Cruising Flight Performance
Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2016

Learning Objectives
- Definitions of airspeed
- Performance parameters
- Steady cruising flight conditions
- Breguet range equations
- Optimize cruising flight for minimum thrust and power
- Flight envelope

Review Questions
- What is “static margin”?
- Is the airplane’s pitching moment sensitivity to angle of attack linear?
- What factors are most important in defining the airplane’s pitching moment sensitivity to angle of attack?
- Which is more important: stability or control?
- Is the airplane’s yawing moment sensitivity to sideslip angle linear?
- What effect does the wing dihedral angle have on airplane stability?
- Why would an airplane have a “twin tail”?
- What are “ventral fins”, and why do airplanes have/not have them?
U.S. Standard Atmosphere, 1976

Dynamic Pressure and Mach Number

\[ \rho = \text{air density}, \text{ function of height} \]
\[ = \rho_{\text{sealevel}} e^{-\beta h} \]
\[ a = \text{speed of sound} \]
\[ = \text{linear function of height} \]

Dynamic pressure \( \bar{q} = \rho \frac{V^2}{2} \)
Mach number \( V/a \)
Definitions of Airspeed

Airspeed is speed of aircraft measured with respect to air mass

Airspeed = Inertial speed if wind speed = 0

• Indicated Airspeed (IAS)

\[
\text{IAS} = \sqrt{2 \left( \frac{P_{\text{stagnation}} - P_{\text{ambient}}}{\rho_{\text{SL}}} \right)} = \sqrt{\frac{2 \left( P_{\text{total}} - P_{\text{static}} \right)}{\rho_{\text{SL}}}}
\]

\( \triangleq \sqrt{\frac{2q_{c}}{\rho_{\text{SL}}}} \), with \( q_{c} \triangleq \text{impact pressure} \)

• Calibrated Airspeed (CAS)*

\[
\text{CAS} = \text{IAS corrected for instrument and position errors} = \sqrt{\frac{2(q_{c})_{\text{corr} \#1}}{\rho_{\text{SL}}}}
\]


Definitions of Airspeed

Airspeed is speed of aircraft measured with respect to air mass

Airspeed = Inertial speed if wind speed = 0

Equivalent Airspeed (EAS)*

\[
\text{EAS} = \text{CAS corrected for compressibility effects} = \sqrt{\frac{2(q_{c})_{\text{corr} \#2}}{\rho_{\text{SL}}}}
\]

True Airspeed (TAS)*

\[
V \triangleq TAS = EAS \sqrt{\frac{\rho_{\text{SL}}}{\rho(z)}} = IAS_{\text{corrected}} \sqrt{\frac{\rho_{\text{SL}}}{\rho(z)}}
\]

Mach number

\[
M = \frac{TAS}{a}
\]

Checklist

- IAS?
- CAS?
- EAS?
- TAS?
- M?
Longitudinal Variables

Longitudinal Point-Mass Equations of Motion

- Assume thrust is aligned with the velocity vector (small-angle approximation for $\alpha$
- Mass = constant

\[
\begin{align*}
\dot{V} &= \frac{(C_T \cos\alpha - C_D) \frac{1}{2} \rho V^2 S - mg \sin \gamma}{m} \\
\dot{V} &\approx \frac{(C_T - C_D) \frac{1}{2} \rho V^2 S - mg \sin \gamma}{m} \\
\dot{h} &= -\dot{z} = -v_z = V \sin \gamma \\
\dot{r} &= \dot{x} = v_x = V \cos \gamma
\end{align*}
\]

$V =$ velocity = Earth-relative airspeed
$\rho =$ True airspeed with zero wind
$\gamma =$ flight path angle
$h =$ height (altitude)
r = range
Conditions for Steady, Level Flight

- Flight path angle = 0
- Altitude = constant
- Airspeed = constant
- Dynamic pressure = constant

\[
0 = \frac{(C_T - C_D) \frac{1}{2} \rho V^2 S}{m}
\]

\[
0 = \frac{C_L \frac{1}{2} \rho V^2 S - mg}{mV}
\]

\[\dot{h} = 0\]
\[\ddot{r} = V\]

- Thrust = Drag
- Lift = Weight

Power and Thrust

Propeller

\[
Power = P = T \times V = C_T \frac{1}{2} \rho V^3 S \approx \text{independent of airspeed}
\]

Turbojet

\[
Thrust = T = C_T \frac{1}{2} \rho V^2 S \approx \text{independent of airspeed}
\]

Throttle Effect

\[
T = T_{max} \delta T = C_T \frac{1}{2} \delta \bar{q} S, \quad 0 \leq \delta T \leq 1
\]
Typical Effects of Altitude and Velocity on Power and Thrust

- **Propeller**

- **Turbojet**

Models for Altitude Effect on Turbofan Thrust

From *Flight Dynamics*, pp.117-118

\[
\text{Thrust} = C_T (V, \delta T) \frac{1}{2} \rho(h)V^2S = (k_o + k_1V^\eta) \frac{1}{2} \rho(h)V^2S\delta T, \text{ N}
\]

- \( k_o \) = Static thrust coefficient at sea level
- \( k_1 \) = Velocity sensitivity of thrust coefficient
- \( \eta \) = Exponent of velocity sensitivity
  - \( \eta = -2 \) for turbojet
- \( \delta T \) = Throttle setting, \((0,1)\)
- \( \rho(h) = \rho_{sl}e^{-\beta h}, \quad \rho_{sl} = 1.225 \text{ kg/m}^3, \quad \beta = (1/9,042) \text{ m}^{-1} \)
Models for Altitude Effect on Turbofan Thrust

From AeroModelMach.m in FLIGHT.m, Flight Dynamics,
http://www.princeton.edu/~stengel/AeroModelMach.m

[airdens,airPres,temp,soundSpeed] = Atmos(-x(6));
Thrust = u(4) * StaticThrust * (airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000));

Atmos(-x(6)) : 1976 U.S. Standard Atmosphere function
-x(6) = h = Altitude, m
airDens = ρ = Air density at altitude h, kg/m^3
u(4) = δT = Throttle setting, (0,1)

Empirical fit to match known characteristics of powerplant for generic business jet

(airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000))

Thrust of a Propeller-Driven Aircraft

With constant rpm, variable-pitch propeller

\[ T = \eta_p \eta_I \frac{P_{engine}}{V} = \eta_{net} \frac{P_{engine}}{V} \]

where
\[ \eta_p = \text{propeller efficiency} \]
\[ \eta_I = \text{ideal propulsive efficiency} \]
\[ \eta_{net, max} \approx 0.85 - 0.9 \]

Efficiencies decrease with airspeed
Engine power decreases with altitude
Proportional to air density, w/o supercharger
Reciprocating-Engine Power and Specific Fuel Consumption (SFC)

\[
\frac{P(h)}{P_{SL}} = 1.132 \frac{\rho(h)}{\rho_{SL}} - 0.132
\]

\[SFC \propto \text{Independent of Altitude}\]

- Engine power decreases with altitude
  - Proportional to air density, w/o supercharger
  - Supercharger increases inlet manifold pressure, increasing power and extending maximum altitude

\[\text{Propeller Efficiency, } \eta_p, \text{ and Advance Ratio, } J\]

\[J = \frac{V}{nD}\]

where
\[V = \text{airspeed, m/s}\]
\[n = \text{rotation rate, revolutions/s}\]
\[D = \text{propeller diameter, m}\]
Thrust of a Turbojet Engine

\[ T = \dot{m}V \left\{ \left[ \frac{\theta_o}{\theta_o - 1} \right] \left[ \frac{\theta_i}{\theta_i - 1} \right] (\tau_c - 1) + \frac{\theta_i}{\theta_o \tau_c} \right\}^{1/2} - 1 \]  

\[ \dot{m} = \dot{m}_{\text{air}} + \dot{m}_{\text{fuel}} \]

\[ \theta_o = \left( \frac{p_{\text{stag}}}{p_{\text{ambient}}} \right)^{(γ-1)/γ} \]; \; \gamma = \text{ratio of specific heats} \approx 1.4 \]

\[ \theta_i = \left( \text{turbine inlet temp./freestream ambient temp.} \right) \]

\[ \tau_c = \left( \text{compressor outlet temp./compressor inlet temp.} \right) \]

from Kerrebrock

Little change in thrust with airspeed below \( M_{\text{crit}} \)

Decrease with increasing altitude

Performance Parameters

Lift-to-Drag Ratio

\[ \frac{L}{D} = \frac{C_L}{C_D} \]

Load Factor

\[ n = \frac{L}{W} = \frac{L}{mg}, "g" s \]

Thrust-to-Weight Ratio

\[ \frac{T}{W} = \frac{T}{mg}, "g" s \]

Wing Loading

\[ \frac{W}{S}, \; N/m^2 \text{ or lb/ft}^2 \]
Checklist

- Flight variables?
- Propeller vs. jet propulsion?
- Variation with airspeed and altitude?
- Advance ratio?
- Wing loading?

Historical Factoid

- Aircraft Flight Distance Records

- Aircraft Flight Endurance Records
Steady, Level Flight

Trimmed Lift Coefficient, $C_L$

- Trimmed lift coefficient, $C_L$
  - Proportional to weight and wing loading factor
  - Decreases with $V^2$
  - At constant true airspeed, increases with altitude

\[
W = C_{L_{\text{trim}}} \left( \frac{1}{2} \rho V^2 \right) S = C_{L_{\text{trim}}} qS
\]

\[
C_{L_{\text{trim}}} = \frac{1}{q} \left( \frac{W}{S} \right) = \frac{2}{\rho V^2} \left( \frac{W}{S} \right) = \left( \frac{2 e^{\beta h}}{\rho_0 V^2} \right) \left( \frac{W}{S} \right)
\]

$\beta = 1/9,042$ m, inverse scale height of air density
Trimmed Angle of Attack, $\alpha$

- Trimmed angle of attack, $\alpha$
  - Constant if dynamic pressure and weight are constant
  - If dynamic pressure decreases, angle of attack must increase

\[
\alpha_{trim} = \frac{2W}{\rho V^2 S - C_{L_o}} = \frac{1}{\frac{q}{q}} \left( \frac{W}{S} \right) - C_{L_o}
\]

Thrust Required for Steady, Level Flight
Thrust Required for Steady, Level Flight

Trimmed thrust

\[ T_{\text{trim}} = D_{\text{cruise}} = C_{D_0} \left( \frac{1}{2} \rho V^2 S \right) + \varepsilon \frac{2W^2}{\rho V^2 S} \]

Minimum required thrust conditions

\[ \frac{\partial T_{\text{trim}}}{\partial V} = C_{D_0} \left( \rho V S \right) - \frac{4\varepsilon W^2}{\rho V^3 S} = 0 \]

Necessary Condition: Slope = 0

Necessary and Sufficient Conditions for Minimum Required Thrust

Necessary Condition = Zero Slope

\[ C_{D_0} \left( \rho V S \right) = \frac{4\varepsilon W^2}{\rho V^3 S} \]

Sufficient Condition for a Minimum = Positive Curvature when slope = 0

\[ \frac{\partial^2 T_{\text{trim}}}{\partial V^2} = C_{D_0} \left( \rho S \right) + \frac{12\varepsilon W^2}{\rho V^4 S} > 0 \]
Airspeed for Minimum Thrust in Steady, Level Flight

Satisfy necessary condition

\[ V^4 = \left( \frac{4 \varepsilon}{C_{D_o} \rho^2} \right) \left( \frac{W}{S} \right)^2 \]

Fourth-order equation for velocity
Choose the positive root

\[ V_{MT} = \sqrt{\frac{2}{\rho} \left( \frac{W}{S} \right) \sqrt{\frac{\varepsilon}{C_{D_o}}} } \]

Lift, Drag, and Thrust Coefficients in Minimum-Thrust Cruising Flight

Lift coefficient

\[ C_{L_{MT}} = \frac{2}{\rho V_{MT}^2} \left( \frac{W}{S} \right) \]

\[ = \sqrt{\frac{C_{D_o}}{\varepsilon}} = (C_L)_{(L/D)_{\text{max}}} \]

Drag and thrust coefficients

\[ C_{D_{MT}} = C_{D_o} + \varepsilon C_{L_{MT}}^2 = C_{D_o} + \varepsilon \frac{C_{D_o}}{\varepsilon} \]

\[ = 2C_{D_o} \equiv C_{T_{MT}} \]
Achievable Airspeeds in Constant-Altitude Flight

- Two equilibrium airspeeds for a given thrust or power setting
  - Low speed, high $C_L$, high $\alpha$
  - High speed, low $C_L$, low $\alpha$
- Achievable airspeeds between minimum and maximum values with maximum thrust or power

Power Required for Steady, Level Flight

\[ P = T \times V \]
Power Required for Steady, Level Flight

Trimmed power

Parasitic Drag

Induced Drag

\[ P_{trim} = T_{trim} V = D_{cruise} V = \left[ C_{D_o} \left( \frac{1}{2} \rho V^2 S \right) + \frac{2\epsilon W^2}{\rho V^2 S} \right] V \]

Minimum required power conditions

\[ \frac{\partial P_{trim}}{\partial V} = C_{D_o} \left( \frac{3}{2} \left( \rho V^2 S \right) - \frac{2\epsilon W^2}{\rho V^2 S} \right) = 0 \]

Airspeed for Minimum Power in Steady, Level Flight

- Satisfy necessary condition

\[ C_{D_o} \frac{3}{2} \left( \rho V^2 S \right) = \frac{2\epsilon W^2}{\rho V^2 S} \]

- Fourth-order equation for velocity
  - Choose the positive root

\[ V_{MP} = \sqrt{\frac{2 \left( \frac{W}{\rho S} \right) \sqrt{\frac{\epsilon}{3C_{D_o}}}}{\rho V^2 S}} \]

- Corresponding lift and drag coefficients

\[ C_{LMP} = \sqrt{\frac{3C_{D_o}}{\epsilon}} \]

\[ C_{DMP} = 4C_{D_o} \]
Achievable Airspeeds for Jet in Cruising Flight

Thrust = constant

\[ T_{avail} = C_D \bar{q} S = C_{D_o} \left( \frac{1}{2} \rho V^2 S \right) + \frac{2 \varepsilon W^2}{\rho V^2 S} \]

\[ C_{D_o} \left( \frac{1}{2} \rho V^4 S \right) - T_{avail} V^2 + \frac{2 \varepsilon W^2}{\rho S} = 0 \]

\[ V^4 - \frac{2 T_{avail}}{C_{D_o} \rho S} V^2 + \frac{4 \varepsilon W^2}{C_{D_o} (\rho S)^2} = 0 \]

4th-order algebraic equation for \( V \)

Achievable Airspeeds for Jet in Cruising Flight

Solutions for \( V^2 \) can be put in quadratic form and solved easily

\[ V^2 \triangleq x; \quad V = \pm \sqrt{x} \]

\[ V^4 - \frac{2 T_{avail}}{C_{D_o} \rho S} V^2 + \frac{4 \varepsilon W^2}{C_{D_o} (\rho S)^2} = 0 \]

\[ x^2 + bx + c = 0 \]

\[ x = -\frac{b}{2} \pm \sqrt{\left( \frac{b}{2} \right)^2 - c} = V^2 \]
Available thrust decreases with altitude
Stall limitation at low speed
Mach number effect on lift and drag increases thrust required at high speed

**Typical Simplified Jet Thrust Model**

\[
T_{\text{max}}(h) = T_{\text{max}}(SL) \left[ \frac{\rho(SL)e^{-\beta h}}{\rho(SL)} \right]^x = T_{\text{max}}(SL)e^{-x\beta h}
\]

Empirical correction to force thrust to zero at a given altitude, \( h_{\text{max}} \).
\( c \) is a convergence factor.

\[
T_{\text{max}}(h) = T_{\text{max}}(SL)e^{-x\beta h} \left[ 1 - \frac{e^{-(h-h_{\text{max}})/c}}{x} \right]
\]
Checklist

- Thrust required vs. airspeed and altitude?
- Minimum-thrust cruise vs. minimum-power cruise?
- Power/thrust available for climb?
- Mach effect?

Flying Qualities Becomes a Science
Chapter 3, Airplane Stability and Control, Abzug and Larrabee

- What are the principal subject and scope of the chapter?
- What technical ideas are needed to understand the chapter?
- During what time period did the events covered in the chapter take place?
- What are the three main "takeaway" points or conclusions from the reading?
- What are the three most surprising or remarkable facts that you found in the reading?
The Flight Envelope

Flight Envelope Determined by Available Thrust

- All altitudes and airspeeds at which an aircraft can fly
  - in steady, level flight
  - at fixed weight

  - Flight ceiling defined by available climb rate
    - Absolute: 0 ft/min
    - Service: 100 ft/min
    - Performance: 200 ft/min

Excess thrust provides the ability to accelerate or climb
Additional Factors Define the Flight Envelope

- Maximum Mach number
- Maximum allowable aerodynamic heating
- Maximum thrust
- Maximum dynamic pressure
- Performance ceiling
- Wing stall
- Flow-separation buffet
  - Angle of attack
  - Local shock waves

Piper Dakota Stall Buffet
http://www.youtube.com/watch?v=mCCjGAthZ4g

Boeing 787 Flight Envelope
(HW #5, 2008)
Lockheed U-2
“Coffin Corner”
Stall buffeting and Mach buffeting are limiting factors
Narrow corridor for safe flight

Climb Schedule

Historical Factoids
Air Commerce Act of 1926
• Airlines formed to carry mail and passengers:
  – Northwest (1926)
  – Eastern (1927), bankruptcy
  – Pan Am (1927), bankruptcy
  – Boeing Air Transport (1927), became United (1931)
  – Delta (1928), consolidated with Northwest, 2010
  – American (1930)
  – TWA (1930), acquired by American
  – Continental (1934), consolidated with United, 2010

http://www.youtube.com/watch?v=3a8G87qnZz4
Commercial Aircraft of the 1930s
Streamlining, engine cowlings

Douglas DC-1, DC-2, DC-3

Lockheed 14 Super Electra, Boeing 247

Comfort and Elegance by the End of the Decade

Boeing 307, 1st pressurized cabin (1936), flight engineer, B-17 pre-cursor, large dorsal fin (exterior and interior)

Sleeping bunks on transcontinental planes (e.g., DC-3)
Full-size dining rooms on flying boats
Seaplanes Became the First TransOceanic Air Transports

• PanAm led the way
  – 1st scheduled TransPacific flights (1935)
  – 1st scheduled TransAtlantic flights (1938)
  – 1st scheduled non-stop Trans-Atlantic flights (VS-44, 1939)
• Boeing B-314, Vought-Sikorsky VS-44, Shorts Solent
• Superseded by more efficient landplanes (lighter, less drag)

http://www.youtube.com/watch?v=x8SkeE1h-A

Checklist

☑ Flight envelope?
**Optimal Cruising Flight**

**Maximum Lift-to-Drag Ratio**

**Lift-to-drag ratio**

\[
\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_o} + \epsilon C_L^2}
\]

Satisfy necessary condition for a maximum

\[
\frac{\partial}{\partial C_L} \left( \frac{C_L}{C_D} \right) = \frac{1}{C_{D_o} + \epsilon C_L^2} - \frac{2\epsilon C_L^2}{\left(C_{D_o} + \epsilon C_L^2 \right)^2} = 0
\]

Lift coefficient for maximum \(L/D\) and minimum thrust are the same

\[
(C_L)_{L/D_{\text{max}}} = \sqrt{\frac{C_{D_o}}{\epsilon}} = C_{L_{\text{MT}}}
\]
Airspeed, Drag Coefficient, and Lift-to-Drag Ratio for $L/D_{\text{max}}$

\[ V_{L/D_{\text{max}}} = V_{MT} = \sqrt{\frac{2(W)}{\rho S}} \sqrt{\frac{\varepsilon}{C_{D_o}}} \]

\[ (C_D)_{L/D_{\text{max}}} = C_{D_o} + C_{D_o} = 2C_{D_o} \]

\[ (L / D)_{\text{max}} = \frac{\sqrt{C_{D_o}/\varepsilon}}{2C_{D_o}} = \frac{1}{2\sqrt{\varepsilon}C_{D_o}} \]

Maximum $L/D$ depends only on induced drag factor and zero-lift drag coefficient. Induced drag factor and zero-lift drag coefficient are functions of Mach number.

Cruising Range and Specific Fuel Consumption

- Thrust = Drag
  \[ 0 = (C_T - C_D)\frac{1}{2} \rho V^2 S/m \]
- Lift = Weight
  \[ 0 = \left(C_L \frac{1}{2} \rho V^2 S - mg\right)/mV \]

- Thrust specific fuel consumption, $TSFC = c_T$
  - Fuel mass burned per sec per unit of thrust
    \[ c_T : \frac{kg/s}{kN} \]
    \[ \dot{m}_f = -c_T T \]

- Power specific fuel consumption, $PSFC = c_P$
  - Fuel mass burned per sec per unit of power
    \[ c_P : \frac{kg/s}{kW} \]
    \[ \dot{m}_f = -c_P P \]
Breguet Range Equation for Jet Aircraft

Rate of change of range with respect to weight of fuel burned

\[ \frac{dr}{dm} = \frac{dr}{dt} \frac{1}{dm/dt} = \frac{\dot{r}}{\dot{m}} = \frac{V}{-c_f T} = \frac{V}{c_f D} = \left( \frac{L}{D} \right) \frac{V}{c_f mg} \]

\[ dr = -\left( \frac{L}{D} \right) \frac{V}{c_f mg} \, dm \]

Range traveled

\[ \text{Range} = R = \int_{0}^{R} dr = -\int_{W_i}^{W_f} \left( \frac{L}{D} \right) \left( \frac{V}{c_f g} \right) \frac{dm}{m} \]

Maximum Range of a Jet Aircraft Flying at Constant Altitude

At constant altitude and $SFC$

\[ V_{\text{cruise}}(t) = \sqrt{\frac{2W(t)}{C_l \rho (\text{fl}) S}} \]

\[ \text{Range} = -\int_{W_i}^{W_f} \left( \frac{C_L}{C_D} \right) \left( \frac{1}{c_f g} \right) \sqrt{\frac{2}{C_l \rho S}} \, dm = \left( \frac{\sqrt{C_L}}{C_D} \right) \left( \frac{2}{c_f g} \right) \sqrt{\frac{2}{\rho S}} \left( m_i^{1/2} - m_f^{1/2} \right) \]

Range is maximized when

\[ \left( \frac{\sqrt{C_L}}{C_D} \right) = \text{maximum} \]
Breguet Range Equation for Jet Aircraft

For constant true airspeed, \( V = V_{\text{cruise}} \) and \( SFC \)

\[
R = -
\left( \frac{L}{D} \right) \left( \frac{V_{\text{cruise}}}{c_T g} \right) \ln \left( \frac{m_f}{m_i} \right)
= \left( \frac{L}{D} \right) \left( \frac{V_{\text{cruise}}}{c_T g} \right) \ln \left( \frac{m_i}{m_f} \right)
= \left( V_{\text{cruise}} \frac{C_L}{C_D} \right) \left( \frac{1}{c_T g} \right) \ln \left( \frac{m_i}{m_f} \right)
\]

- \( V_{\text{cruise}}(C_L/C_D) \) as large as possible
- Respect \( M_{\text{crit}} \)
- \( \rho \) as small as possible
- \( h \) as high as possible

Maximize Jet Aircraft Range Using Optimal Cruise-Climb

\[
\frac{\partial R}{\partial C_L} \propto \frac{\partial \left( V_{\text{cruise}} \frac{C_L}{C_D} \right)}{\partial C_L} = \frac{\partial \left( V_{\text{cruise}} \frac{C_L}{C_{D0} + \varepsilon C_L^2} \right)}{\partial C_L} = 0
\]

\[
V_{\text{cruise}} = \sqrt{2W/C_L\rho S}
\]

Assume \( \sqrt{2W(t)/\rho(h)S} = \text{constant} \)

\( i.e., \) airplane climbs at constant \( \text{TAS} \) as fuel is burned
Maximize Jet Aircraft Range Using Optimal Cruise-Climb

\[ \frac{\partial V_{cruise} C_L}{\partial C_L} \left( \frac{C_{D_o} + \varepsilon C_L^2}{C_{D_o} + \varepsilon C_L^2} \right) \sqrt{2W \frac{C_{L}^{1/2}}{C_{D_o} + \varepsilon C_L^2} \frac{CL}{CL}} = 0 \]

Optimal values: (see Supplemental Material)

\[ C_{L_M} = \sqrt{\frac{C_{D_o}}{3\varepsilon}} : \text{Lift Coefficient for Maximum Range} \]

\[ C_{D_M} = C_{D_o} + \frac{C_{D_o}}{3} = \frac{4}{3} C_{D_o} \]

\[ V_{cruise-climb} = \sqrt{2W(t) / C_{L_M} \rho(h) S} = a(h) M_{cruise-climb} \]

\( a(h) \): Speed of sound; \( M_{cruise-climb} \): Mach number

Step-Climb Approximates Optimal Cruise-Climb

- Cruise-climb usually violates air traffic control rules
- Constant-altitude cruise does not
- **Compromise**: Step climb from one allowed altitude to the next as fuel is burned
**Historical Factoid**

- Louis Breguet (1880-1955), aviation pioneer
  - Gyroplane (1905), flew vertically in 1907
  - Breguet Type 1 (1909), fixed-wing aircraft
  - Formed Compagnie des messageries aériennes (1919), predecessor of Air France
- Breguet Aviation manufactured numerous military and commercial aircraft until after World War II; teamed with BAC in SEPECAT
- Merged with Dassault in 1971

---

**Checklist**

- *Specific fuel consumption?*
- *Breguet equation?*
  - *Constant altitude?*
  - *Cruise-climb?*
Next Time:
Gliding, Climbing, and Turning Flight

Reading:
*Flight Dynamics*
*Aerodynamic Coefficients, 130-141, 147-155*

**Learning Objectives**
- Conditions for gliding flight
- Parameters for maximizing climb angle and rate
- Review the V-n diagram
- Energy height and specific excess power
- Alternative expressions for steady turning flight
- The Herbst maneuver

Supplemental Material
Achievable Airspeeds in Propeller-Driven Cruising Flight

Power = constant

\[ P_{\text{avail}} = T_{\text{avail}}V \]

\[ V^4 - \frac{P_{\text{avail}}}{C_{D_0} \rho S} V + \frac{4 \epsilon W^2}{C_{D_0} (\rho S)^2} = 0 \]

Solutions for \( V \) cannot be put in quadratic form; solution is more difficult, e.g., Ferrari’s method

\[ aV^4 + (0)V^3 + (0)V^2 + dV + e = 0 \]

Best bet: roots in MATLAB

P-51 Mustang Minimum-Thrust Example

Wing Span = 37 ft (9.83 m)

Wing Area = 235 ft² (21.83 m²)

Loaded Weight = 9,200 lb (3,465 kg)

\[ C_{D_0} = 0.0163 \]

\[ \epsilon = 0.0576 \]

\[ \frac{W}{S} = 39.3 \text{ lb/ft}^2 \text{ (1555.7 N/m}^2\text{)} \]

Airspeed for minimum thrust

\[ V_{MT} = \sqrt{\frac{2}{\rho} \left( \frac{W}{S} \right)} \sqrt{\frac{\epsilon}{C_{D_0}}} = \sqrt{\frac{2}{\rho} \left(1555.7\right) \left(0.0163\right)} = \frac{76.49}{\sqrt{\rho}} \text{ m/s} \]

Air Density, \( \text{kg/m}^3 \)

<table>
<thead>
<tr>
<th>Altitude, m</th>
<th>Air Density, kg/m³</th>
<th>VMT, m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.23</td>
<td>69.11</td>
</tr>
<tr>
<td>2,500</td>
<td>0.96</td>
<td>78.20</td>
</tr>
<tr>
<td>5,000</td>
<td>0.74</td>
<td>89.15</td>
</tr>
<tr>
<td>10,000</td>
<td>0.41</td>
<td>118.87</td>
</tr>
</tbody>
</table>
**P-51 Mustang**

**Maximum L/D Example**

\[
(C_D)_{L/D_{\text{max}}} = 2C_{D_0} = 0.0326
\]

\[
(C_L)_{L/D_{\text{max}}} = \sqrt{\frac{C_{D_0}}{\varepsilon}} = C_{L_{\text{max}}} = 0.531
\]

\[
(L / D)_{\text{max}} = \frac{1}{2\sqrt{\varepsilon C_{D_0}}} = 16.31
\]

\[
V_{L/D_{\text{max}}} = \frac{V_{MT}}{\sqrt{\rho}} = \frac{76.49}{m / s}
\]

<table>
<thead>
<tr>
<th>Altitude, m</th>
<th>Air Density, kg/m³</th>
<th>VMT, m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.23</td>
<td>66.11</td>
</tr>
<tr>
<td>2,500</td>
<td>0.96</td>
<td>78.20</td>
</tr>
<tr>
<td>5,000</td>
<td>0.74</td>
<td>89.15</td>
</tr>
<tr>
<td>10,000</td>
<td>0.41</td>
<td>118.87</td>
</tr>
</tbody>
</table>

\[
\text{Wing Span} = 37 \text{ ft (9.83 m)}
\]

\[
\text{Wing Area} = 235 \text{ ft}^2 (21.83 \text{ m}^2)
\]

\[
\text{Loaded Weight} = 9,200 \text{ lb (3,465 kg)}
\]

\[
C_{D_0} = 0.0163
\]

\[
\varepsilon = 0.0576
\]

\[
W / S = 1555.7 \text{ N/m}^2
\]

---

**Breguet Range Equation for Propeller-Driven Aircraft**

Rate of change of range with respect to weight of fuel burned

\[
\frac{dr}{dw} = \frac{\dot{r}}{\dot{w}} = -\frac{V}{c_p P} = -\frac{V}{c_p TV} = -\frac{V}{c_p DV} = -\left(\frac{L}{D}\right)\frac{1}{c_p W}
\]

Range traveled

\[
\text{Range} = R = \int_0^R dr = -\int_{W_f}^{W_i} \left(\frac{L}{D}\right)\frac{1}{c_p} dw
\]
Breguet Range Equation for Propeller-Driven Aircraft

For constant true airspeed, \( V = V_{\text{cruise}} \)

\[
R = -\left( \frac{L}{D} \right) \left( \frac{1}{c_p} \right) \ln \left( \frac{w}{W_f} \right) = \left( \frac{C_L}{C_D} \right) \left( \frac{1}{c_p} \right) \ln \left( \frac{W_i}{W_f} \right)
\]

Range is maximized when

\[
\left( \frac{C_L}{C_D} \right) = \text{maximum} = \left( \frac{L}{D} \right)_{\text{max}}
\]

---

P-51 Mustang Maximum Range (Internal Tanks only)

\[
W = C_{t_{\text{vis}}} qS
\]

\[
C_{t_{\text{vis}}} = \frac{1}{q} (W/S) = \frac{2}{\rho V^2} (W/S) = \left( \frac{2 e^{\rho h}}{\rho_0 V^2} \right) (W/S)
\]

\[
R = \left( \frac{C_L}{C_D} \right)_{\text{max}} \left( \frac{1}{c_p} \right) \ln \left( \frac{W_i}{W_f} \right)
\]

\[
= (16.31) \left( \frac{1}{0.0017} \right) \ln \left( \frac{3,465 + 600}{3,465} \right)
\]

\[
= 1,530 \text{ km} \ (825 \text{ nm})
\]
Maximize Jet Aircraft Range
Using Optimal Cruise-Climb

\[ \frac{\partial}{\partial C_L} \left( \frac{C_L}{(C_{D_o} + \varepsilon C_L^2)} \right) = \sqrt{\frac{2w}{\rho S}} \frac{\partial}{\partial C_L} \left( \frac{C_L^{1/2}}{(C_{D_o} + \varepsilon C_L^2)} \right) = 0 \]

\[ \sqrt{\frac{2w}{\rho S}} = \text{Constant}; \text{ let } C_L^{1/2} = x, \ C_L = x^2 \]

\[ \frac{\partial}{\partial x} \left[ \frac{x}{(C_{D_o} + \varepsilon x^4)} \right] = \frac{(C_{D_o} + \varepsilon x^4) - x(4\varepsilon x^3)}{(C_{D_o} + \varepsilon x^4)^2} = \frac{(C_{D_o} - 3\varepsilon x^4)}{(C_{D_o} + \varepsilon x^4)^2} \]

Optimal values:

\[ C_{L_{\text{opt}}} = \sqrt{\frac{C_{D_o}}{3\varepsilon}} : \ C_{D_{\text{opt}}} = C_{D_o} + \frac{C_{D_o}}{3} = \frac{4}{3} C_{D_o} \]