Spacecraft Guidance
Space System Design, MAE 342, Princeton University
Robert Stengel

- Oberth’s “Synergy Curve”
- Explicit ascent guidance
- Impulsive $\Delta V$ maneuvers
- Hohmann transfer between circular orbits
- Sphere of gravitational influence
- Synodic periods and launch windows
- Hyperbolic orbits and escape trajectories
- Battin’s universal formulas
- Lambert’s time-of-flight theorem (hyperbolic orbit)
- Fly-by (swingby) trajectories for gravity assist

Guidance, Navigation, and Control

- **Navigation**: Where are we?
- **Guidance**: How do we get to our destination?
- **Control**: What do we tell our vehicle to do?
Energy Gained from Propellant

Specific energy = energy per unit weight

\[ \mathcal{E} = h + \frac{V^2}{2g} \]

\( h \): height; \( V \): velocity

Rate of change of specific energy per unit of expended propellant mass

\[
\frac{d\mathcal{E}}{dm} = \frac{dh}{dm} + \frac{V}{g} \frac{dV}{dm} = \frac{1}{(dm/dt)} \left( \frac{dh}{dt} + \frac{V}{g} \frac{dV}{dt} \right)
\]

\[
= \frac{1}{(dm/dt)} \left( \frac{dh}{dt} + \frac{1}{g} V^2 \frac{dV}{dt} \right) = \frac{1}{(dm/dt)} \left( V \sin \gamma + \frac{1}{g} V^2 (T - mg) \right)
\]

\[
= \frac{1}{(dm/dt)} \left( V \sin \gamma + \frac{VT}{mg} \cos \alpha - V \sin \gamma \right)
\]

Oberth’s Synergy Curve

\( \gamma \): Flight Path Angle
\( \theta \): Pitch Angle
\( \alpha \): Angle of Attack

\( d\mathcal{E}/dm \) maximized when \( \alpha = 0 \), or \( \theta = \gamma \), i.e., thrust along the velocity vector

Approximate round-earth equations of motion

\[
\frac{dV}{dt} = \frac{T}{m} \cos \alpha - \frac{\text{Drag}}{m} - g \sin \gamma
\]

\[
\frac{dy}{dt} = \frac{T}{mV} \sin \alpha + \left( \frac{V}{r} - \frac{g}{V} \right) \cos \gamma
\]
Gravity-Turn Pitch Program

With angle of attack, $\alpha = 0$

$$\frac{d\gamma}{dt} = \frac{d\theta}{dt} = \left(\frac{V}{r} - \frac{g}{V}\right) \cos \gamma$$

Locally optimal flight path
Minimizes aerodynamic loads
Feedback controller minimizes $\alpha$ or load factor

Gravity-Turn Flight Path

• Gravity-turn flight path is function of 3 variables
  – Initial pitchover angle (from vertical launch)
  – Velocity at pitchover
  – Acceleration profile, $T(t)/m(t)$

Gravity-turn program closely approximated by tangent steering laws (see Supplemental Material)
Feedback Control Law

Errors due to disturbances and modeling errors corrected by feedback control

Motor Gimbal Angle\( (t) \triangleq \delta_c(t) = c_\theta \left[ \theta_{des}(t) - \theta(t) \right] - c_q q(t) \)

\( \theta_{des} = \) Desired pitch angle; \( q = \frac{d\theta}{dt} = \) pitch rate

\( c_\theta, c_q \): Feedback control law gains

Thrust Vector Control During Launch Through Wind Profile

- **Attitude control**
  - Attitude and rate feedback
- **Drift-minimum control**
  - Attitude and accelerometer feedback
  - Increased loads
- **Load relief control**
  - Rate and accelerometer feedback
  - Increased drift
Effect of Launch Latitude on Orbital Parameters

- Launch latitude establishes **minimum** orbital inclination (without “dogleg” maneuver)
- Time of launch establishes line of nodes
- Argument of perigee established by
  - Launch trajectory
  - On-orbit adjustment

Guidance Law for Launch to Orbit

(Brand, Brown, Higgins, and Pu, CSDL, 1972)

- **Initial conditions**
  - End of pitch program, outside atmosphere
- **Final condition**
  - Insertion in desired orbit
- **Initial inputs**
  - Desired radius
  - Desired velocity magnitude
  - Desired flight path angle
  - Desired inclination angle
  - Desired longitude of the ascending/descending node
- **Continuing outputs**
  - Unit vector describing desired thrust direction
  - Throttle setting, % of maximum thrust
Guidance Program Initialization

- Thrust acceleration estimate
- Mass/mass flow rate
- Acceleration limit (~ 3g)
- Effective exhaust velocity
- Various coefficients
- Unit vector normal to desired orbital plane, $i_q$

\[
   \mathbf{i}_q = \begin{bmatrix}
   \sin i_d \sin \Omega_d \\
   \sin i_d \cos \Omega_d \\
   \cos i_d
   \end{bmatrix}
\]

$i_d$: Desired inclination angle of final orbit
$\Omega_d$: Desired longitude of descending node

Guidance Program Operation: Position and Velocity

- Obtain thrust acceleration estimate, $a_T$, from guidance system
- Compute corresponding mass, mass flow rate, and throttle setting, $\delta T$

\[
   \mathbf{i}_r = \frac{\mathbf{r}}{r} : \text{Unit vector aligned with local vertical}
\]

\[
   \mathbf{i}_c = \mathbf{i}_r \times \mathbf{i}_q : \text{Downrange direction}
\]

\[
   \begin{bmatrix}
   r \\
   y \\
   z
   \end{bmatrix} = \begin{bmatrix}
   |r| \\
   r \sin^{-1}(\mathbf{i} \cdot \mathbf{i}_q) \\
   \text{open}
   \end{bmatrix}
\]

\[
   \begin{bmatrix}
   \dot{r} \\
   \dot{y} \\
   \dot{z}
   \end{bmatrix} = \begin{bmatrix}
   \mathbf{v}_{\text{IMU}} \cdot \mathbf{i}_r \\
   \mathbf{v}_{\text{IMU}} \cdot \mathbf{i}_q \\
   \mathbf{v}_{\text{IMU}} \cdot \mathbf{i}_c
   \end{bmatrix}
\]

$v_{\text{IMU}}$: Velocity estimate in IMU frame
Guidance Program: Velocity and Time to Go

Effective gravitational acceleration

\[ g_{\text{eff}} = -\frac{\mu}{r^2} + \frac{|\mathbf{r} \times \mathbf{v}|^2}{r^3} \]

Time to go (to motor burnout)

\[ t_{g_{\text{new}}} = t_{g_{\text{old}}} - \Delta t \]
\[ \Delta t: \text{Guidance command interval} \]

Velocity to be gained

\[ \mathbf{v}_{g_0} = \begin{bmatrix} \dot{r}_d - r \frac{g_{\text{eff}} t_{g_0}}{2} \\ -\dot{y} \\ \dot{z}_d - \dot{z} \end{bmatrix} \]

Time to go prediction (prior to acceleration limiting)

\[ t_{g_0} = \frac{m}{\dot{m}} \left( 1 - e^{-v_{g_0} / c_{\text{eff}}} \right) \]
\[ c_{\text{eff}}: \text{Effective exhaust velocity} \]

Guidance Program Commands

Guidance law: required radial and cross-range accelerations

\[ a_r = a_T \left[ A + B(t - t_x) \right] - g_{\text{eff}} \]
\[ a_r = a_T \left[ C + D(t - t_y) \right] \]
\[ a_T = \text{Net available acceleration} \]

Guidance coefficients, \( A, B, C, \) and \( D \) are functions of

\[ \left( r_d, \dot{r}, \dot{r}_{g_0}, r_d, \dot{r}, \dot{r}_{g_0} \right) \]
\[ \left( y, \dot{y}, t_{g_0} \right) \]
\[ \text{plus } c_{\text{eff}}, m/\dot{m}, \text{Acceleration limit} \]

Required thrust direction, \( \mathbf{i}_T \) (i.e., vehicle orientation in \( \mathbf{i}_n, \mathbf{i}_q, \mathbf{i}_z \) frame)

\[ \mathbf{a}_T = \begin{bmatrix} a_{T_r} \\ a_{T_t} \\ \text{what's left over} \end{bmatrix} \]
\[ i_T = \frac{\mathbf{a}_T}{\|\mathbf{a}_T\|} \]

Throttle command is a function of \( a_T \) (i.e., acceleration magnitude) and acceleration limit
Impulsive $\Delta V$ Orbital Maneuver

- If rocket burn time is short compared to orbital period (e.g., seconds compared to hours), \textit{impulsive $\Delta V$} approximation can be made
  - Change in position during burn is $\sim$ zero
  - Change in velocity is $\sim$ instantaneous

### Vector diagram of velocity change

*Velocity impulse at apogee*

*Orbit Change due to Impulsive $\Delta V$*

- Maximum energy change accrues when $\Delta V$ is aligned with the instantaneous orbital velocity vector
  - Energy change $\rightarrow$ Semi-major axis change
  - Maneuver at perigee raises or lowers apogee
  - Maneuver at apogee raises or lowers perigee
- Optimal transfer from one circular orbit to another involves two impulses [*Hohmann transfer*]
- Other maneuvers
  - In-plane parameter change
  - Orbital plane change
Assumptions for Impulsive Maneuver

Instantaneous change in velocity vector

\[ \mathbf{v}_2 = \mathbf{v}_1 + \Delta \mathbf{v}_{\text{rocket}} \]

Negligible change in radius vector

\[ \mathbf{r}_2 = \mathbf{r}_1 \]

Therefore, new orbit intersects old orbit
Velocities different at the intersection

Geometry of Impulsive Maneuver

Change in velocity magnitude, \(|\mathbf{v}|\), vertical flight path angle, \(\gamma\), and horizontal flight path angle, \(\xi\)

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} V \cos \gamma \cos \xi \\ V \cos \gamma \sin \xi \\ -V \sin \gamma \end{bmatrix} \\
\mathbf{v}_2 &= \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_2 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_2 = \begin{bmatrix} V \cos \gamma \cos \xi \\ V \cos \gamma \sin \xi \\ -V \sin \gamma \end{bmatrix}_2 \\
\Delta \mathbf{v} &= \left[ \begin{array}{c} \Delta v_x \\
\Delta v_y \\
\Delta v_z \end{array} \right] = \left[ \begin{array}{c} (v_{x_2} - v_{x_1}) \\
(v_{y_2} - v_{y_1}) \\
(v_{z_2} - v_{z_1}) \end{array} \right]
\end{align*}
\]
Required $\Delta v$ for Impulsive Maneuver

$$
\Delta v = \begin{bmatrix}
\Delta v_x \\
\Delta v_y \\
\Delta v_z
\end{bmatrix} = \begin{bmatrix}
(v_{x_2} - v_{x_1}) \\
(v_{y_2} - v_{y_1}) \\
(v_{z_2} - v_{z_1})
\end{bmatrix}
$$

$$
\Delta v_{\text{rocket}} = \begin{bmatrix}
\Delta V_{\text{rocket}} \\
\delta_{\text{rocket}} \\
\gamma_{\text{rocket}}
\end{bmatrix} = \begin{bmatrix}
\left(\Delta v_x^2 + \Delta v_y^2 + \Delta v_z^2\right)^{1/2} \\
\sin^{-1}\left(\frac{\Delta v_y}{\left(\Delta v_x^2 + \Delta v_y^2\right)^{1/2}}\right) \\
\sin^{-1}\left(\frac{\Delta v_z}{\Delta V}\right)_{\text{rocket}}
\end{bmatrix}
$$

Single Impulse Orbit Adjustment

Coplanar (i.e., in-plane) maneuvers

- Change energy
- Change angular momentum
- Change eccentricity

- Required velocity increment

$$
\mathcal{E} = \frac{1}{2}v^2 - \mu/r = (e^2 - 1)\mu^2/h^2
$$

$$
h = \sqrt{\frac{\mu^2(e^2 - 1)}{\mathcal{E}}} = \sqrt{\frac{\mu^2(e^2 - 1)}{v^2/2 - \mu/r}}
$$

$$
e = \sqrt{1 + 2\mathcal{E}h^2/\mu^2}
$$

$$
v_{\text{new}} \triangleq v_{\text{old}} + \Delta v_{\text{rocket}} = \sqrt{2\left(\mathcal{E}_{\text{new}} + \mu/r\right)}
$$

$$= \sqrt{2\left( (e_{\text{new}}^2 - 1)\mu^2/h_{\text{new}}^2 + \mu/r \right)}
$$

$$
\Delta v_{\text{rocket}} = v_{\text{new}} - v_{\text{old}}
$$
Single Impulse Orbit Adjustment
Coplanar (i.e., in-plane) maneuvers

• Change semi-major axis
  – magnitude
  – orientation (i.e., argument of perigee); in-plane isosceles triangle

\[ a_{\text{new}} = \frac{h_{\text{new}}^2}{\mu} \left( \frac{1}{1 - e_{\text{new}}^2} \right) \]

• Change apogee or perigee
  – radius
  – velocity

\[ r_{\text{perigee}} = a (1 - e) \]
\[ r_{\text{apogee}} = a (1 + e) \]
\[ v_{\text{perigee}} = \sqrt{\frac{\mu}{a}} \left( \frac{1 + e}{1 - e} \right) \]
\[ v_{\text{apogee}} = \sqrt{\frac{\mu}{a}} \left( \frac{1 - e}{1 + e} \right) \]

In-Plane Orbit Circularization

Initial orbit is elliptical, with apogee radius equal to desired circular orbit radius

\[ a = \left( r_{\text{cir(target)}} + r_{\text{insertion}} \right) / 2 \]
\[ e = \left( r_{\text{cir(target)}} - r_{\text{insertion}} \right) / 2a \]
\[ v_{\text{apogee}} = \sqrt{\frac{\mu}{a}} \left( \frac{1 - e}{1 + e} \right) \]

Velocity in circular orbit is a function of the radius

“Vis viva” equation:

\[ v_{\text{cir}} = \sqrt{\frac{2}{r_{\text{cir}}} - \frac{1}{a_{\text{cir}}}} = \sqrt{\frac{2}{r_{\text{cir}}}} - \frac{1}{a_{\text{cir}}} = \sqrt{\frac{\mu}{r_{\text{cir}}}} \]

Rocket must provide the difference

\[ \Delta v_{\text{rocket}} = v_{\text{cir}} - v_{\text{apogee}} \]
Single Impulse Orbit Adjustment
Out-of-plane maneuvers

- Change orbital inclination
- Change longitude of the ascending node
- $\nu_1, \Delta \nu$, and $\nu_2$ form isosceles triangle perpendicular to the orbital plane to leave in-plane parameters unchanged

Change in Inclination and Longitude of Ascending Node

Sellers, 2005
Two Impulse Maneuvers

Transfer to Non-Intersecting Orbit

1st $\Delta v$ produces target orbit intersection
2nd $\Delta v$ matches target orbit
Minimize $(|\Delta v_1| + |\Delta v_2|)$ to minimize propellant use
Rendezvous with trailing spacecraft in same orbit
At perigee, increase speed to increase orbital period
At future perigee, decrease speed to resume original orbit

Hohmann Transfer between Coplanar Circular Orbits
(Outward transfer example)

Thrust in direction of motion at transfer perigee and apogee

Transfer Orbit

\[
\begin{align*}
a & = \frac{(r_{c_{ir_1}} + r_{c_{ir_2}})}{2} \\
e & = \frac{(r_{c_{ir_2}} - r_{c_{ir_1}})}{2a} \\
v_{p_{transfer}} & = \sqrt{\frac{\mu}{a}} \frac{1 + e}{1 - e} \\
v_{a_{transfer}} & = \sqrt{\frac{\mu}{a}} \frac{1 - e}{1 + e}
\end{align*}
\]
Outward Transfer Orbit
Velocity Requirements

\[
\Delta v_{1} = v_{p\text{transfer}} - v_{\text{circ}_1}
= v_{\text{circ}_1} \left( \sqrt{\frac{2r_{\text{circ}_2}}{r_{\text{circ}_1} + r_{\text{circ}_2}}} - 1 \right)
\]

\[
\Delta v_{2} = v_{\text{circ}_1} - v_{p\text{transfer}}
= v_{\text{circ}_2} \left( 1 - \sqrt{\frac{2r_{\text{circ}_1}}{r_{\text{circ}_1} + r_{\text{circ}_2}}} \right)
\]

\[v_{\text{circ}_2} = v_{\text{circ}_1} \sqrt{\frac{r_{\text{circ}_1}}{r_{\text{circ}_2}}}\]

Hohmann Transfer is energy-optimal for 2-impulse transfer between circular orbits and \( r_2/r_1 < 11.94 \)

\[
\Delta v_{\text{total}} = v_{\text{circ}_1} \left[ \sqrt{\frac{2r_{\text{circ}_2}}{r_{\text{circ}_1} + r_{\text{circ}_2}}} \left( 1 - \frac{r_{\text{circ}_1}}{r_{\text{circ}_2}} \right) + \sqrt{\frac{r_{\text{circ}_1}}{r_{\text{circ}_2}}} - 1 \right]
\]

Rendezvous Requires Phasing of the Maneuver
Transfer orbit time equals target’s time to reach rendezvous point

Sellers, 2005
Solar Orbits

• Same equations used for Earth-referenced orbits
  – Dimensions of the orbit
  – Position and velocity of the spacecraft
  – Period of elliptical orbits
  – Different gravitational constant

\[ \mu_{\text{Sun}} = 1.3327 \times 10^{11} \text{ km}^3/\text{s}^2 \]

Escape from a Circular Orbit
Minimum escape trajectory shape is a parabola
In-plane Parameters of Earth Escape Trajectories

Dimensions of the orbit

\[ p = \frac{h^2}{\mu} \] = "The parameter" or semi-latus rectum

\[ h = \text{Angular momentum about center of mass} \]

\[ e = \sqrt{1 + \frac{\mathcal{E} p}{\mu}} = \text{Eccentricity} \geq 1 \]

\[ \mathcal{E} = \text{Specific energy}, \geq 0 \]

\[ a = \frac{p}{1 - e^2} = \text{Semi-major axis}, < 0 \]

\[ r_{\text{perigee}} = a(1 - e) = \text{Perigee radius} \]

Position and velocity of the spacecraft

\[ r = \frac{p}{1 + e \cos \theta} = \text{Radius of the spacecraft} \]

\[ \theta = \text{True anomaly} \]

\[ V = \sqrt{2\mu \left(\frac{1}{r} - \frac{1}{2a}\right)} = \text{Velocity of the spacecraft} \]

\[ V_{\text{perigee}} \geq \sqrt{\frac{2\mu}{r_{\text{perigee}}}} \]
**Escape from Circular Orbit**

**Velocity in circular orbit**

\[
V_c = \sqrt{\frac{2}{r_c} - 1} = \sqrt{\frac{\mu}{r_c}}
\]

**Velocity at perigee of parabolic orbit**

\[
V_\text{perigee} = \sqrt{\mu \left( \frac{2}{r_c} - \frac{1}{(a \to \infty)} \right)} = \sqrt{\frac{2\mu}{r_c}}
\]

**Velocity increment required for escape**

\[
\Delta V_\text{escape} = V_\text{perigee, parabola} - V_c = \sqrt{\frac{2\mu}{r_c}} - \sqrt{\frac{\mu}{r_c}} \approx 0.414V_c
\]

---

**Earth Escape Trajectory**

- \(\Delta v_1\) to increase speed to escape velocity
- Velocity required for transfer at sphere of influence
Transfer Orbits and Spheres of Influence

• **Sphere of Influence (Laplace):**
  – Radius within which gravitational effects of planet are more significant than those of the Sun

• **Patched-conic section approximation**
  – Sequence of 2-body orbits
  – Outside of planet’s sphere of influence, Sun is the center of attraction
  – Within planet’s sphere of influence, planet is the center of attraction

• **Fly-by (swingby) trajectories** dip into intermediate object’s sphere of influence for gravity assist

Solar System Spheres of Influence

\[
\text{for } \frac{m_{\text{Planet}}}{m_{\text{Sun}}} \ll 1, \quad r_{SI} \approx r_{\text{Planet-Sun}} \left( \frac{m_{\text{Planet}}}{m_{\text{Sun}}} \right)^{2/5}
\]

<table>
<thead>
<tr>
<th>Planet</th>
<th>Sphere of Influence, km</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>112,000</td>
</tr>
<tr>
<td>Venus</td>
<td>616,000</td>
</tr>
<tr>
<td>Earth</td>
<td>929,000</td>
</tr>
<tr>
<td>Mars</td>
<td>578,000</td>
</tr>
<tr>
<td>Jupiter</td>
<td>48,200,000</td>
</tr>
<tr>
<td>Saturn</td>
<td>54,500,000</td>
</tr>
<tr>
<td>Uranus</td>
<td>51,800,000</td>
</tr>
<tr>
<td>Neptune</td>
<td>86,800,000</td>
</tr>
<tr>
<td>Pluto</td>
<td>27,000,000-45,000,000</td>
</tr>
</tbody>
</table>
Interplanetary Mission Planning

- Example: Direct Hohmann Transfer from Earth Orbit to Mars Orbit (No fly-bys)
  1) Calculate required perigee velocity for transfer orbit - Sun as center of attraction: Elliptical orbit
  2) Calculate $\Delta v$ required to reach Earth’s sphere of influence with velocity required for transfer – Earth as center of attraction: Hyperbolic orbit
  3) Calculate $\Delta v$ required to enter circular orbit about Mars, given transfer apogee velocity – Mars as center of attraction: Hyperbolic orbit

Launch Opportunities for Fixed Transit Time: The Synodic Period

- Synodic Period, $S_n$: The time between conjunctions
  - $P_A$: Period of Planet A
  - $P_B$: Period of Planet B
- Conjunction: Two planets, $A$ and $B$, in a line or at some fixed angle

$$S_n = \frac{P_A P_B}{P_A - P_B}$$
Launch Opportunities for Fixed Transit Time: The Synodic Period

<table>
<thead>
<tr>
<th>Planet</th>
<th>Synodic Period with respect to Earth, days</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>116</td>
<td>88 days</td>
</tr>
<tr>
<td>Venus</td>
<td>584</td>
<td>225 days</td>
</tr>
<tr>
<td>Earth</td>
<td>-</td>
<td>365 days</td>
</tr>
<tr>
<td>Mars</td>
<td>780</td>
<td>687 days</td>
</tr>
<tr>
<td>Jupiter</td>
<td>399</td>
<td>11.9 yr</td>
</tr>
<tr>
<td>Saturn</td>
<td>378</td>
<td>29.5 yr</td>
</tr>
<tr>
<td>Uranus</td>
<td>370</td>
<td>84 yr</td>
</tr>
<tr>
<td>Neptune</td>
<td>367</td>
<td>165 yr</td>
</tr>
<tr>
<td>Pluto</td>
<td>367</td>
<td>248 yr</td>
</tr>
</tbody>
</table>

Hyperbolic Orbits

<table>
<thead>
<tr>
<th>Orbit Shape</th>
<th>Eccentricity, e</th>
<th>Energy, E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>0</td>
<td>&lt;0</td>
</tr>
<tr>
<td>Ellipse</td>
<td>0 &lt; e &lt;1</td>
<td>&lt;0</td>
</tr>
<tr>
<td>Parabola</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>&gt;1</td>
<td>&gt;0</td>
</tr>
</tbody>
</table>

\[ E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}, \quad \therefore a < 0 \]

Velocity remains positive as radius approaches \( \infty \)

\[ v \xrightarrow{r \to \infty} v_\infty \]

\[ E_\infty = \frac{v_\infty^2}{2}, \quad \text{and} \quad v_\infty = \sqrt{-\frac{\mu}{a}} \quad \text{or} \quad a = -\frac{\mu}{v_\infty^2} \]
Hyperbolic Encounter with a Planet

- Trajectory is deflected by target planet’s gravitational field
  - In-plane
  - Out-of-plane
- Velocity w.r.t. Sun is increased or decreased

\[ \Delta : \text{Miss Distance, km} \]
\[ \delta : \text{Deflection Angle, deg or rad} \]

Hyperbolic Orbits

Asymptotic Value of True Anomaly

Polar Equation for a Conic Section

\[ r = \frac{p}{1 + e \cos \theta} = \frac{a(1 - e^2)}{1 + e \cos \theta} \]

\[ \cos \theta = \frac{1}{e} \left[ \frac{a(1 - e^2)}{r} - 1 \right] \]

\[ \theta \xrightarrow{r \to \infty} \theta_\infty \]
\[ \theta_\infty = \cos^{-1} \left( -\frac{1}{e} \right) \]
Hyperbolic Orbits

**Angular Momentum**

\[ h = \text{Constant} = v_\infty \Delta \]

\[ = \sqrt{\mu p} = \sqrt{\mu a(1 - e^2)} = \sqrt{\frac{\mu^2 (e^2 - 1)}{v_\infty^2}} \]

**Eccentricity**

\[ e = \sqrt{1 + \frac{2h^2 E}{\mu^2}} = \sqrt{1 + \frac{v_\infty^4 \Delta^2}{\mu^2}} \]

**Perigee Radius**

\[ r_p = a(1 - e) = \frac{\mu}{v_\infty^2} (e - 1) \]

Hyperbolic Mean and Eccentric Anomalies

\[ H : \text{Hyperbolic Eccentric Anomaly} \]

\[ M = e \sinh H - H \]

*Newton’s method of successive approximation* to find \( H \) from \( M \), similar to solution for \( E \) (Lecture 2)

\[ \theta(t) = 2 \tan^{-1} \left[ \sqrt{\frac{e+1}{e-1}} \tanh \frac{H(t)}{2} \right] \]

\[ r = a \left(1 - e \cosh H\right) \]

*see Ch. 7, Kaplan, 1976*
Battin’s Universal Formulas for Conic Section Position and Velocity as Functions of Time

\[ r(t_2) = \left[ 1 - \frac{\chi^2}{r(t_1)} \right] C\left( \frac{\chi^2}{a} \right) r(t_1) + \left[ t_2 - \frac{\chi^3}{\sqrt{\mu}} S\left( \frac{\chi^2}{a} \right) \right] v(t_1) \]

\[ v(t_2) = \frac{\sqrt{\mu}}{r(t_1) r(t_2)} \left[ \frac{\chi^3}{a} S\left( \frac{\chi^2}{a} \right) - \chi \right] r(t_1) + \left[ 1 - \frac{\chi^2}{r(t_2)} C\left( \frac{\chi^2}{a} \right) \right] v(t_1) \]

\[ \chi = \begin{cases} \sqrt{a} \left[ E(t_2) - E(t_1) \right], & \text{Ellipse} \\ \sqrt{a} \left[ H(t_2) - H(t_1) \right], & \text{Hyperbola} \\ \sqrt{p} \left[ \tan \frac{\theta(t_2)}{2} - \tan \frac{\theta(t_1)}{2} \right], & \text{Parabola} \end{cases} \]

see Ch. 7, Kaplan, 1976; also Battin, 1964

Lambert’s Time-of-Flight Theorem

(\textit{Hyperbolic Orbit})

\[(t_2 - t_1) = \sqrt{\frac{-a^3}{\mu}} \left[ (\sinh \gamma - \gamma) + (\sinh \delta - \delta) \right] \]

where

\[ \gamma \triangleq 2 \sinh^{-1} \sqrt{\frac{r_1 + r_2 + c}{-4a}}; \quad \delta \triangleq 2 \sinh^{-1} \sqrt{\frac{r_1 + r_2 - c}{-4a}} \]

see Ch. 7, Kaplan, 1976; also Battin, 1964


Asteroid Encounter

Swing-By/Fly-By Trajectories

- Hyperbolic encounters with planets and the moon provide gravity assist
  - Shape, energy, and duration of transfer orbit altered
  - Potentially large reduction in rocket $\Delta V$ required to accomplish mission

Why does gravity assist work?
Effect of Target Planet’s Gravity on Probe’s Sun-Relative Velocity

**Deflection – Velocity Reduction**

Effect of Target Planet’s Gravity on Probe’s Sun-Relative Velocity

**Deflection – Velocity Addition**
Planet Capture Trajectory

Hyperbolic approach to planet’s sphere of influence
\( \Delta v \) to decrease speed to circular velocity

Effect of Target Planet’s Gravity on Probe’s Velocity
Next Time:
Spacecraft Environment

Supplemental Material
Phases of Ascent Guidance

- Vertical liftoff
- Roll to launch azimuth
- Pitch program to atmospheric “exit”
  - Jet stream penetration
  - Booster cutoff and staging
- Explicit guidance to desired orbit
  - Booster separation
  - Acceleration limiting
  - Orbital insertion

Tangent Steering Laws

- Neglecting surface curvature
  \[ \tan \theta(t) = \tan \theta_0 \left( 1 - \frac{t}{t_{BO}} \right) \]
- “Open-loop” command, i.e., no feedback of vehicle state
- Accounting for effect of Earth surface curvature on burnout flight path angle
  \[ \tan \theta(t) = \tan \theta_0 \left( 1 - \frac{t}{t_{BO}} - \tan \beta \frac{t}{t_{BO}} \right) \]
Jet Stream Profiles

- Launch vehicle must be able to fly through strong wind profiles
- Design profiles assume 95th-99th-percentile worst winds and wind shear

Longitudinal (2-D) Equations of Motion for Re-Entry

Differential equations for velocity ($x_1 = V$), flight path angle ($x_2 = \gamma$), altitude ($x_3 = h$), and range ($x_4$)

Angle of attack ($\alpha$) is optimization control variable

\[
\begin{align*}
\dot{x}_1 &= -D(x_1, x_3, \alpha)/m - g \cos x_2 \\
\dot{x}_2 &= \left[g/x_1 - x_1 / (R + x_3)\right] \sin x_2 - L(x_1, x_3, \alpha) / mx_1 \\
\dot{x}_3 &= x_1 \cos x_2 \\
\dot{x}_4 &= x_1 \sin x_2 / (1 + x_3 / R)
\end{align*}
\]

Equations of motion define the dynamic constraint

\[
\dot{x}(t) = f[x(t), \alpha(t)]
\]
A Different Approach to Guidance: Optimizing a Cost Function

- **Minimize a scalar function**, $J$, of terminal and integral costs

\[
    J = \phi[x(t_f)] + \int_{t_0}^{t_f} L[x(t),u(t)] dt
\]

with respect to the control, $u(t)$, in $(t_0, t_f)$, subject to a dynamic constraint

\[
    \dot{x}(t) = f[x(t),u(t)], \quad x(t_0) \text{ given}
\]

Guidance Cost Function

- **Terminal cost**, e.g., in final position and velocity
- **Integral cost**, e.g., tradeoff between control usage and trajectory error

- Minimization of cost function determines the optimal state and control, $x^*$ and $u^*$, over the flight path duration

\[
    \min_u J = \min_u \left\{ \phi[x(t_f)] + \int_{t_0}^{t_f} L[x(t),u(t)] dt \right\}
\]

\[
    = \phi[x^*(t_f)] + \int_{t_0}^{t_f} L[x^*(t),u^*(t)] dt \rightarrow [x^*(t),u^*(t)]
\]
Example of Re-Entry Flight Path Cost Function

\[ J = a[V(t_f) - V_d]^2 + b[r(t_f) - r_d]^2 + \int_{t_o}^{t_f} c[\alpha(t)]^2 dt \]

- Cost function includes
  - Terminal range and velocity
  - Penalty on control use
  - \( a, b, \) and \( c \) tradeoff importance of each factor
- Minimization of this cost function
  - Defines the optimal path, \( x^*(t) \), from \( t_o \) to \( t_f \)
  - Defines the optimal control, \( \alpha^*(t) \), from \( t_o \) to \( t_f \)

Extension to Three Dimensions

- Add roll angle as a control; add crossrange as a state
- For the guidance law, replace range and crossrange from the starting point by distance to go and azimuth to go to the destination
Optimal Trajectories for Space Shuttle Transition

Altitude vs. Velocity

Range vs. Cross-Range ("footprint")

Optimal Controls for Space Shuttle Transition

Angle of Attack and Roll Angle vs. Specific Energy
Optimal Guidance System Derived from Optimal Trajectories

Angle of Attack and Roll Angle vs. Specific Energy

Diagram of Energy-Guidance Law

Guidance Functions for Space Shuttle Transition

Angle of Attack Guidance Function

Roll Angle Guidance Function
Lunar Module Navigation, Guidance, and Control Configuration

Lunar Module Transfer Ellipse to Powered Descent Initiation
Lunar Module Powered Descent

<table>
<thead>
<tr>
<th>PHASE</th>
<th>INITIAL EVENT</th>
<th>DESIGN CRITERIA</th>
</tr>
</thead>
<tbody>
<tr>
<td>BRAKING</td>
<td>PDI</td>
<td>MINIMIZE PROPELLANT USAGE</td>
</tr>
<tr>
<td>APPROACH</td>
<td>HIGH GATE</td>
<td>CREW VISIBILITY</td>
</tr>
<tr>
<td>LANDING</td>
<td>LOW GATE</td>
<td>MANUAL CONTROL</td>
</tr>
</tbody>
</table>

Lunar Module Descent Events

<table>
<thead>
<tr>
<th>Event</th>
<th>TFI, min</th>
<th>Inertial velocity, fps</th>
<th>Altitude rate, fps</th>
<th>Altitude, ft</th>
<th>ΔV, fps</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Ulage</td>
<td>-00:07</td>
<td>5360</td>
<td>-4</td>
<td>48 814</td>
<td>0</td>
</tr>
<tr>
<td>B Pow-red descent initiation</td>
<td>00:14</td>
<td>5529</td>
<td>-3</td>
<td>48 725</td>
<td>31</td>
</tr>
<tr>
<td>C Throttle to maximum thrust</td>
<td>00:26</td>
<td>5000</td>
<td>-50</td>
<td>44 924</td>
<td>1572</td>
</tr>
<tr>
<td>D Rotate to window-up position</td>
<td>02:56</td>
<td>3065</td>
<td>-89</td>
<td>29 201</td>
<td>2535</td>
</tr>
<tr>
<td>E LR altitude update</td>
<td>04:18</td>
<td>1456</td>
<td>-106</td>
<td>24 635</td>
<td>4239</td>
</tr>
<tr>
<td>F Throttle recovery</td>
<td>06:24</td>
<td>1315</td>
<td>-127</td>
<td>22 644</td>
<td>4399</td>
</tr>
<tr>
<td>G LR velocity update</td>
<td>06:42</td>
<td>506</td>
<td>-145</td>
<td>7 515</td>
<td>5375</td>
</tr>
<tr>
<td>H High gate</td>
<td>08:26</td>
<td>55 (a)</td>
<td>-16</td>
<td>512</td>
<td>6176</td>
</tr>
<tr>
<td>I Low gate</td>
<td>10:06</td>
<td>15 (b)</td>
<td>12</td>
<td>8775</td>
<td></td>
</tr>
</tbody>
</table>

(a) Time from ignition of the DPS.
(b) Horizontal velocity relative to surface.
Lunar Module Descent
Targeting Sequence

Braking Phase (P63)
Approach Phase (P64)
Terminal Descent Phase (P66)

Characterize Braking Phase
By Five Points
Lunar Module Descent Guidance Logic
(Klumpp, *Automatica*, 1974)

- Reference (nominal) trajectory, \( r(t) \), from target position back to starting point (Braking Phase example)
  - Three 4th-degree polynomials in time
  - 5 points needed to specify each polynomial

\[
r_r(t) = r_t + v_t t + a_t \frac{t^2}{2} + j_t \frac{t^3}{6} + s_t \frac{t^4}{24}
\]

\[
r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
\]

Coefficients of the Polynomials

- \( r \) = position vector
- \( v \) = velocity vector
- \( a \) = acceleration vector
- \( j \) = jerk vector (time derivative of acceleration)
- \( s \) = snap vector (time derivative of jerk)
Corresponding Reference Velocity and Acceleration Vectors

\[ \mathbf{v}_r(t) = \mathbf{v}_t + \mathbf{a}_t t + \mathbf{j}_t \frac{t^2}{2} + \mathbf{s}_t \frac{t^3}{6} \]

\[ \mathbf{a}_r(t) = \mathbf{a}_t + \mathbf{j}_t t + \mathbf{s}_t \frac{t^2}{2} \]

- \( \mathbf{a}_r(t) \) is the reference control vector
  - Descent engine thrust / mass = total acceleration
  - Vector components controlled by orienting yaw and pitch angles of the Lunar Module

Guidance Logic Defines Desired Acceleration Vector

- If initial conditions, dynamic model, and thrust control were perfect, \( \mathbf{a}_r(t) \) would produce \( \mathbf{r}_r(t) \)

\[ \mathbf{a}_r(t) = \mathbf{a}_t + \mathbf{j}_t t + \mathbf{s}_t \frac{t^2}{2} \implies \mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24} \]

- ... but they are not
- Therefore, feedback control is required to follow the reference trajectory
Guidance Law for the Lunar Module Descent

Linear feedback guidance law

\[ a_{\text{command}}(t) = a_r(t) + K_v [v_{\text{measured}}(t) - v_r(t)] + K_r [r_{\text{measured}}(t) - r_r(t)] \]

- \( K_v \): velocity error gain
- \( K_r \): position error gain

Nominal acceleration profile is corrected for measured differences between actual and reference flight paths

Considerable modifications made in actual LM implementation (see Klumpp’s original paper on Blackboard)