
\[ f_3 = \frac{M(\delta T, \delta E, \theta, h, V)}{I_{yy}} \]

- \( f_1, f_2, \) and \( f_3 = 0 \) in equilibrium, but not for arbitrary \( \delta T_0, \delta E_0, \) and \( \theta_0 \)
- Define a scalar, positive-definite trim error cost function, e.g.,

\[ J(\delta T, \delta E, \theta) = a(f_1^2) + b(f_2^2) + c(f_3^2) \]
Minimize the Cost Function with Respect to the Trim Parameters

Error cost is “bowl-shaped”

\[ J(\delta T, \delta E, \theta) = a\left(f_1^2\right) + b\left(f_2^2\right) + c\left(f_3^2\right) \]

Cost is minimized at bottom of bowl, i.e., when

\[ \begin{bmatrix} \frac{\partial J}{\partial \delta T} & \frac{\partial J}{\partial \delta E} & \frac{\partial J}{\partial \theta} \end{bmatrix} = 0 \]

Search to find the minimum value of \( J \)

Example of Search for Trimmed Condition (Fig. 3.6-9, Flight Dynamics)

In MATLAB, use \texttt{fminsearch} to find trim settings

\[ (\delta T^*, \delta E^*, \theta^*) = \text{fminsearch}[J,(\delta T, \delta E, \theta)] \]
Elements of the Stability Matrix

Stability derivatives portray acceleration sensitivities to state perturbations

\[ \frac{\partial f_1}{\partial V} = -D_V; \quad \frac{\partial f_1}{\partial \gamma} = -g \cos \gamma; \quad \frac{\partial f_1}{\partial q} = -D_q; \quad \frac{\partial f_1}{\partial \alpha} = -D_\alpha \]

\[ \frac{\partial f_2}{\partial V} = \frac{L_v}{V_N}; \quad \frac{\partial f_2}{\partial \gamma} = \frac{g}{V_N} \sin \gamma; \quad \frac{\partial f_2}{\partial q} = \frac{L_q}{V_N}; \quad \frac{\partial f_2}{\partial \alpha} = \frac{L_\alpha}{V_N} \]

\[ \frac{\partial f_3}{\partial V} = M_V; \quad \frac{\partial f_3}{\partial \gamma} = 0; \quad \frac{\partial f_3}{\partial q} = M_q; \quad \frac{\partial f_3}{\partial \alpha} = M_\alpha \]

\[ \frac{\partial f_4}{\partial V} = \frac{L_v}{V_N}; \quad \frac{\partial f_4}{\partial \gamma} = -\frac{g}{V_N} \sin \gamma; \quad \frac{\partial f_4}{\partial q} = 1 - \frac{L_q}{V_N}; \quad \frac{\partial f_4}{\partial \alpha} = -\frac{L_\alpha}{V_N} \]

Control and Disturbance Sensitivities in Flight Path Angle, Pitch Rate, and Angle-of-Attack Dynamics

\[ \frac{\partial f_2}{\partial \delta E} = \frac{1}{mV_N} \left[ C_{l_{ae}} \frac{\rho V_N^2}{2} S \right] \]

\[ \frac{\partial f_2}{\partial \delta T} = \frac{1}{mV_N} \left[ C_{I_{ae}} \sin \alpha \frac{\rho V_N^2}{2} S \right] \]

\[ \frac{\partial f_2}{\partial \delta F} = \frac{1}{mV_N} \left[ C_{l_{af}} \frac{\rho V_N^2}{2} S \right] \]

\[ \frac{\partial f_3}{\partial \delta E} = \frac{1}{I_{yy}} \left[ C_{m_{ae}} \frac{\rho V_N^2}{2} S \right] \]

\[ \frac{\partial f_3}{\partial \delta T} = \frac{1}{I_{yy}} \left[ C_{m_{st}} \frac{\rho V_N^2}{2} S \right] \]

\[ \frac{\partial f_3}{\partial \delta F} = \frac{1}{I_{yy}} \left[ C_{m_{af}} \frac{\rho V_N^2}{2} S \right] \]

\[ \frac{\partial f_4}{\partial \delta E} = \frac{\partial f_2}{\partial \delta E}; \quad \frac{\partial f_4}{\partial \delta T} = \frac{\partial f_2}{\partial \delta T}; \quad \frac{\partial f_4}{\partial \delta F} = \frac{\partial f_2}{\partial \delta F} \]
Velocity-Dependent Derivative Definitions

Air compressibility effects are a principal source of velocity dependence

\[ C_{DM} \equiv \frac{\partial C_D}{\partial M} = \frac{\partial C_D}{\partial (V/a)} = a \frac{\partial C_D}{\partial V} \]

\( a = \) Speed of Sound

\( M = \) Mach number = \( V/\alpha \)

\[ C_{D_L} = \frac{\partial C_D}{\partial V} = \left( \frac{1}{\alpha} \right) C_{D_L} \]

\[ C_{L_{\alpha}} = \frac{\partial C_L}{\partial \alpha} = \left( \frac{1}{\alpha} \right) C_{L_{\alpha}} \]

\[ C_{m_{\alpha}} = \frac{\partial C_{m_{\alpha}}}{\partial \alpha} = \left( \frac{1}{\alpha} \right) C_{m_{\alpha}} \]

Wing Lift and Moment Coefficient Sensitivity to Pitch Rate

Straight-wing incompressible flow estimate (Etkin)

\[ C_{L_{\alpha}}^{\text{avg}} = -2C_{L_{\alpha}} \left( h_{\text{cm}} - 0.75 \right) \]

\[ C_{m_{\alpha}}^{\text{avg}} = -2C_{L_{\alpha}} \left( h_{\text{cm}} - 0.5 \right)^2 \]

Straight-wing supersonic flow estimate (Etkin)

\[ C_{L_{\alpha}}^{\text{avg}} = -2C_{L_{\alpha}} \left( h_{\text{cm}} - 0.5 \right) \]

\[ C_{m_{\alpha}}^{\text{avg}} = \frac{2}{3\sqrt{M^2 - 1}} - 2C_{L_{\alpha}} \left( h_{\text{cm}} - 0.5 \right)^2 \]

Triangular-wing estimate (Bryson, Nielsen)

\[ C_{L_{\alpha}}^{\text{avg}} = \frac{2\pi}{3} C_{L_{\alpha}}^{\text{avg}} \]

\[ C_{m_{\alpha}}^{\text{avg}} = -\frac{\pi}{3AR} \]
Control- and Disturbance-Effect Matrices

- Control-effect derivatives portray acceleration sensitivities to control input perturbations

\[
G_{Lon} = \begin{bmatrix}
-D_{\delta E} & T_{\delta T} & -D_{\delta F} \\
L_{\delta E} / V_N & L_{\delta T} / V_N & L_{\delta F} / V_N \\
M_{\delta E} & M_{\delta T} & M_{\delta F} \\
-L_{\delta E} / V_N & -L_{\delta T} / V_N & -L_{\delta F} / V_N
\end{bmatrix}
\]

- Disturbance-effect derivatives portray acceleration sensitivities to disturbance input perturbations

\[
L_{Lon} = \begin{bmatrix}
-D_{\ell_{\text{wind}}} & -D_{\alpha_{\text{wind}}} \\
L_{\ell_{\text{wind}}} / V_N & L_{\alpha_{\text{wind}}} / V_N \\
M_{\ell_{\text{wind}}} & M_{\alpha_{\text{wind}}} \\
-L_{\ell_{\text{wind}}} / V_N & -L_{\alpha_{\text{wind}}} / V_N
\end{bmatrix}
\]

Primary Longitudinal Stability Derivatives

\[
D_v \triangleq \frac{-1}{m} \left[ (C_{T_v} - C_{D_v}) \frac{\rho V_N^2}{2} S + (C_{T_\alpha} - C_{D_\alpha}) \rho V_N S \right]
\]

\[
L_{\ell_{V_N}} = \frac{1}{m V_N} \left[ C_{\ell_{V_N}} \frac{\rho V_N^2}{2} S + C_{\alpha_{V_N}} \rho V_N S \right] - \frac{1}{m V_N^2} \left[ C_{\alpha_{V_N}} \frac{\rho V_N^2}{2} S - mg \right]
\]

\[
M_q = \frac{1}{I_{xy}} \left[ C_{m_q} \frac{\rho V_N^2}{2} SC \right]
\]

\[
M_{\alpha} = \frac{1}{I_{yy}} \left[ C_{m_{\alpha}} \frac{\rho V_N^2}{2} SC \right]
\]

\[
L_{\alpha_{V_N}} = \frac{1}{m V_N} \left[ (C_{T_\alpha} + C_{L_\alpha}) \frac{\rho V_N^2}{2} S \right]
\]

Small angle assumptions
Primary Phugoid Control Derivatives

\[ D_{\delta T} = \frac{-1}{m} \left[ C_{T_{\delta T}} \frac{\rho V_N^2}{2} S \right] \]

\[ L_{\delta F} \frac{\delta F}{V_N} = \frac{1}{mV_N} \left[ C_{L_{\delta F}} \frac{\rho V_N^2}{2} S \right] \]

Primary Short-Period Control Derivatives

\[ M_{\delta E} = C_{m_{\delta E}} \left( \frac{\rho_N V_N^2}{2I_{yy}} \right) S \bar{c} \]

\[ L_{\delta E} \frac{\delta E}{V} = C_{L_{\delta E}} \left( \frac{\rho_N V_N^2}{2m} \right) S \]
Flight Motions

Simulator Demonstration of Short-Period Response to Elevator Deflection
http://www.youtube.com/watch?v=1Q7ZqBSO_B8

Simulator Demonstration of Phugoid Response
http://www.youtube.com/watch?v=DEOGM_9NGTI

Dornier Do-128 Short-Period Demonstration
http://www.youtube.com/watch?v=JhLXE0rc9Q

Dornier Do-128 Phugoid Demonstration
http://www.youtube.com/watch?v=jzxpQJ0nLg&feature=related