Aircraft Equations of Motion: Translation and Rotation
Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2016

Learning Objectives
• What use are the equations of motion?
• How is the angular orientation of the airplane described?
• What is a cross-product-equivalent matrix?
• What is angular momentum?
• How are the inertial properties of the airplane described?
• How is the rate of change of angular momentum calculated?

Reading: Flight Dynamics 155-161

Review Questions
• What characteristic(s) provide maximum gliding range?
• Do gliding heavy airplanes fall out of the sky faster than light airplanes?
• Are the factors for maximum gliding range and minimum sink rate the same?
• How does the maximum climb rate vary with altitude?
• What are “energy height“ and “specific excess power”?
• What is an “energy climb“?
• How is the “maneuvering envelope” defined?
• What factors determine the maximum steady turning rate?
Dynamic Systems

Dynamic Process: Current state depends on prior state
- \( x \) = dynamic state
- \( u \) = input
- \( w \) = exogenous disturbance
- \( p \) = parameter
- \( t \) or \( k \) = time or event index

Observation Process: Measurement may contain error or be incomplete
- \( y \) = output (error-free)
- \( z \) = measurement
- \( n \) = measurement error

\[
\frac{dx(t)}{dt} = f[x(t),u(t),w(t),p(t),t]
\]

\[
y(t) = h[x(t),u(t)]
\]

\[
z(t) = y(t) + n(t)
\]

Ordinary Differential Equations Fall Into 4 Categories

- Nonlinear, time-varying models (NTV)
- Linear, time-varying models (LTV)

\[
\frac{dx(t)}{dt} = f[x(t),u(t),w(t)]
\]

\[
\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t) + L(t)w(t)
\]
What Use are the Equations of Motion?

- **Nonlinear equations of motion**
  - Compute “exact” flight paths and motions
    - Simulate flight motions
    - Optimize flight paths
    - Predict performance
  - Provide basis for approximate solutions

\[
\frac{dx(t)}{dt} = f[x(t), u(t), w(t), p(t), t]
\]

\[
\frac{dx(t)}{dt} = Fx(t) + Gu(t) + Lw(t)
\]

- **Linear equations of motion**
  - Simplify computation of flight paths and solutions
  - Define modes of motion
  - Provide basis for control system design and flying qualities analysis

Examples of Airplane Dynamic System Models

- **Nonlinear, Time-Varying**
  - Large amplitude motions
  - Significant change in mass

- **Nonlinear, Time-Invariant**
  - Large amplitude motions
  - Negligible change in mass

- **Linear, Time-Varying**
  - Small amplitude motions
  - Perturbations from a dynamic flight path

- **Linear, Time-Invariant**
  - Small amplitude motions
  - Perturbations from an equilibrium flight path
Translational Position

Position of a Particle

Projections of vector magnitude on three axes

\[ \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos \delta_x \\ \cos \delta_y \\ \cos \delta_z \end{bmatrix} \]

\[ \begin{bmatrix} \cos \delta_x \\ \cos \delta_y \\ \cos \delta_z \end{bmatrix} = \text{Direction cosines} \]
Cartesian Frames of Reference

- Two reference frames of interest
  - \( I \): Inertial frame (fixed to inertial space)
  - \( B \): Body frame (fixed to body)

- Translation
  - Relative linear positions of origins

- Rotation
  - Orientation of the body frame with respect to the inertial frame

Common convention (\( z \) up)  
Aircraft convention (\( z \) down)

Measurement of Position in Alternative Frames - 1

- Two reference frames of interest
  - \( I \): Inertial frame (fixed to inertial space)
  - \( B \): Body frame (fixed to body)

\[
\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

\[
\mathbf{r}_{\text{particle}} = \mathbf{r}_{\text{origin}} + \Delta \mathbf{r}_{\text{w.r.t. origin}}
\]

- Differences in frame orientations must be taken into account in adding vector components
Measurement of Position in Alternative Frames - 2

**Inertial-axis view**

\[ \mathbf{r}_{\text{particle}_I} = \mathbf{r}_{\text{origin}-B_I} + \mathbf{H}_B^I \Delta \mathbf{r}_B \]

**Body-axis view**

\[ \mathbf{r}_{\text{particle}_B} = \mathbf{r}_{\text{origin}-I_B} + \mathbf{H}_I^B \Delta \mathbf{r}_I \]

Rotational Orientation
Direction Cosine Matrix

\[ H^B_I = \begin{bmatrix}
\cos \delta_{11} & \cos \delta_{21} & \cos \delta_{31} \\
\cos \delta_{12} & \cos \delta_{22} & \cos \delta_{32} \\
\cos \delta_{13} & \cos \delta_{23} & \cos \delta_{33}
\end{bmatrix} \]

- Projections of unit vector components of one reference frame on another
- Rotational orientation of one reference frame with respect to another
- Cosines of angles between each \( I \) axis and each \( B \) axis

\[ \mathbf{r}_B = H^B_I \mathbf{r}_I \]

Properties of the Rotation Matrix

\[ H^B_I = \begin{bmatrix}
\cos \delta_{11} & \cos \delta_{21} & \cos \delta_{31} \\
\cos \delta_{12} & \cos \delta_{22} & \cos \delta_{32} \\
\cos \delta_{13} & \cos \delta_{23} & \cos \delta_{33}
\end{bmatrix} = \begin{bmatrix}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{bmatrix} \]

\[ \mathbf{r}_B = H^B_I \mathbf{r}_I \quad \mathbf{s}_B = H^B_I \mathbf{s}_I \]

Orthonormal transformation

Angles between vectors are preserved

Lengths are preserved

\[ |\mathbf{r}_I| = |\mathbf{r}_B| \quad ; \quad |\mathbf{s}_I| = |\mathbf{s}_B| \]

\[ \angle(\mathbf{r}_I, \mathbf{s}_I) = \angle(\mathbf{r}_B, \mathbf{s}_B) = x \text{ deg} \]
Euler Angles

- Body attitude measured with respect to inertial frame
- Three-angle orientation expressed by sequence of three orthogonal single-angle rotations

\[\text{Inertial} \rightarrow \text{Intermediate}_1 \rightarrow \text{Intermediate}_2 \rightarrow \text{Body}\]

- 24 (±12) possible sequences of single-axis rotations
- Aircraft convention: 3-2-1, z positive down

\[\psi : \text{Yaw angle}\]
\[\theta : \text{Pitch angle}\]
\[\phi : \text{Roll angle}\]

Euler Angles Measure the Orientation of One Frame with Respect to the Other

- Conventional sequence of rotations from inertial to body frame
  - Each rotation is about a single axis
  - Right-hand rule
  - Yaw, then pitch, then roll
  - These are called Euler Angles

Other sequences of 3 rotations can be chosen; however, once sequence is chosen, it must be retained
Reference Frame Rotation from Inertial to Body: Aircraft Convention (3-2-1)

### Yaw rotation (ψ) about z_i axis

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{i} =
\begin{bmatrix}
  \cos \psi & \sin \psi & 0 \\
  -\sin \psi & \cos \psi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{i} =
\begin{bmatrix}
  x_i \cos \psi + y_i \sin \psi \\
  -x_i \sin \psi + y_i \cos \psi \\
  z_i
\end{bmatrix}
\]

\( r_i = H^i \rightarrow r_i \)

### Pitch rotation (θ) about y_i axis

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{i} =
\begin{bmatrix}
  \cos \theta & 0 & -\sin \theta \\
  0 & 1 & 0 \\
  \sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{i} =
\begin{bmatrix}
  r_2 = H^i \rightarrow r_i = H^i \rightarrow r_i
\end{bmatrix}
\]

### Roll rotation (φ) about x_2 axis

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{i} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \phi & \sin \phi \\
  0 & -\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}_{i} =
\begin{bmatrix}
  r_B = H^i \rightarrow r_i = H^i \rightarrow r_i
\end{bmatrix}
\]

#### The Rotation Matrix

The three-angle rotation matrix is the product of 3 single-angle rotation matrices:

\[
H^B_i(\phi, \theta, \psi) = H^B_2(\phi)H^2_1(\theta)H^1_i(\psi)
\]

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \phi & \sin \phi \\
  0 & -\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
  \cos \theta & 0 & -\sin \theta \\
  0 & 1 & 0 \\
  \sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  \cos \psi & \sin \psi & 0 \\
  -\sin \psi & \cos \psi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi & \cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi & \sin \phi \cos \theta \\
\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi & -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi & \cos \phi \cos \theta
\end{bmatrix}
\]

an expression of the Direction Cosine Matrix
Rotation Matrix Inverse

Inverse relationship: interchange sub- and superscripts

\[ r_B = H_I^B r_I \]
\[ r_I = (H_I^B)^{-1} r_B = H_B^I r_B \]

Because transformation is orthonormal
Inverse = transpose
Rotation matrix is always non-singular

\[ [H_B^I (\phi, \theta, \psi)]^{-1} = [H_I^B (\phi, \theta, \psi)]^T = H_b^I (\psi, \theta, \phi) \]

\[ H_B^I = (H_I^B)^{-1} = (H_I^B)^T = H_1^I H_2^1 H_2^B \]

\[ H_B^I H_I^B = H_I^B H_B^I = I \]

Checklist

- What are direction cosines?
- What are Euler angles?
- What rotation sequence is used to describe airplane attitude?
- What are properties of the rotation matrix?
Angular Momentum of a Particle

- **Moment of linear momentum of differential particles that make up the body**
  - (Differential masses) x components of the velocity that are perpendicular to the moment arms

\[
d\mathbf{h} = \mathbf{r} \times d\mathbf{m} \mathbf{v} = \left( \mathbf{r} \times \mathbf{v}_m \right) dm = \left[ \mathbf{r} \times \left( \mathbf{v}_o + \mathbf{\omega} \times \mathbf{r} \right) \right] dm
\]

- **Cross Product**: Evaluation of a determinant with unit vectors \((i, j, k)\) along axes, \((x, y, z)\) and \((v_x, v_y, v_z)\) projections on to axes

\[
\mathbf{r} \times \mathbf{v} = \begin{vmatrix}
    i & j & k \\
    x & y & z \\
v_x & v_y & v_z 
\end{vmatrix} = \left( v_y v_z - v_z v_y \right) i + \left( v_z v_x - v_x v_z \right) j + \left( v_x v_y - v_y v_x \right) k
\]
Cross-Product-Equivalent Matrix

\[
\mathbf{r} \times \mathbf{v} = \begin{vmatrix}
    i & j & k \\
    x & y & z \\
    v_x & v_y & v_z \\
\end{vmatrix}
= \left( yv_z - vz_y \right) i + \left( zv_x - xv_z \right) j + \left( xv_y - yv_z \right) k
\]

Cross-product-equivalent matrix

\[
\mathbf{\tilde{r}} = \begin{bmatrix}
    0 & -z & y \\
    z & 0 & -x \\
    -y & x & 0 \\
\end{bmatrix}
\]

Angular Momentum of the Aircraft

• Integrate moment of linear momentum of differential particles over the body

\[
\mathbf{h} = \int_{\text{Body}} \left[ \mathbf{r} \times (\mathbf{v}_o + \mathbf{\omega} \times \mathbf{r}) \right] dm = \int_{\text{Body}} \int_{\gamma_{\min}}^{\gamma_{\max}} \int_{\zeta_{\min}}^{\zeta_{\max}} (\mathbf{r} \times \mathbf{v}) \rho(x,y,z) dx dy dz =
\begin{bmatrix}
    h_x \\
    h_y \\
    h_z \\
\end{bmatrix}
\]

\[
\rho(x,y,z) = \text{Density of the body}
\]

• Choose the center of mass as the rotational center

\[
\mathbf{h} = \int_{\text{Body}} \left( \mathbf{r} \times \mathbf{v}_o \right) dm + \int_{\text{Body}} \left[ \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}) \right] dm
= 0 - \int_{\text{Body}} \left[ \mathbf{r} \times (\mathbf{r} \times \mathbf{\omega}) \right] dm
= -\int_{\text{Body}} (\mathbf{r} \times \mathbf{r}) dm \times \mathbf{\omega} \equiv -\int_{\text{Body}} (\mathbf{\tilde{r}} \mathbf{r}) dm \mathbf{\omega}
\]
Location of the Center of Mass

\[ r_{cm} = \frac{1}{m} \int \int \int r \rho(x, y, z) dx dy dz = \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix} \]

The Inertia Matrix
The Inertia Matrix

\[ h = - \int_{\text{Body}} \hat{r} \times \omega \, dm = - \int_{\text{Body}} \hat{r} \, d\mathbf{r} \, \omega = I \omega \]

\[ \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \]

where

\[ I = - \int_{\text{Body}} \hat{r} \times \hat{r} \, dm = \int_{\text{Body}} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \, dm \]

\[ = \int_{\text{Body}} \begin{bmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{bmatrix} \, dm \]

**Inertia matrix** derives from equal effect of angular rate on all particles of the aircraft

Moments and Products of Inertia

\[ I = \int_{\text{Body}} \begin{bmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{bmatrix} \, dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \]

**Inertia matrix**
- Moments of inertia on the diagonal
- Products of inertia off the diagonal

- If products of inertia are zero, \((x, y, z)\) are principal axes --->
- All rigid bodies have a set of principal axes

**Ellipsoid of Inertia**

\[ I_{xx} x^2 + I_{yy} y^2 + I_{zz} z^2 = 1 \]
Inertia Matrix of an Aircraft with Mirror Symmetry

\[
\mathbf{I} = \int_{\text{Body}} \begin{bmatrix}
(y^2 + z^2) & 0 & -xz \\
0 & (x^2 + z^2) & 0 \\
-xz & 0 & (x^2 + y^2)
\end{bmatrix} \, dm = \begin{bmatrix}
\mathbf{I}_{xx} & 0 & -\mathbf{I}_{xz} \\
0 & \mathbf{I}_{yy} & 0 \\
-\mathbf{I}_{xz} & 0 & \mathbf{I}_{zz}
\end{bmatrix}
\]

Nose high/low product of inertia, \( I_{xz} \)

North American XB-70

Nominal Configuration
Tips folded, 50% fuel, \( W = 38,524 \) lb
\( x_{cm} @ 0.218 c \)
\( \mathbf{I}_{xx} = 1.8 \times 10^6 \) slug-ft\(^2\)
\( \mathbf{I}_{yy} = 19.9 \times 10^6 \) slug-ft\(^2\)
\( \mathbf{I}_{xx} = 22.1 \times 10^6 \) slug-ft\(^2\)
\( \mathbf{I}_{xz} = -0.88 \times 10^6 \) slug-ft\(^2\)

Checklist

- How is the location of the center of mass found?
- What is a cross-product-equivalent matrix?
- What is the inertia matrix?
- What is an ellipsoid of inertia?
- What does the “nose-high” product of inertia represent?
Historical Factoids
Technology of World War II Aviation

- **1938-45**: Analytical and experimental approach to design
  - Many configurations designed and flight-tested
  - Increased specialization; radar, navigation, and communication
  - Approaching the "sonic barrier"

- **Aircraft Design**
  - Large, powerful, high-flying aircraft
  - Turbocharged engines
  - Oxygen and Pressurization

Power Effects on Stability and Control

- **Brewster Buffalo**: over- armored and under-powered
- During W.W.II, the size of fighters remained about the same, but installed horsepower doubled (*F4F* vs. *F8F*)
- Use of flaps means high power at low speed, increasing relative significance of thrust effects
World War II Carrier-Based Airplanes

- Takeoff without catapult, relatively low landing speed
  [http://www.youtube.com/watch?v=4dySbhK1vNk](http://www.youtube.com/watch?v=4dySbhK1vNk)
- Tailhook and arresting gear
- Carrier steams into wind
- Design for storage (short tail length, folding wings) affects stability and control

Multi-Engine Aircraft of World War II

- Large W.W.II aircraft had unpowered controls:
  - High foot-pedal force
  - Rudder stability problems arising from balancing to reduce pedal force
- Severe engine-out problem for twin-engine aircraft
Managing Control Forces

Chapter 5, Airplane Stability and Control, Abzug and Larrabee

- What are the principal subject and scope of the chapter?
- What technical ideas are needed to understand the chapter?
- During what time period did the events covered in the chapter take place?
- What are the three main "takeaway" points or conclusions from the reading?
- What are the three most surprising or remarkable facts that you found in the reading?
Rate of Change of Angular Momentum

Newton’s 2nd Law, Applied to Rotational Motion

In inertial frame, rate of change of angular momentum = applied moment (or torque), \( M \)

\[
\frac{d\mathbf{h}}{dt} = \frac{d(\mathbf{I}\omega)}{dt} = \mathbf{I}\frac{d\omega}{dt} + \mathbf{I}\frac{d\mathbf{\omega}}{dt} = \mathbf{I}\mathbf{\omega} + \mathbf{I}\mathbf{\dot{\omega}} = \mathbf{M} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}
\]
Angular Momentum and Rate

Angular momentum and rate vectors are not necessarily aligned

\[ h = I \omega \]

How Do We Get Rid of \( dI/dt \) in the Angular Momentum Equation?

Chain Rule ... and in an inertial frame

\[ \frac{d(I \omega)}{dt} = \dot{I} \omega + I \dot{\omega} \]

\[ \dot{I} \neq 0 \]

- Dynamic equation in a body-referenced frame
  - Inertial properties of a constant-mass, rigid body are unchanging in a body frame of reference
  - ... but a body-referenced frame is “non-Newtonian” or “non-inertial”
  - Therefore, dynamic equation must be modified for expression in a rotating frame
Angular Momentum Expressed in Two Frames of Reference

- Angular momentum and rate are vectors
  - Expressed in either the inertial or body frame
  - Two frames related algebraically by the rotation matrix

\[
\mathbf{h}_B(t) = \mathbf{H}^B_I(t)\mathbf{h}_I(t); \quad \mathbf{h}_I(t) = \mathbf{H}^I_B(t)\mathbf{h}_B(t)
\]

\[
\mathbf{\omega}_B(t) = \mathbf{H}^B_I(t)\mathbf{\omega}_I(t); \quad \mathbf{\omega}_I(t) = \mathbf{H}^I_B(t)\mathbf{\omega}_B(t)
\]

Vector Derivative Expressed in a Rotating Frame

Chain Rule

\[
\mathbf{h}_I = \mathbf{H}^I_B\mathbf{h}_B + \mathbf{H}^I_B\dot{\mathbf{h}}_B
\]

Alternatively

\[
\dot{\mathbf{h}}_I = \mathbf{H}^I_B\mathbf{\omega}_B + \mathbf{\omega}_I \times \mathbf{h}_I = \mathbf{H}^I_B\dot{\mathbf{h}}_B + \mathbf{\tilde{\omega}}_I\mathbf{h}_I
\]

Consequently, the 2\textsuperscript{nd} term is

\[
\mathbf{\hat{H}}^I_B\mathbf{h}_B = \mathbf{\tilde{\omega}}_I\mathbf{h}_I = \mathbf{\tilde{\omega}}_I\mathbf{H}^I_B\mathbf{h}_B
\]

... where the cross-product equivalent matrix of angular rate is

\[
\mathbf{\tilde{\omega}} = \begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0
\end{bmatrix}
\]
External Moment Causes Change in Angular Rate

Positive rotation of Frame B w.r.t. Frame A is a negative rotation of Frame A w.r.t. Frame B

In the body frame of reference, the angular momentum change is

\[
\dot{\mathbf{h}}_B = H_B^B \dot{\mathbf{h}}_I + \dot{H}_B^B \mathbf{h}_I = H_B^B \dot{\mathbf{h}}_I - \mathbf{\omega}_B \times \mathbf{h}_B = H_B^B \dot{\mathbf{h}}_I - \bar{\mathbf{\omega}}_B \mathbf{h}_B \\
= H_B^B \mathbf{M}_I - \bar{\mathbf{\omega}}_B \mathcal{I}_B \mathbf{\omega}_B = \mathbf{M}_B - \bar{\mathbf{\omega}}_B \mathcal{I}_B \mathbf{\omega}_B
\]

Moment = torque = force x moment arm

\[
\mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} ; \quad \mathbf{M}_B = H_B^B \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}
\]

Rate of Change of Body-Referenced Angular Rate due to External Moment

In the body frame of reference, the angular momentum change is

\[
\dot{\mathbf{h}}_B = H_B^B \dot{\mathbf{h}}_I + \dot{H}_B^B \mathbf{h}_I = H_B^B \dot{\mathbf{h}}_I - \mathbf{\omega}_B \times \mathbf{h}_B \\
= H_B^B \dot{\mathbf{h}}_I - \bar{\mathbf{\omega}}_B \mathbf{h}_B = H_B^B \mathbf{M}_I - \bar{\mathbf{\omega}}_B \mathcal{I}_B \mathbf{\omega}_B \\
= \mathbf{M}_B - \bar{\mathbf{\omega}}_B \mathcal{I}_B \mathbf{\omega}_B
\]

For constant body-axis inertia matrix

\[
\dot{\mathbf{h}}_B = \mathcal{I}_B \bar{\mathbf{\omega}}_B = \mathbf{M}_B - \bar{\mathbf{\omega}}_B \mathcal{I}_B \mathbf{\omega}_B
\]

Consequently, the differential equation for angular rate of change is

\[
\dot{\mathbf{\omega}}_B = \mathcal{I}_B^{-1} \left( \mathbf{M}_B - \bar{\mathbf{\omega}}_B \mathcal{I}_B \mathbf{\omega}_B \right)
\]
Checklist

- Why is it inconvenient to solve momentum rate equations in an inertial reference frame?
- Are angular rate and momentum vectors aligned?
- How are angular rate equations transformed from an inertial to a body frame?

Next Time:
Aircraft Equations of Motion: Flight Path Computation

Reading:
*Flight Dynamics*
161-180

Learning Objectives

- How is a rotating reference frame described in an inertial reference frame?
- Is the transformation singular?
- What adjustments must be made to expressions for forces and moments in a non-inertial frame?
- How are the 6-DOF equations implemented in a computer?
  - Damping effects
Moments and Products of Inertia

(Bedford & Fowler)

Moments and products of inertia tabulated for geometric shapes with uniform density

Construct aircraft moments and products of inertia from components using parallel-axis theorem

Model in CREO, etc.

Thin circular plate

\[
I_{x\text{,min}} = I_{y\text{,min}} = \frac{1}{4} mR^2, \quad I_{z\text{,min}} = \frac{1}{2} mR^2, \\
I_{xy} = I_{yx} = I_{x'y'} = 0.
\]

Slender bar

\[
I_{x\text{,min}} = 0, \quad I_{y\text{,min}} = I_{z\text{,min}} = \frac{1}{3} mR^2, \\
I_{xy} = I_{yz} = I_{zx} = 0, \\
I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.
\]

Rectangular prism

\[
I_{x\text{,min}} = \frac{1}{12} m(a^2 + b^2), \quad I_{y\text{,min}} = \frac{1}{12} m(a^2 + c^2), \\
I_{z\text{,min}} = \frac{1}{12} m(b^2 + c^2), \quad I_{xy} = I_{yz} = I_{zx} = 0.
\]

Circular cylinder

\[
I_{x\text{,min}} = I_{y\text{,min}} = m\left(\frac{3}{4}R^2 + \frac{1}{8} r^2\right), \quad I_{z\text{,min}} = \frac{1}{2} mR^2, \\
I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.
\]