Root Locus Analysis of Parameter Variations and Feedback Control
Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2016

Learning Objectives
• Effects of system parameter variations on modes of motion
• Root locus analysis
  – Evans’s rules for construction
  – Application to longitudinal dynamic models

Reading:
Flight Dynamics
357-361, 465-467, 488-490, 509-514

Review Questions

- What is a Laplace transform?
- What are some characteristics of Laplace transforms?
- What is the significance of \((sI - F)^{-1}\)?
- Why is the partial fraction expansion of \((sI - F)^{-1}\) important?
- What is the dimension of the transfer function matrix with 4 outputs and 3 inputs?
- Why must every complex root (or eigenvalue) be accompanied by its complex conjugate?
- Why is it bad for roots to lie in the right half of the \(s\) plane plot?
- Of what use is a Bode plot?
- What are some important rules for constructing a Bode plot?
Characteristic Equation

\[ \Delta x(s) = [sI - F]^{-1}[\Delta x(0) + G \Delta u(s) + L \Delta w(s)] \]

\[ [sI - F]^{-1} = \frac{\text{Adj}(sI - F)}{|sI - F|} = \frac{C^T(s)}{|sI - F|} (n \times n) \quad (1 \times 1) \]

Characteristic equation defines the modes of motion

\[ |sI - F| = \Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \]
\[ = (s - \lambda_1)(s - \lambda_2)(\ldots)(s - \lambda_n) = 0 \]

Recall: \( s \) is a complex variable

\[ s = \sigma + j\omega \]

Real Roots

- On real axis in \( s \) Plane
- Represents convergent or divergent time response
- Time constant, \( \tau = -1/\lambda = -1/\mu, \text{ sec} \)

\[ \lambda_i = \mu_i \quad \text{(Real number)} \]
\[ x(t) = x(0)e^{\mu t} \]
Complex Roots

- Occur only in complex-conjugate pairs
- Represent oscillatory modes
- Natural frequency, \( \omega_n \), and damping ratio, \( \zeta \), as shown

\[
\lambda_1 = \mu_1 + j\nu_1 = -\zeta \omega_n + j\omega_n \sqrt{1-\zeta^2}
\]

\[
\lambda_2 = \mu_2 + j\nu_2 = \mu_1 - j\nu_1 \equiv \lambda_1^*
\]

\[
= -\zeta \omega_n - j\omega_n \sqrt{1-\zeta^2}
\]

Natural Frequency, Damping Ratio, and Damped Natural Frequency

\[
(s - \lambda_1)(s - \lambda_1^*) = [s - (\mu_1 + j\nu_1)][s - (\mu_1 - j\nu_1)]
\]

\[
= s^2 - 2\mu_1 s + (\mu_1^2 + \nu_1^2) \equiv s^2 + 2\zeta \omega_n s + \omega_n^2
\]

\[
\mu_1 = -\zeta \omega_n = -1/\text{Time constant}
\]

\[
(\mu_1^2 + \nu_1^2)^{1/2} = \omega_n = \text{Natural frequency}
\]

\[
\nu_1 = \omega_n \sqrt{1-\zeta^2} \equiv \omega_n \text{damped} = \text{Damped natural frequency}
\]
Longitudinal Characteristic Equation

How do the roots vary when we change $M_a$?

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$= s^4 + \left( D_V + \frac{L_\alpha}{V_N} - M_q \right) s^3$$

$$+ \left[ \left( g - D_\alpha \right) \frac{L_V}{V_N} + D_V \left( \frac{L_\alpha}{V_N} - M_q \right) - M_q \frac{L_\alpha}{V_N} - M_a \right] s^2$$

$$+ \left\{ M_q \left[ \left( D_\alpha - g \right) \frac{L_v}{V_N} - D_V L_\alpha + D_\alpha M_V - D_\alpha M_a \right] \right\} s$$

$$+ g \left( M_V L_\alpha - M_\alpha \frac{L_V}{V_N} \right) = 0$$

with $L_q = D_q = 0$

... or pitch-rate damping, $M_q$?

$$\Delta_{Lon}(s) = s^4 + \left( D_V + \frac{L_\alpha}{V_N} - M_q \right) s^3$$

$$+ \left[ \left( g - D_\alpha \right) \frac{L_V}{V_N} + D_V \left( \frac{L_\alpha}{V_N} - M_q \right) - M_q \frac{L_\alpha}{V_N} - M_a \right] s^2$$

$$+ \left\{ M_q \left[ \left( D_\alpha - g \right) \frac{L_v}{V_N} - D_V L_\alpha + D_\alpha M_V - D_\alpha M_a \right] \right\} s$$

$$+ g \left( M_V L_\alpha - M_\alpha \frac{L_V}{V_N} \right) = 0$$
Evans’s Rules for Root Locus Analysis

Walter R. Evans (1920-1999)

Root Locus Analysis of Parametric Effects on Aircraft Dynamics

- Parametric variations alter eigenvalues of F
- Graphical technique for finding the roots as a parameter ("gain") varies

**Locus**: “the set of all points whose location is determined by stated conditions”
Parameters of Characteristic Polynomial

\[ \Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \]
\[ = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) \]
\[ = \left( s^2 + 2\zeta_p \omega_{n_p} s + \omega_{n_p}^2 \right) \left( s^2 + 2\zeta_p \omega_{n_p} s + \omega_{n_p}^2 \right) = 0 \]

Break the polynomial into 2 parts to examine the effect of a single parameter

\[ \Delta(s) = d(s) + k n(s) = 0 \]

Effect of \( a_0 \) Variation on Longitudinal Root Location

Example: Root locus gain, \( k = a_0 \)

\[ \Delta_{Lon}(s) = \left[ s^4 + a_3 s^3 + a_2 s^2 + a_1 s \right] + k \frac{\Delta}{d(s) + kn(s)} \]
\[ = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) = 0 \]

\( d(s) \): Polynomial in \( s \), degree = \( n \); there are \( n \) (\( = 4 \)) poles

\( n(s) \): Polynomial in \( s \), degree = \( q \); there are \( q \) (\( = 0 \)) zeros

\[ d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s \]
\[ = \left[ s - (0) \right](s - \lambda'_2)(s - \lambda'_3)(s - \lambda'_4); \text{ four poles, one at 0} \]
\[ n(s) = 1; \text{ no zeros} \]
**Effect of \( a_1 \) Variation on Longitudinal Root Location**

**Example: Root locus gain, \( k = a_1 \)**

\[
\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + k s + a_0 \triangleq d(s) + k n(s)
\]

\[
= (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) = 0
\]

- \( d(s) \): Polynomial in \( s \), degree = \( n \); there are \( n \) (\( = 4 \)) poles
- \( n(s) \): Polynomial in \( s \), degree = \( q \); there are \( q \) (\( = 1 \)) zeros

\[
d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_0
\]

\[
= (s - \lambda'_1)(s - \lambda'_2)(s - \lambda'_3)(s - \lambda'_4); \text{ four poles}
\]

\[
n(s) = s; \text{ one zero at 0}
\]

**Three Equivalent Equations for Evaluating Locations of Roots**

\[
\Delta(s) = d(s) + k n(s) = 0
\]

\[
1 + k \frac{n(s)}{d(s)} = 0
\]

\[
k \frac{n(s)}{d(s)} = -1 = (1)e^{\pm j \pi \text{(rad)}} = (1)e^{\pm j180^\circ}
\]
Evan’s Root Locus Criterion

All points on the locus of roots must satisfy the equation

\[ k \frac{n(s)}{d(s)} = -1 \]

i.e., all points on root locus must have phase angle = ±180°

\[ k \frac{n(s)}{d(s)} = (1)e^{±j180°} \]

Longitudinal Equation Example

Typical flight condition

\[ \Delta_{Lon}(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 \]

\[ = s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145 \]

\[ = \left[ s^2 + 2(0.0678)0.124s + (0.124)^2 \right] \left[ s^2 + 2(0.411)3.1s + (3.1)^2 \right] = 0 \]
Effect of $a_0$ Variation

$\Delta_{Lon}(s) = s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145 = 0$

$k = a_0$

$\Delta(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$

$= s\left(s^3 + a_3s^2 + a_2s + a_1\right) + k$

$= s(s + 0.21)\left(s^2 + 2.55s + 9.62\right) + k$

Rearrange

$$\frac{k}{s(s + 0.21)\left(s^2 + 2.55s + 9.62\right)} = -1$$

Effect of $a_1$ Variation

$k = a_1$

$\Delta(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$

$= s^4 + a_3s^3 + a_2s^2 + ks + a_0$

$= \left(s^4 + a_3s^3 + a_2s^2 + a_0\right) + ks$

$= \left[s^2 - 0.00041s + 0.015\right]s^2 + 2.57s + 9.67 + ks$

Rearrange

$$\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]s^2 + 2.57s + 9.67} = -1$$
Origins of Roots \((k = 0)\)

\[ n \text{ poles of } d(s) \]

\[ \Delta(s) = d(s) + kn(s) \xrightarrow[k \to 0]{} d(s) \]

\[
\frac{k}{s(s + 0.21)} \left[ s^2 + 2.55s + 9.62 \right] = -1
\]

\[
\frac{ks}{s^2 - 0.00041s + 0.015} \left[ s^2 + 2.57s + 9.67 \right] = -1
\]

Destinations of Roots as \(k\) Becomes Large

1) \(q\) roots go to the zeros of \(n(s)\)

\[
\frac{d(s) + kn(s)}{k} = \frac{d(s)}{k} + k n(s) \xrightarrow[k \to \pm \infty]{} n(s) = (s - z_1)(s - z_2) \cdots
\]

2) \((n - q)\) roots go to infinity

\[
\left[ \frac{d(s) + kn(s)}{n(s)} \right] = \left[ \frac{d(s)}{n(s)} + k \right] \xrightarrow[k \to \pm R \to \pm \infty]{} \left[ \frac{s^n}{s^q + k} \right] \to s^{(n-q)} \pm R \to \pm \infty
\]
Destinations of Roots for Large $k$

\[
k \frac{s(s + 0.21)[s^2 + 2.55s + 9.62]}{-1}
\]

\[
k_5 \frac{(s^2 - 0.00041s + 0.015)[s^2 + 2.57s + 9.67]}{-1}
\]

(n – q) Roots Approach Asymptotes as $k \rightarrow \pm \infty$

Asymptote angles for positive $k$

\[
\theta(rad) = \frac{\pi + 2m\pi}{n - q}, \quad m = 0, 1, \ldots, (n - q) - 1
\]

Asymptote angles for negative $k$

\[
\theta(rad) = \frac{2m\pi}{n - q}, \quad m = 0, 1, \ldots, (n - q) - 1
\]
Origin of Asymptotes = “Center of Gravity”

(Sum of real parts of poles minus sum of real parts of zeros) ÷ (# of poles minus # of zeros)

\[
\text{"c.g."} = \frac{\sum_{i=1}^{n} \sigma_{\lambda_i} - \sum_{j=1}^{q} \sigma_{z_j}}{n - q}
\]

Root Locus on Real Axis

- **Locus on real axis**
  - \( k > 0 \): Any segment with **odd** number of poles and zeros to the right on the axis
  - \( k < 0 \): Any segment with **even** number of poles and zeros to the right on the axis
Roots for Positive and Negative Variations of $k = a_0$

\[
\frac{k}{s(s + 0.21)[s^2 + 2.55s + 9.62]} = -1
\]

Roots for Positive and Negative Variations of $k = a_1$

\[
\frac{ks}{[s^2 - 0.00041s + 0.015][s^2 + 2.57s + 9.67]} = -1
\]
Root Locus Analysis of Simplified Longitudinal Modes

Approximate Phugoid Model

2nd-order equation

\[
\Delta \dot{x}_{Ph} = \begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -D_v & -g \\ L_v/V_N & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} T_{\delta T} \\ L_{\delta T}/V_N \end{bmatrix} \Delta \delta T
\]

Characteristic polynomial

\[
|sI - F_{Ph}| = \det(sI - F_{Ph}) = \\
\Delta(s) = s^2 + D_v s + gL_v/V_N \\
= s^2 + 2\zeta \omega_n s + \omega_n^2
\]

\[
\omega_n = \sqrt{gL_v/V_N} \\
\zeta = \frac{D_v}{2\sqrt{gL_v/V_N}}
\]
Effect of Airspeed on Natural Frequency and Period

Neglecting compressibility effects

\[ g \frac{L_N}{V_N} \approx \frac{g}{m} \left[ C_{L_N} \rho_N S \right] \]
\[ = \frac{2g}{mV_N^2} \left[ C_{L_N} \frac{1}{2} \rho_N V_N^2 S \right] = \frac{2g}{mV_N^2} [mg] = \frac{2g^2}{V_N^2} \]

\[ \omega_n \approx \sqrt{\frac{2}{V_N}} g \frac{13.87}{V_N(m/s)}, \text{rad/s} \]
\[ \text{Period, } T = \frac{2\pi}{\omega_n} \approx 0.45V_N(m/s), \text{sec} \]

Effect of L/D on Damping Ratio

Neglecting compressibility effects and thrust sensitivity to velocity, \( T_V \)

\[ D_V \approx \frac{1}{m} \left[ C_{D_N} \rho_N V_N S \right] \]
\[ = \frac{C_{D_N} \rho_N V_N^2 S}{2\sqrt{2} \ g \ N} = \frac{1}{\sqrt{2}} \left( \frac{C_{D_N}}{C_{t_g}} \right) \]
\[ \zeta = \frac{D_V}{2g \ N / V_N} \approx \frac{1}{\sqrt{2} (L/D)_N} \]

<table>
<thead>
<tr>
<th>Natural</th>
<th>Damping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Frequency</td>
</tr>
<tr>
<td>m/s</td>
<td>rad/s</td>
</tr>
<tr>
<td>50</td>
<td>0.28</td>
</tr>
<tr>
<td>100</td>
<td>0.14</td>
</tr>
<tr>
<td>200</td>
<td>0.07</td>
</tr>
<tr>
<td>400</td>
<td>0.035</td>
</tr>
</tbody>
</table>
Effect of $L/V_N$ Variation

\[ k = gL/V_N \]

\[ \Delta(s) = \left( s^2 + D_V s \right) + k \]

\[ = \left[ s \left( s + D_V \right) \right] + [k] \]

Change in damped natural frequency

\[ \omega_{n_{damped}} \triangleq \omega_n \sqrt{1 - \xi^2} \]

Effect of $D_V$ Variation

\[ k = D_V \]

\[ \Delta(s) = \left( s^2 + gL + V_N \right) + ks \]

\[ = \left[ \left( s + j\sqrt{gL} / V_N \right) \left( s - j\sqrt{gL} / V_N \right) \right] + [ks] \]

Change in damping ratio

\[ \xi \]
Approximate Short-Period Model

Approximate Short-Period Equation \((L_q = 0)\)

\[
\Delta \tilde{x}_s = \begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha/V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -L_{\delta E}/V_N \end{bmatrix} \Delta \delta E
\]

Characteristic polynomial

\[
\Delta(s) = s^2 + \left( \frac{L_\alpha}{V_N} - M_q \right) s - \left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right) = s^2 + 2\zeta \omega_n s + \omega_n^2
\]

\[
\omega_n = \sqrt{\left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right)}; \quad \zeta = \frac{\left( \frac{L_\alpha}{V_N} - M_q \right)}{2 \sqrt{\left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right)}}
\]

Effect of \(M_\alpha\) on Approximate Short-Period Roots

\[
k = M_\alpha
\]

\[
\Delta(s) = s^2 + \left( \frac{L_\alpha}{V_N} - M_q \right) s - \left( M_q \frac{L_\alpha}{V_N} \right) - k = 0
\]

\[
= \left( s + \frac{L_\alpha}{V_N} \right) \left( s - M_q \right) - k = 0
\]

Change in damped natural frequency
Effect of $M_q$ on Approximate Short-Period Roots

$$k = M_q$$

Change primarily in damping ratio

$$\Delta(s) = s^2 + \frac{L_\alpha}{V_N} s - M_\alpha - M_q \left( s + \frac{L_\alpha}{V_N} \right)$$

Effect of $L_\alpha/V_N$ on Approximate Short-Period Roots

$$k = L_\alpha/V_N$$

• Change primarily in damping ratio

$$\Delta(s) = s^2 - M_q s - M_\alpha + \frac{L_\alpha}{V_N} \left( s - M_q \right)$$

$$= \left( s + \frac{M_q}{2} - \sqrt{\left( \frac{M_q}{2} \right)^2 + M_\alpha} \right) \left( s + \frac{M_q}{2} + \sqrt{\left( \frac{M_q}{2} \right)^2 + M_\alpha} \right) + k \left( s - M_q \right) = 0$$
Flight Control Systems

SAS = Stability Augmentation System

Effect of Scalar Feedback Control on Roots of the System

\[ \Delta y(s) = H'(s) \Delta u(s) = \frac{kn(s)}{d(s)} \Delta u(s) = \frac{kn(s)}{d(s)} K \Delta \epsilon(s) \]

\[ = K H'(s) \left[ \Delta y_c(s) - \Delta y(s) \right] \]

\[ \Delta y(s) = K H'(s) \Delta y_c(s) - K H'(s) \Delta y(s) \]
Scalar Closed-Loop Transfer Function

\[ [1 + K \mathcal{H}(s)] \Delta y(s) = K \mathcal{H}(s) \Delta y_c(s) \]

\[ \frac{\Delta y(s)}{\Delta y_c(s)} = \frac{K \mathcal{H}(s)}{[1 + K \mathcal{H}(s)]} \]

Roots of the Closed-Loop Control System

\[ \Delta y(s) = \frac{K \frac{kn(s)}{d(s)}}{[1 + K \frac{kn(s)}{d(s)}]} = \frac{K kn(s)}{d(s) + K kn(s)} = \frac{K kn(s)}{\Delta_{\text{closed loop}}(s)} \]

Closed-loop roots are solutions to

\[ \Delta_{\text{closed loop}}(s) = d(s) + K kn(s) = 0 \]

or

\[ K \frac{kn(s)}{d(s)} = -1 \]
Pitch Rate Feedback to Elevator

\[ K \mathcal{H}(s) = K \frac{\Delta q(s)}{\Delta \delta E(s)} = \frac{k^q_{\delta E} \left( s - z^q_{\delta E} \right)}{s^2 + 2\zeta_{SP} \omega_{nSP} s + \omega_{nSP}^2} = -1 \]

- \# of roots = 2
- \# of zeros = 1
- Destinations of roots (for \( k = \pm\infty \)):
  - 1 root goes to zero of \( n(s) \)
  - 1 root goes to infinite radius
- Angles of asymptotes, \( \theta \), for the roots going to \( \infty \)
  - \( K \to +\infty \): –180 deg
  - \( K \to -\infty \): 0 deg

Pitch Rate Feedback to Elevator

- “Center of gravity” on real axis
- Locus on real axis
  - \( K > 0 \): Segment to the left of the zero
  - \( K < 0 \): Segment to the right of the zero

Feedback effect is analogous to changing \( M_q \)
Next Time: Advanced Longitudinal Dynamics

Learning Objectives
Angle-of-attack-rate aero effects
Fourth-order dynamics
Steady-state response to control
Transfer functions
Frequency response
Root locus analysis of parameter variations
Nichols chart
Pilot-aircraft interactions

Supplemental Material
Corresponding 2\textsuperscript{nd}-Order Initial Condition Response

Same envelopes for displacement and rate

\begin{align*}
x_1(t) &= Ae^{-\xi_0 t} \sin[\omega_n \sqrt{1-\zeta^2} t + \varphi] \\
x_2(t) &= Ae^{-\xi_0 t} \left[ \omega_n \sqrt{1-\zeta^2} \right] \cos\left[ \omega_n \sqrt{1-\zeta^2} t + \varphi \right]
\end{align*}

Multi-Modal LTI Responses Superpose Individual Modal Responses

- With distinct roots, \((n = 4)\) for example, \textit{partial fraction expansion} for each state element is

\[ \Delta x_i(s) = \frac{d_{1i}}{(s - \lambda_1)} + \frac{d_{2i}}{(s - \lambda_2)} + \frac{d_{3i}}{(s - \lambda_3)} + \frac{d_{4i}}{(s - \lambda_4)}, \quad i = 1, 4 \]

Corresponding 4\textsuperscript{th}-order time response is

\[ \Delta x_i(t) = d_{1i} e^{\lambda_1 t} + d_{2i} e^{\lambda_2 t} + d_{3i} e^{\lambda_3 t} + d_{4i} e^{\lambda_4 t}, \quad i = 1, 4 \]
Magnitudes of Roots Going to Infinity

\[ s^{(n-q)} = R e^{-j180^\circ} \rightarrow \infty \quad \text{or} \quad R e^{-j360^\circ} \rightarrow -\infty \]

Magnitudes of roots are the same for given \( k \)

Angles from the origin are different

Asymptotes of Roots (for \( k \rightarrow \pm\infty \))

4 roots to infinite radius

Asymptotes = \( \pm45^\circ, \pm135^\circ \)

3 roots to infinite radius

Asymptotes = \( \pm60^\circ, -180^\circ \)
Summary of Root Locus Concepts

Effects of Airspeed, Altitude, Mass, and Moment of Inertia on Fighter Aircraft Short Period

Airspeed variation at constant altitude

<table>
<thead>
<tr>
<th>Airspeed m/s</th>
<th>Dynamic Pressure P</th>
<th>Angle of Attack deg</th>
<th>Natural Frequency rad/s</th>
<th>Period sec</th>
<th>Damping Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>91</td>
<td>2540</td>
<td>14.6</td>
<td>1.34</td>
<td>4.7</td>
<td>0.3</td>
</tr>
<tr>
<td>152</td>
<td>7040</td>
<td>5.8</td>
<td>2.3</td>
<td>2.74</td>
<td>0.31</td>
</tr>
<tr>
<td>213</td>
<td>13790</td>
<td>3.2</td>
<td>3.21</td>
<td>1.96</td>
<td>0.3</td>
</tr>
<tr>
<td>274</td>
<td>22790</td>
<td>2.2</td>
<td>3.84</td>
<td>1.64</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Mass variation at constant altitude

<table>
<thead>
<tr>
<th>Mass Variation %</th>
<th>Natural Frequency rad/s</th>
<th>Period sec</th>
<th>Damping Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50</td>
<td>2.4</td>
<td>2.62</td>
<td>0.44</td>
</tr>
<tr>
<td>0</td>
<td>2.3</td>
<td>2.74</td>
<td>0.31</td>
</tr>
<tr>
<td>50</td>
<td>2.25</td>
<td>2.78</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Altitude variation with constant dynamic pressure

<table>
<thead>
<tr>
<th>Airspeed m/s</th>
<th>Altitude m</th>
<th>Natural Frequency rad/s</th>
<th>Period sec</th>
<th>Damping Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>122</td>
<td>2235</td>
<td>2.36</td>
<td>2.67</td>
<td>0.39</td>
</tr>
<tr>
<td>152</td>
<td>6095</td>
<td>2.3</td>
<td>2.74</td>
<td>0.31</td>
</tr>
<tr>
<td>213</td>
<td>11915</td>
<td>2.24</td>
<td>2.8</td>
<td>0.23</td>
</tr>
<tr>
<td>274</td>
<td>16260</td>
<td>2.18</td>
<td>2.88</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Moment of inertia variation at constant altitude

<table>
<thead>
<tr>
<th>Moment of Inertia Variation %</th>
<th>Natural Frequency rad/s</th>
<th>Period sec</th>
<th>Damping Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50</td>
<td>3.25</td>
<td>1.94</td>
<td>0.33</td>
</tr>
<tr>
<td>0</td>
<td>2.3</td>
<td>2.74</td>
<td>0.31</td>
</tr>
<tr>
<td>50</td>
<td>1.87</td>
<td>3.35</td>
<td>0.31</td>
</tr>
</tbody>
</table>
Wind Shear Encounter

- Inertial Frames
  - Earth-Relative
  - Wind-Relative (Constant Wind)
- Non-Inertial Frames
  - Body-Relative
  - Wind-Relative (Varying Wind)

Pitch Angle and Normal Velocity Frequency Response to Axial Wind

- Pitch angle resonance at phugoid natural frequency
- Normal velocity (~ angle of attack) resonance at phugoid and short period natural frequencies

MacRuer, Ashkenas, and Graham, 1973
Pitch Angle and Normal Velocity Frequency Response to Vertical Wind

- Pitch angle resonance at phugoid and short period natural frequencies
- Normal velocity (~ angle of attack) resonance

\[ \frac{\Delta \theta(j\omega)}{V_n \Delta \alpha_{\text{wind}}(j\omega)} \]

MacRuer, Ashkenas, and Graham, 1973

Microbursts

1/2-3-km-wide "Jet" impinges on surface

High-speed outflow from jet core

Ring vortex forms in outflow

Outflow strong enough to knock down trees

http://en.wikipedia.org/wiki/Microburst
The Insidious Nature of Microburst Encounter

The wavelength of the phugoid mode and the disturbance input are comparable.

Importance of Proper Response to Microburst Encounter

- Stormy evening July 2, 1994
- USAir Flight 1016, Douglas DC-9, Charlotte
- Windshear alert issued as 1016 began descent along glideslope

- DC-9 encountered 61-kt windshear, executed missed approach
- Go-around procedure begun correctly -- aircraft's nose rotated up -- but power was not advanced
- Together with increasing tailwind, aircraft stalled
- Crew lowered nose to eliminate stall, but descent rate increased, causing ground impact
- Plane continued to descend, striking trees and telephone poles before impact
Optimal Flight Path Through Worst JAWS Profile

- Graduate research of Mark Psiaki
- Joint Aviation Weather Study (JAWS) measurements of microbursts (Colorado High Plains, 1983)
- Negligible deviation from intended path using available controllability
- Aircraft has sufficient performance margins to stay on the flight path

Optimal and 15° Pitch Angle Recovery during Microburst Encounter

Graduate Research of Sandeep Mulgund

Altitude vs. Time

Airspeed vs. Time

Angle of Attack vs. Time

FAA Windshear Training Aid, 1987, addresses proper operating procedures for suspected windshear