

Spacecraft Guidance

Space System Design, MAE 342, Princeton University
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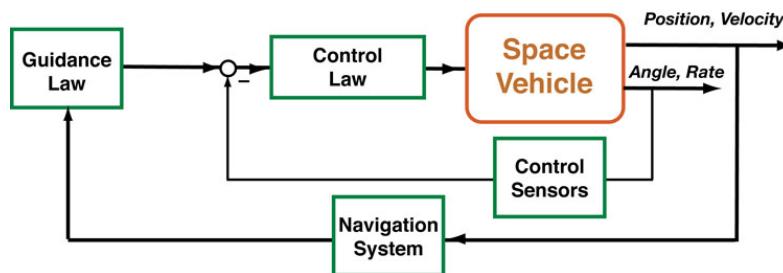
- Oberth's "Synergy Curve"
- Explicit ascent guidance
- Impulsive ΔV maneuvers
- Hohmann transfer between circular orbits
- Sphere of gravitational influence
- Synodic periods and launch windows
- Hyperbolic orbits and escape trajectories
- Battin's universal formulas
- Lambert's time-of-flight theorem (hyperbolic orbit)
- Fly-by (swingby) trajectories for gravity assist

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<http://www.princeton.edu/~stengel/MAE342.html>*

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Guidance, Navigation, and Control

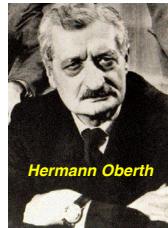


- **Navigation:** Where are we?
- **Guidance:** How do we get to our destination?
- **Control:** What do we tell our vehicle to do?

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Energy Gained from Propellant

Specific energy = energy per unit weight

$$E = h + \frac{V^2}{2g}$$

h : height; V : velocity

Rate of change of specific energy per unit of expended propellant mass

$$\begin{aligned} \frac{dE}{dm} &= \frac{dh}{dm} + \frac{V}{g} \frac{dV}{dm} = \frac{1}{(dm/dt)} \left(\frac{dh}{dt} + \frac{V}{g} \frac{dV}{dt} \right) \\ &= \frac{1}{(dm/dt)} \left(\frac{dh}{dt} + \frac{1}{g} \mathbf{v}^T \frac{d\mathbf{v}}{dt} \right) = \frac{1}{(dm/dt)} \left(V \sin \gamma + \frac{1}{g} \mathbf{v}^T (\mathbf{T} - m\mathbf{g}) \right) \\ &= \frac{1}{(dm/dt)} \left(V \sin \gamma + \frac{VT}{mg} \cos \alpha - V \sin \gamma \right) \end{aligned}$$

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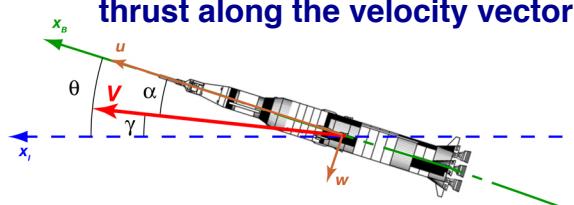
Oberth's Synergy Curve

γ : Flight Path Angle

θ : Pitch Angle

α : Angle of Attack

dE/dm maximized when $\alpha = 0$, or $\theta = \gamma$, i.e., thrust along the velocity vector



Approximate round-earth equations of motion

$$\begin{aligned} \frac{dV}{dt} &= \frac{T}{m} \cos \alpha - \frac{\text{Drag}}{m} - g \sin \gamma \\ \frac{d\gamma}{dt} &= \frac{T}{mV} \sin \alpha + \left(\frac{V}{r} - \frac{g}{V} \right) \cos \gamma \end{aligned}$$

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Gravity-Turn Pitch Program

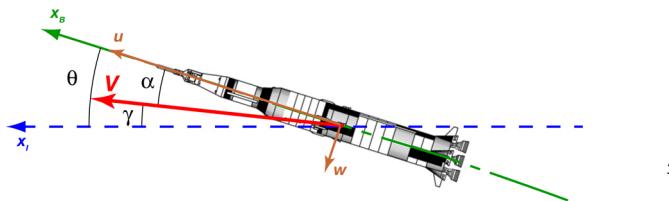
With angle of attack, $\alpha = 0$

$$\frac{d\gamma}{dt} = \frac{d\theta}{dt} = \left(\frac{V}{r} - \frac{g}{V} \right) \cos \gamma$$

Locally optimal flight path

Minimizes aerodynamic loads

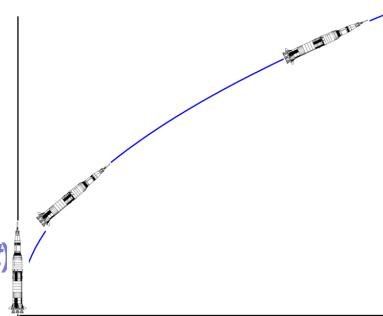
Feedback controller minimizes a or load factor



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Gravity-Turn Flight Path

- Gravity-turn flight path is function of 3 variables
 - Initial pitchover angle (from vertical launch)
 - Velocity at pitchover
 - Acceleration profile, $T(t)/m(t)$



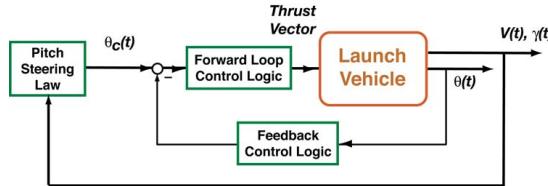
Gravity-turn program closely approximated by tangent steering laws (see Supplemental Material)

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Feedback Control Law

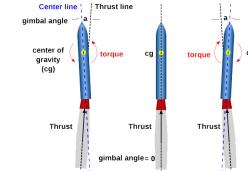
Errors due to disturbances and modeling errors
corrected by feedback control



$$\text{Motor Gimbal Angle}(t) \triangleq \delta_G(t) = c_\theta [\theta_{des}(t) - \theta(t)] - c_q q(t)$$

θ_{des} = Desired pitch angle; $q = \frac{d\theta}{dt}$ = pitch rate

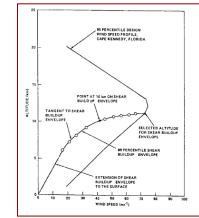
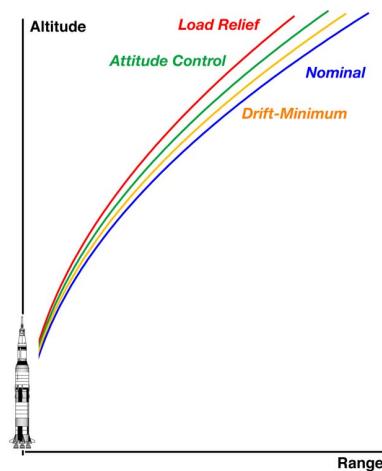
c_θ, c_q : Feedback control law gains



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Thrust Vector Control During Launch Through Wind Profile

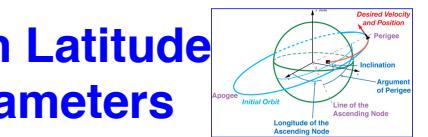


- **Attitude control**
 - Attitude and rate feedback
- **Drift-minimum control**
 - Attitude and accelerometer feedback
 - Increased loads
- **Load relief control**
 - Rate and accelerometer feedback
 - Increased drift

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Effect of Launch Latitude on Orbital Parameters



- Launch latitude establishes minimum orbital inclination (without “dogleg” maneuver)
- Time of launch establishes line of nodes
- Argument of perigee established by
 - Launch trajectory
 - On-orbit adjustment

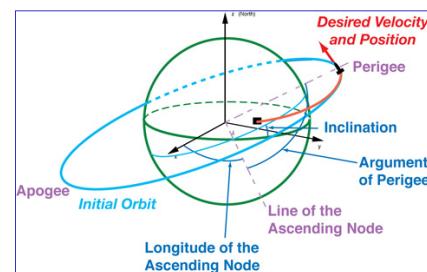
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Guidance Law for Launch to Orbit

(Brand, Brown, Higgins, and Pu, CSDL, 1972)

- Initial conditions**
 - End of pitch program, outside atmosphere
- Final condition**
 - Insertion in desired orbit
- Initial inputs**
 - Desired radius
 - Desired velocity magnitude
 - Desired flight path angle
 - Desired inclination angle
 - Desired longitude of the ascending/descending node
- Continuing outputs**
 - Unit vector describing desired thrust direction
 - Throttle setting, % of maximum thrust



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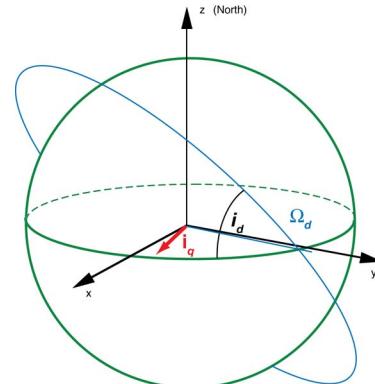
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Guidance Program Initialization

- Thrust acceleration estimate
- Mass/mass flow rate
- Acceleration limit ($\sim 3g$)
- Effective exhaust velocity
- Various coefficients
- Unit vector normal to desired orbital plane, \mathbf{i}_q

$$\mathbf{i}_q = \begin{bmatrix} \sin i_d \sin \Omega_d \\ \sin i_d \cos \Omega_d \\ \cos i_d \end{bmatrix}$$

i_d : Desired inclination angle of final orbit
 Ω_d : Desired longitude of descending node



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Guidance Program Operation: Position and Velocity

- Obtain thrust acceleration estimate, \mathbf{a}_T , from guidance system
- Compute corresponding mass, mass flow rate, and throttle setting, δT

$$\mathbf{i}_r = \frac{\mathbf{r}}{r} : \text{Unit vector aligned with local vertical}$$

$$\mathbf{i}_z = \mathbf{i}_r \times \mathbf{i}_q : \text{Downrange direction}$$

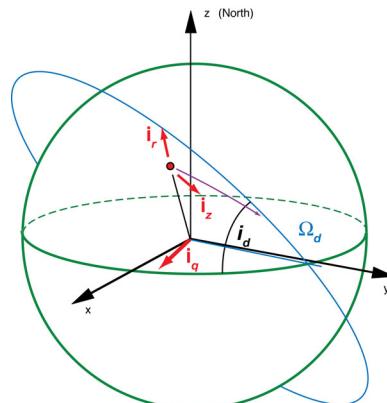
Position

$$\begin{bmatrix} r \\ y \\ z \end{bmatrix} = \begin{bmatrix} |\mathbf{r}| \\ r \sin^{-1}(\mathbf{i}_r \bullet \mathbf{i}_q) \\ \text{open} \end{bmatrix}$$

Velocity

$$\begin{bmatrix} \dot{r} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{IMU} \bullet \mathbf{i}_r \\ \mathbf{v}_{IMU} \bullet \mathbf{i}_q \\ \mathbf{v}_{IMU} \bullet \mathbf{i}_z \end{bmatrix}$$

\mathbf{v}_{IMU} : Velocity estimate in IMU frame



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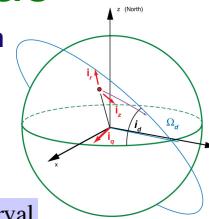
Guidance Program: Velocity and Time to Go

Effective gravitational acceleration

$$g_{eff} = -\frac{\mu}{r^2} + \frac{|\mathbf{r} \times \mathbf{v}|^2}{r^3}$$

Time to go (to motor burnout)

$$t_{go_new} = t_{go_old} - \Delta t \quad \Delta t : \text{Guidance command interval}$$



Velocity to be gained

$$\mathbf{v}_{go} = \begin{bmatrix} (\dot{r}_d - \dot{r}) - g_{eff} t_{go} / 2 \\ -\dot{y} \\ \dot{z}_d - \dot{z} \end{bmatrix}$$

Time to go prediction (prior to acceleration limiting)

$$t_{go} = \frac{m}{\dot{m}} \left(1 - e^{-v_{go}/c_{eff}} \right) \quad c_{eff} : \text{Effective exhaust velocity}$$

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Guidance Program Commands

Guidance law: required radial and cross-range accelerations

$$\begin{aligned} a_{T_r} &= \mathbf{a}_T \left[A + B(t - t_o) \right] - g_{eff} \\ a_{T_y} &= \mathbf{a}_T \left[C + D(t - t_o) \right] \end{aligned}$$

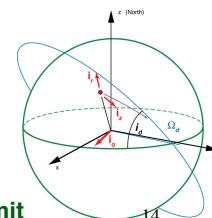
\mathbf{a}_T = Net available acceleration

Guidance coefficients, A , B , C , and D are functions of

$$\begin{pmatrix} (r_d, r, \dot{r}, t_{go}) \\ (y, \dot{y}, t_{go}) \end{pmatrix} \quad \text{plus } c_{eff}, m/\dot{m}, \text{ Acceleration limit}$$

Required thrust direction, \mathbf{i}_T (i.e., vehicle orientation in (i_r, i_q, i_z) frame)

$$\mathbf{a}_T = \begin{bmatrix} a_{T_r} \\ a_{T_y} \\ \text{what's left over} \end{bmatrix}; \quad i_T = \frac{\mathbf{a}_T}{|\mathbf{a}_T|}$$



Throttle command is a function of a_T (i.e., acceleration magnitude) and acceleration limit

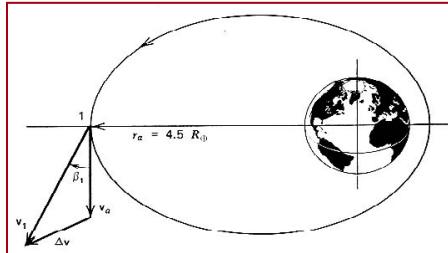
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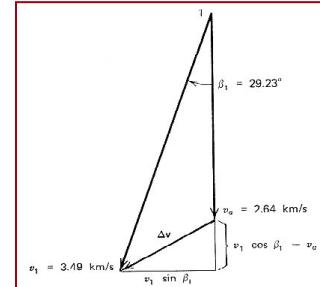
Impulsive ΔV Orbital Maneuver

- If rocket burn time is short compared to orbital period (e.g., seconds compared to hours), impulsive ΔV approximation can be made
 - Change in position during burn is \sim zero
 - Change in velocity is \sim instantaneous

Velocity impulse at apogee



Vector diagram of velocity change

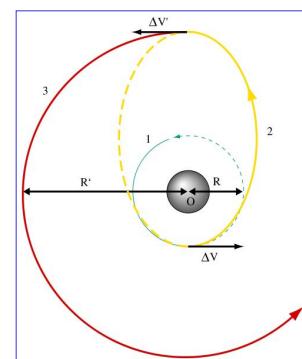


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Orbit Change due to Impulsive ΔV

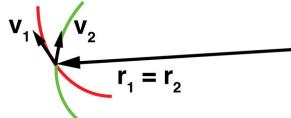
- Maximum energy change accrues when ΔV is aligned with the instantaneous orbital velocity vector
 - Energy change \rightarrow Semi-major axis change
 - Maneuver at perigee raises or lowers apogee
 - Maneuver at apogee raises or lowers perigee
- Optimal transfer from one circular orbit to another involves two impulses [*Hohmann transfer*]
- Other maneuvers
 - In-plane parameter change
 - Orbital plane change



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Assumptions for Impulsive Maneuver



Instantaneous change in velocity vector

$$\mathbf{v}_2 = \mathbf{v}_1 + \Delta \mathbf{v}_{rocket}$$

Negligible change in radius vector

$$\mathbf{r}_2 = \mathbf{r}_1$$

Therefore, new orbit intersects old orbit

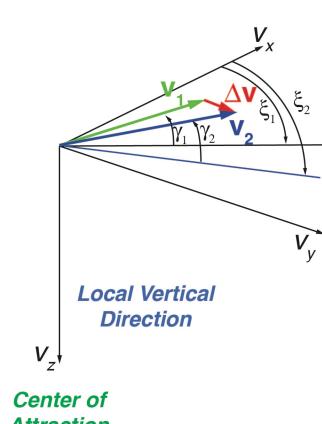
Velocities different at the intersection

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Geometry of Impulsive Maneuver

Change in velocity magnitude, $\|\Delta \mathbf{v}\|$, vertical flight path angle, γ , and horizontal flight path angle, ξ



$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_1 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_1 = \begin{bmatrix} V \cos \gamma \cos \xi \\ V \cos \gamma \sin \xi \\ -V \sin \gamma \end{bmatrix}_1$$

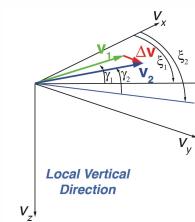
$$\mathbf{v}_2 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_2 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_2 = \begin{bmatrix} V \cos \gamma \cos \xi \\ V \cos \gamma \sin \xi \\ -V \sin \gamma \end{bmatrix}_2$$

$$\Delta \mathbf{v} = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \begin{bmatrix} (v_{x_2} - v_{x_1}) \\ (v_{y_2} - v_{y_1}) \\ (v_{z_2} - v_{z_1}) \end{bmatrix}$$

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Required Δv for Impulsive Maneuver



$$\Delta \mathbf{v} = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \begin{bmatrix} (v_{x_2} - v_{x_1}) \\ (v_{y_2} - v_{y_1}) \\ (v_{z_2} - v_{z_1}) \end{bmatrix}$$

$$\Delta \mathbf{v}_{rocket} = \begin{bmatrix} \Delta V_{rocket} \\ \xi_{rocket} \\ \gamma_{rocket} \end{bmatrix} = \begin{bmatrix} (\Delta v_x^2 + \Delta v_y^2 + \Delta v_z^2)^{1/2}_{rocket} \\ \sin^{-1}\left(\frac{\Delta v_y}{(\Delta v_x^2 + \Delta v_y^2)^{1/2}}\right)_{rocket} \\ \sin^{-1}\left(\frac{\Delta v_z}{\Delta V}\right)_{rocket} \end{bmatrix}$$

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Single Impulse Orbit Adjustment Coplanar (i.e., in-plane) maneuvers

- Change energy
- Change angular momentum
- Change eccentricity

$$\begin{aligned} \mathbb{E} &= \frac{1}{2} v^2 - \mu/r = (e^2 - 1) \mu^2 / h^2 \\ h &= \sqrt{\frac{\mu^2 (e^2 - 1)}{\mathbb{E}}} = \sqrt{\frac{\mu^2 (e^2 - 1)}{v^2/2 - \mu/r}} \\ e &= \sqrt{1 + 2 \mathbb{E} h^2 / \mu^2} \end{aligned}$$

- Required velocity increment

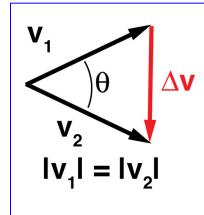
$$\begin{aligned} v_{new} &\triangleq v_{old} + \Delta v_{rocket} = \sqrt{2(\mathbb{E}_{new} + \mu/r)} \\ &= \sqrt{2[(e_{new}^2 - 1)\mu^2 / h_{new}^2 + \mu/r]} \\ \Delta v_{rocket} &= v_{new} - v_{old} \end{aligned}$$

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Single Impulse Orbit Adjustment Coplanar (i.e., in-plane) maneuvers

- Change semi-major axis
 - magnitude
 - orientation (i.e., argument of perigee); in-plane isosceles triangle



$$a_{new} = \frac{h_{new}^2 / \mu}{1 - e_{new}^2}$$

- Change apogee or perigee
 - radius
 - velocity

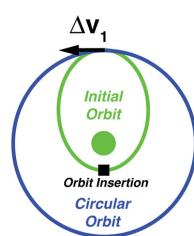
$$\begin{aligned} r_{perigee} &= a(1-e) \\ r_{apogee} &= a(1+e) \\ v_{perigee} &= \sqrt{\frac{\mu(1+e)}{a(1-e)}} \\ v_{apogee} &= \sqrt{\frac{\mu(1-e)}{a(1+e)}} \end{aligned}$$

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In-Plane Orbit Circularization

Initial orbit is elliptical, with apogee radius equal to desired circular orbit radius



$$\begin{aligned} \text{Initial Orbit} \\ a &= (r_{cir(target)} + r_{insertion})/2 \\ e &= (r_{cir(target)} - r_{insertion})/2a \\ v_{apogee} &= \sqrt{\frac{\mu(1-e)}{a(1+e)}} \end{aligned}$$

Velocity in circular orbit is a function of the radius
“Vis viva” equation:

$$v_{cir} = \sqrt{\mu \left(\frac{2}{r_{cir}} - \frac{1}{a_{cir}} \right)} = \sqrt{\mu \left(\frac{2}{r_{cir}} - \frac{1}{r_{cir}} \right)} = \sqrt{\frac{\mu}{r_{cir}}}$$

Rocket must provide
the difference

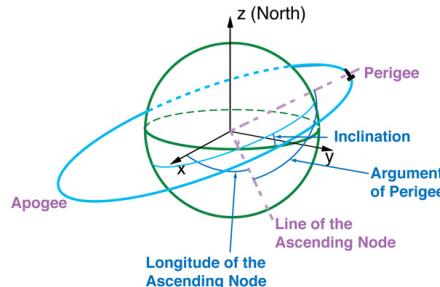
$$\Delta v_{rocket} = v_{cir} - v_{apogee}$$

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Single Impulse Orbit Adjustment Out-of-plane maneuvers

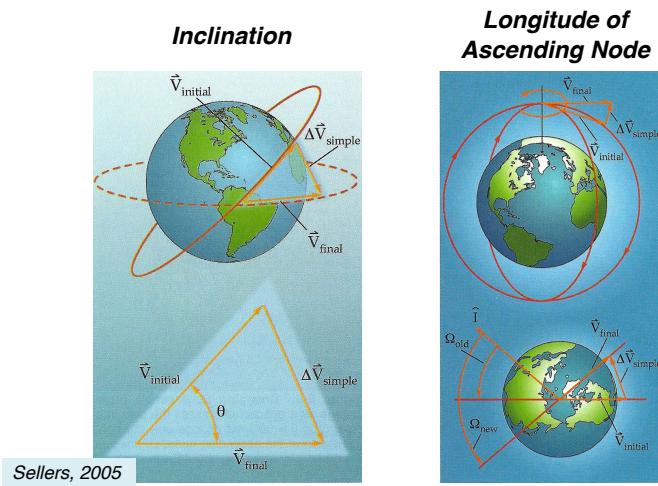
- Change orbital inclination
- Change longitude of the ascending node
- v_1 , Δv , and v_2 form isosceles triangle perpendicular to the orbital plane to leave in-plane parameters unchanged



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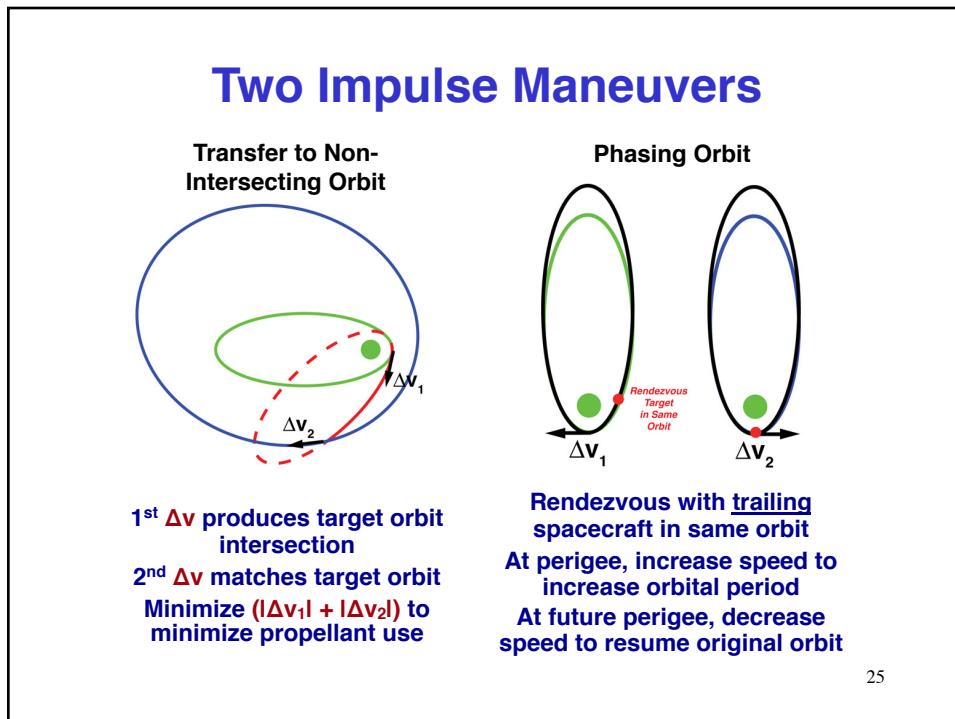
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Change in Inclination and Longitude of Ascending Node

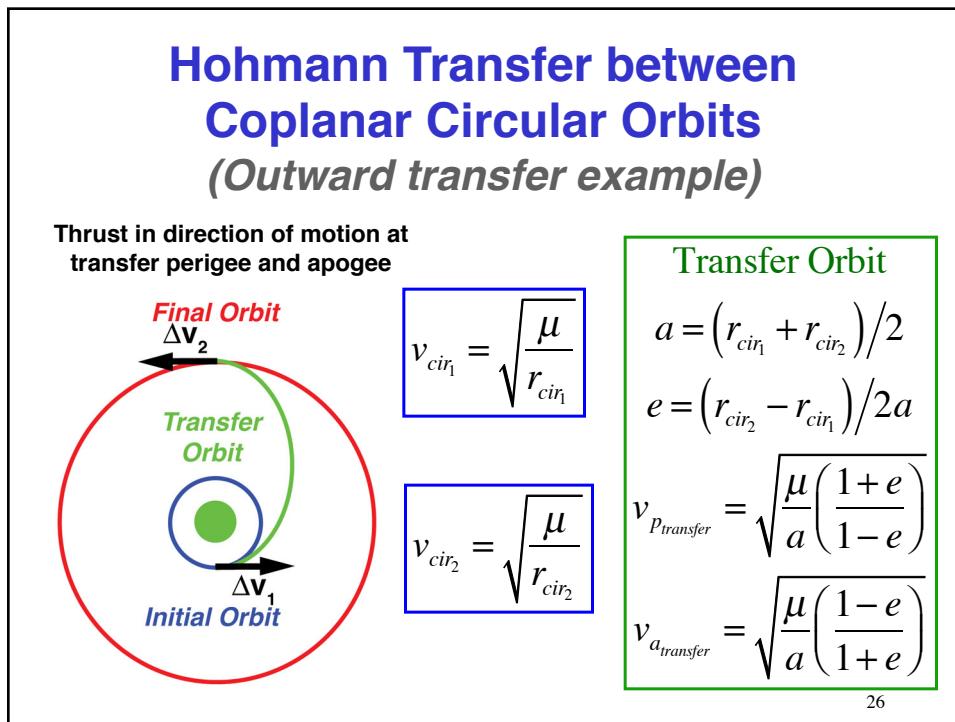


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Outward Transfer Orbit Velocity Requirements

Δv at 1st Burn

$$\begin{aligned}\Delta v_1 &= v_{p_{transfer}} - v_{cir_1} \\ &= v_{cir_1} \left(\sqrt{\frac{2r_{cir_2}}{r_{cir_1} + r_{cir_2}}} - 1 \right)\end{aligned}$$

Δv at 2nd Burn

$$\begin{aligned}\Delta v_2 &= v_{cir_1} - v_{a_{transfer}} \\ &= v_{cir_2} \left(1 - \sqrt{\frac{2r_{cir_1}}{r_{cir_1} + r_{cir_2}}} \right)\end{aligned}$$

$$v_{cir_2} = v_{cir_1} \sqrt{\frac{r_{cir_1}}{r_{cir_2}}}$$

Hohmann Transfer is **energy-optimal** for 2-impulse transfer between circular orbits and $r_2/r_1 < 11.94$

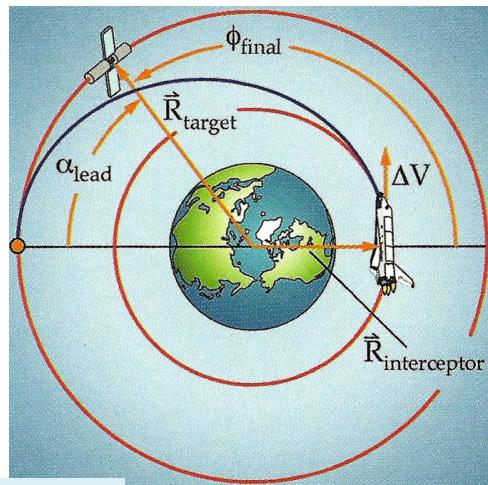
$$\Delta v_{total} = v_{cir_1} \left[\sqrt{\frac{2r_{cir_2}}{r_{cir_1} + r_{cir_2}}} \left(1 - \frac{r_{cir_1}}{r_{cir_2}} \right) + \sqrt{\frac{r_{cir_1}}{r_{cir_2}}} - 1 \right]$$

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Rendezvous Requires Phasing of the Maneuver

Transfer orbit time equals target's time to reach rendezvous point

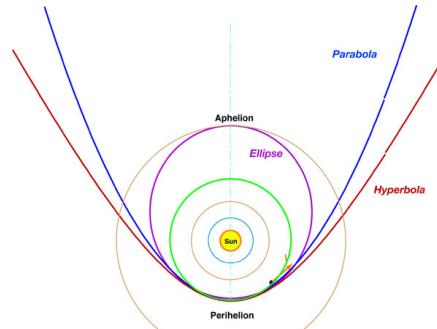


Sellers, 2005

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Solar Orbits



- Same equations used for Earth-referenced orbits
 - Dimensions of the orbit
 - Position and velocity of the spacecraft
 - Period of elliptical orbits
 - Different gravitational constant

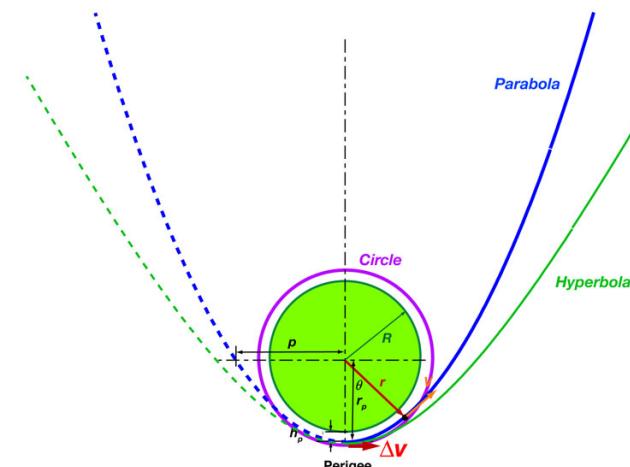
$$\mu_{Sun} = 1.3327 \times 10^{11} \text{ km}^3/\text{s}^2$$

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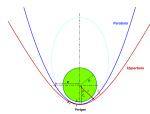
Escape from a Circular Orbit

Minimum escape trajectory shape is a **parabola**



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In-plane Parameters of Earth Escape Trajectories

Dimensions of the orbit

$$p = h^2 / \mu = \text{"The parameter" or semi-latus rectum}$$

h = Angular momentum about center of mass

$$e = \sqrt{1 + 2E p / \mu} = \text{Eccentricity} \geq 1$$

E = Specific energy, ≥ 0

$$a = \frac{p}{1 - e^2} = \text{Semi-major axis, } < 0$$

$$r_{\text{perigee}} = a(1 - e) = \text{Perigee radius}$$

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In-plane Parameters of Earth Escape Trajectories

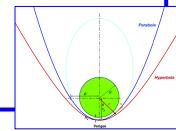
Position and velocity of the spacecraft

$$r = \frac{p}{1 + e \cos \theta} = \text{Radius of the spacecraft}$$

θ = True anomaly

$$V = \sqrt{2\mu \left(\frac{1}{r} - \frac{1}{2a} \right)} = \text{Velocity of the spacecraft}$$

$$V_{\text{perigee}} \geq \sqrt{\frac{2\mu}{r_{\text{perigee}}}}$$



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Escape from Circular Orbit

Velocity in circular orbit

$$V_c = \sqrt{\mu \left(\frac{2}{r_c} - \frac{1}{r_c} \right)} = \sqrt{\frac{\mu}{r_c}}$$

Velocity at perigee of parabolic orbit

$$V_{perigee} = \sqrt{\mu \left(\frac{2}{r_c} - \frac{1}{(a \rightarrow \infty)} \right)} = \sqrt{\frac{2\mu}{r_c}}$$

Velocity increment required for escape

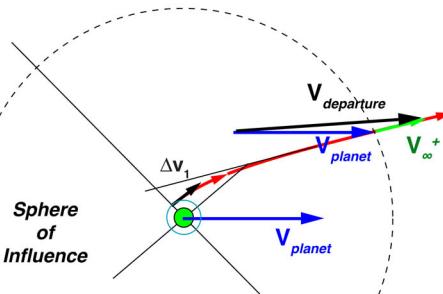
$$\Delta V_{escape} = V_{perigee_{parabola}} - V_c = \sqrt{\frac{2\mu}{r_c}} - \sqrt{\frac{\mu}{r_c}} \approx 0.414 V_c$$

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Earth Escape Trajectory

Δv_1 to increase speed to escape velocity
 Velocity required for transfer at sphere of influence

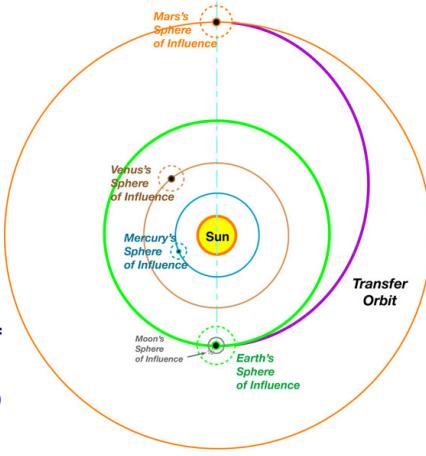


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Transfer Orbits and Spheres of Influence

- **Sphere of Influence (Laplace):**
 - Radius within which gravitational effects of planet are more significant than those of the Sun
- **Patched-conic section approximation**
 - Sequence of 2-body orbits
 - Outside of planet's sphere of influence, Sun is the center of attraction
 - Within planet's sphere of influence, planet is the center of attraction
- **Fly-by (swingby) trajectories dip into intermediate object's sphere of influence for gravity assist**



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Solar System Spheres of Influence

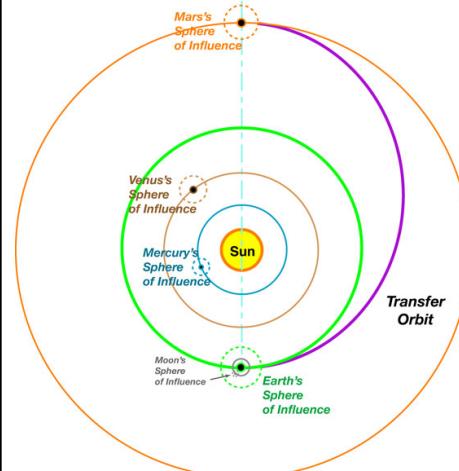
$$\text{for } \frac{m_{\text{Planet}}}{m_{\text{Sun}}} \ll 1, \quad r_{SI} \approx r_{\text{Planet-Sun}} \left(\frac{m_{\text{Planet}}}{m_{\text{Sun}}} \right)^{2/5}$$

Planet	Sphere of Influence, km
Mercury	112,000
Venus	616,000
Earth	929,000
Mars	578,000
Jupiter	48,200,000
Saturn	54,500,000
Uranus	51,800,000
Neptune	86,800,000
Pluto	27,000,000-45,000,000

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Interplanetary Mission Planning



- Example: Direct Hohmann Transfer from Earth Orbit to Mars Orbit (No fly-bys)
- 1) Calculate required perigee velocity for transfer orbit - Sun as center of attraction: Elliptical orbit
 - 2) Calculate Δv required to reach Earth's sphere of influence with velocity required for transfer – Earth as center of attraction: Hyperbolic orbit
 - 3) Calculate Δv required to enter circular orbit about Mars, given transfer apogee velocity – Mars as center of attraction: Hyperbolic orbit

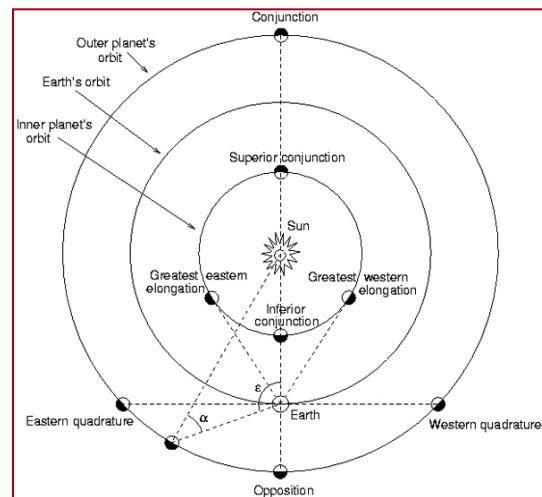
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Launch Opportunities for Fixed Transit Time: The Synodic Period

- **Synodic Period, S_n :** The time between conjunctions
 - P_A : Period of Planet A
 - P_B : Period of Planet B
- **Conjunction:** Two planets, A and B, in a line or at some fixed angle

$$S_n = \frac{P_A P_B}{P_A - P_B}$$



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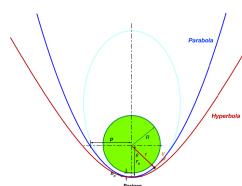
Launch Opportunities for Fixed Transit Time: The Synodic Period

Planet	Synodic Period with respect to Earth, days	Period
Mercury	116	88 days
Venus	584	225 days
Earth	-	365 days
Mars	780	687 days
Jupiter	399	11.9 yr
Saturn	378	29.5 yr
Uranus	370	84 yr
Neptune	367	165 yr
Pluto	367	248 yr

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Hyperbolic Orbits



Orbit Shape	Eccentricity, e	Energy, E
Circle	0	<0
Ellipse	0 < e < 1	<0
Parabola	1	0
Hyperbola	>1	>0

$$\bar{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}, \quad \therefore a < 0$$

Velocity remains positive as radius approaches ∞

$$\begin{aligned} v &\xrightarrow[r \rightarrow \infty]{} v_\infty \\ \therefore \bar{E}_\infty &= \frac{v_\infty^2}{2}, \text{ and } v_\infty = \sqrt{-\frac{\mu}{a}} \text{ or } a = -\frac{\mu}{v_\infty^2} \end{aligned}$$

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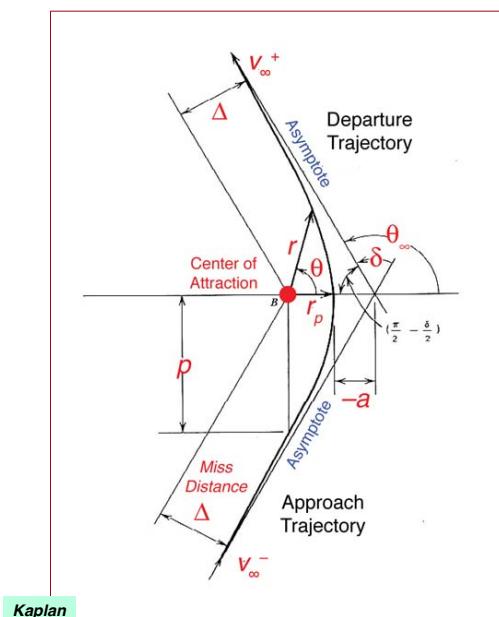
40

Hyperbolic Encounter with a Planet

- Trajectory is deflected by target planet's gravitational field
 - In-plane
 - Out-of-plane
- Velocity w.r.t. Sun is increased or decreased

Δ : Miss Distance, km

δ : Deflection Angle, deg or rad



Kaplan

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Hyperbolic Orbits

Asymptotic Value of True Anomaly

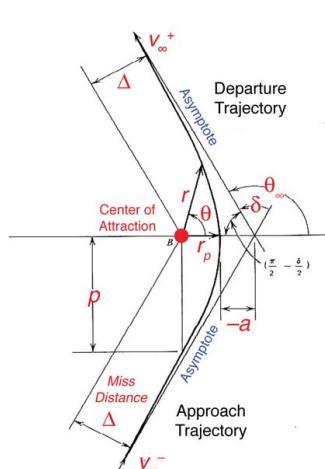
Polar Equation for a Conic Section

$$r = \frac{p}{1 + e \cos \theta} = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

$$\cos \theta = \frac{1}{e} \left[\frac{a(1 - e^2)}{r} - 1 \right]$$

$$\theta \xrightarrow[r \rightarrow \infty]{} \theta_\infty$$

$$\theta_\infty = \cos^{-1} \left(-\frac{1}{e} \right)$$

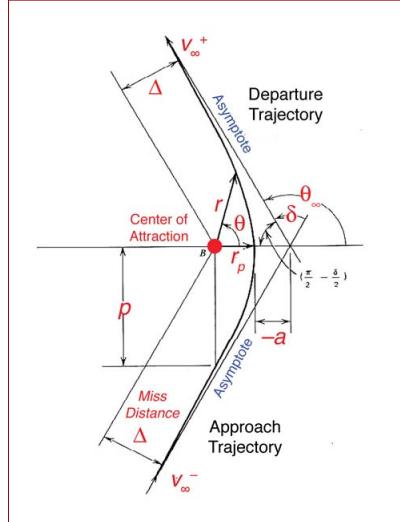


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Hyperbolic Orbits

Angular Momentum



$$h = \text{Constant} = v_\infty \Delta$$

$$= \sqrt{\mu p} = \sqrt{\mu a(1-e^2)} = \sqrt{\frac{\mu^2(e^2-1)}{v_\infty^2}}$$

Eccentricity

$$e = \sqrt{1 + \frac{2h^2 E}{\mu^2}} = \sqrt{1 + \frac{v_\infty^4 \Delta^2}{\mu^2}}$$

Perigee Radius

$$r_p = a(1-e) = \frac{\mu}{v_\infty^2}(e-1)$$

Eccentricity

$$e = \left[1 + \frac{r_p v_\infty^2}{\mu} \right]$$

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Hyperbolic Mean and Eccentric Anomalies

H : Hyperbolic Eccentric Anomaly

$$M = e \sinh H - H$$

Newton's method of successive approximation
to find H from M , similar to solution for E (Lecture 2)

$$\theta(t) = 2 \tan^{-1} \left[\sqrt{\frac{e+1}{e-1}} \tanh \frac{H(t)}{2} \right]$$

$$r = a(1 - e \cosh H)$$

see Ch. 7, Kaplan, 1976

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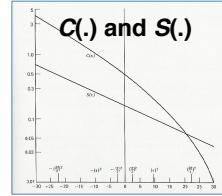
44

Battin's Universal Formulas for Conic Section Position and Velocity as Functions of Time

$$\mathbf{r}(t_2) = \left[1 - \frac{\chi^2}{r(t_1)} C\left(\frac{\chi^2}{a}\right) \right] \mathbf{r}(t_1) + \left[t_2 - \frac{\chi^3}{\sqrt{\mu}} S\left(\frac{\chi^2}{a}\right) \right] \mathbf{v}(t_1)$$

$$\mathbf{v}(t_2) = \frac{\sqrt{\mu}}{r(t_1)r(t_2)} \left[\frac{\chi^3}{a} S\left(\frac{\chi^2}{a}\right) - \chi \right] \mathbf{r}(t_1) + \left[1 - \frac{\chi^2}{r(t_2)} C\left(\frac{\chi^2}{a}\right) \right] \mathbf{v}(t_1)$$

$$\chi = \begin{cases} \sqrt{a}[E(t_2) - E(t_1)], & \text{Ellipse} \\ \sqrt{-a}[H(t_2) - H(t_1)], & \text{Hyperbola} \\ \sqrt{P} \left[\tan \frac{\theta(t_2)}{2} - \tan \frac{\theta(t_1)}{2} \right], & \text{Parabola} \end{cases}$$



see Ch. 7, Kaplan, 1976; also Battin, 1964

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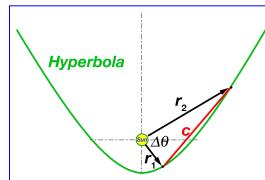
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Lambert's Time-of-Flight Theorem (Hyperbolic Orbit)

$$(t_2 - t_1) = \sqrt{\frac{-a^3}{\mu}} [(\sinh \gamma - \gamma) + (\sinh \delta - \delta)]$$

where

$$\gamma \triangleq 2 \sinh^{-1} \sqrt{\frac{r_1 + r_2 + c}{-4a}}; \quad \delta \triangleq 2 \sinh^{-1} \sqrt{\frac{r_1 + r_2 - c}{-4a}}$$



see Ch. 7, Kaplan, 1976; also Battin, 1964

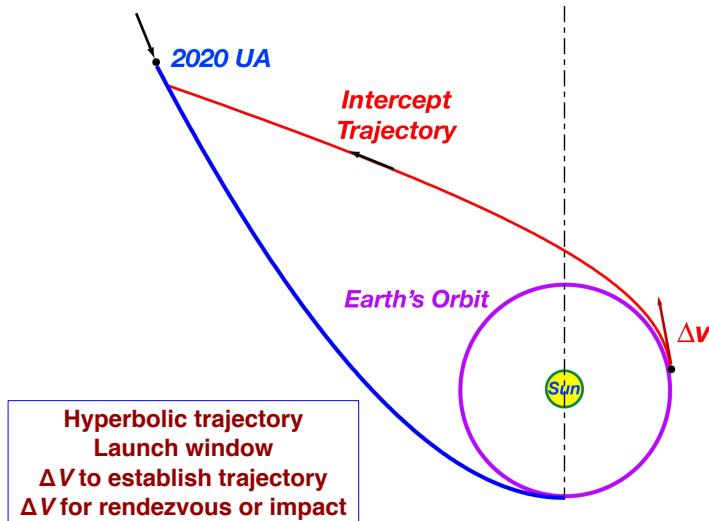
<http://www.mathworks.com/matlabcentral/fileexchange/39530-lambert-s-problem>

<http://www.mathworks.com/matlabcentral/fileexchange/26348-robust-solver-for-lambert-s-orbital-boundary-value-problem>

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Asteroid Encounter

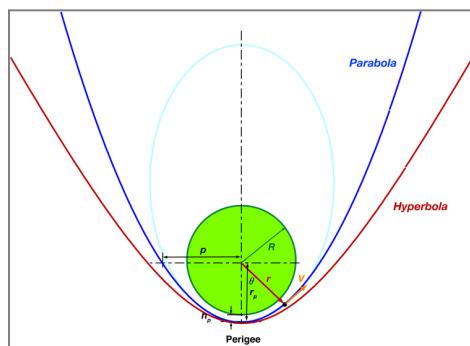


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Swing-By/Fly-By Trajectories

- Hyperbolic encounters with planets and the moon provide gravity assist
 - Shape, energy, and duration of transfer orbit altered
 - Potentially large reduction in rocket ΔV required to accomplish mission



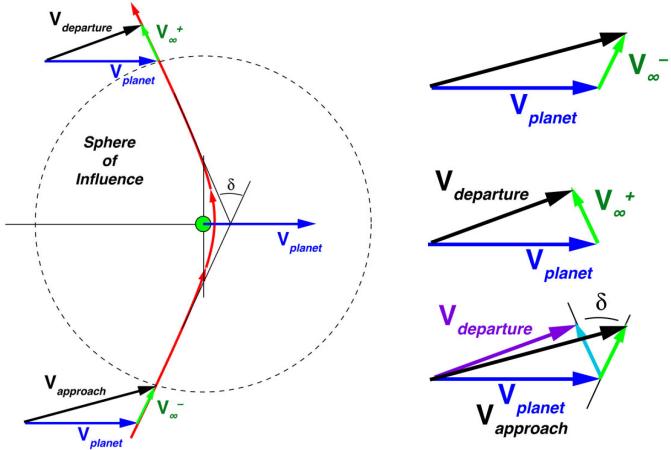
Why does gravity assist work?

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Effect of Target Planet's Gravity on Probe's Sun-Relative Velocity

Deflection – Velocity Reduction

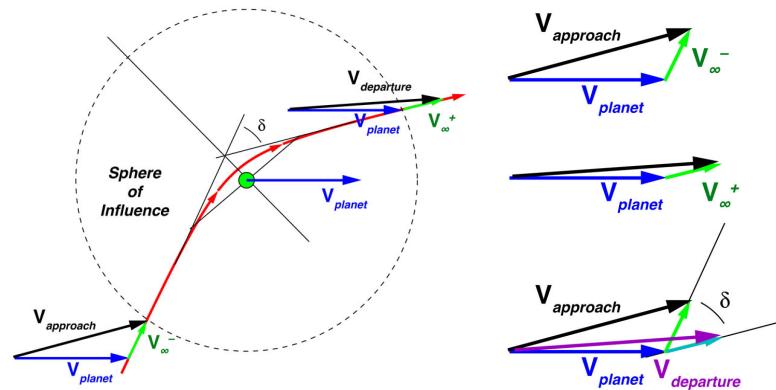


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Effect of Target Planet's Gravity on Probe's Sun-Relative Velocity

Deflection – Velocity Addition

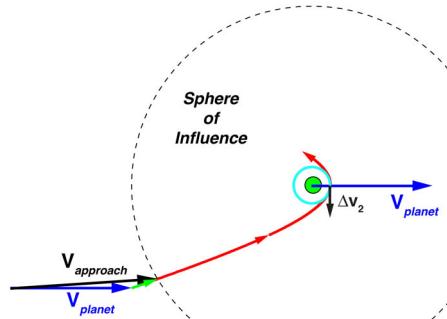


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Planet Capture Trajectory

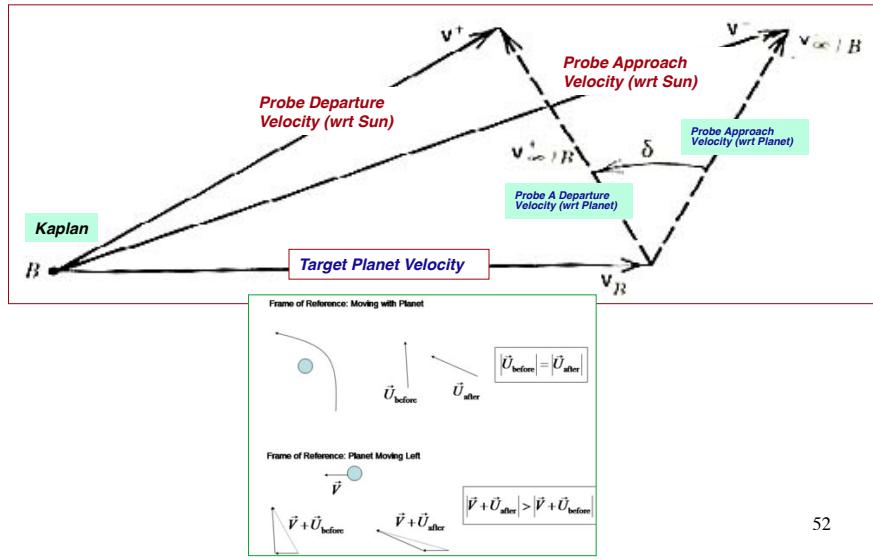
Hyperbolic approach to planet's sphere of influence
 Δv to decrease speed to circular velocity



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Effect of Target Planet's Gravity on Probe's Velocity



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*Next Time:
Spacecraft Environment*

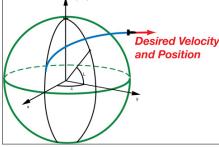
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Supplemental Material

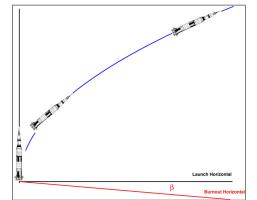
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Phases of Ascent Guidance

- Vertical liftoff
- Roll to launch azimuth
- Pitch program to atmospheric “exit”
 - Jet stream penetration
 - Booster cutoff and staging
- Explicit guidance to desired orbit
 - Booster separation
 - Acceleration limiting
 - Orbital insertion

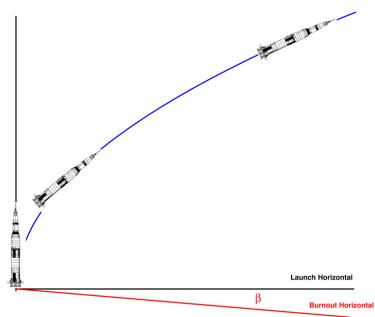



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Tangent Steering Laws

- Neglecting surface curvature
- “Open-loop” command, i.e., no feedback of vehicle state
- Accounting for effect of Earth surface curvature on burnout flight path angle

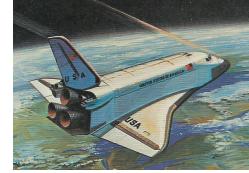
$$\tan \theta(t) = \tan \theta_o \left(1 - \frac{t}{t_{BO}} \right)$$


$$\tan \theta(t) = \tan \theta_o \left[1 - \frac{t}{t_{BO}} - \tan \beta \left(\frac{t}{t_{BO}} \right) \right]$$

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Longitudinal (2-D) Equations of Motion for Re-Entry



Differential equations for velocity ($x_1 = V$), flight path angle ($x_2 = y$), altitude ($x_3 = h$), and range (x_4)

Angle of attack (α) is optimization control variable

$$\begin{aligned}\dot{x}_1 &= -D(x_1, x_3, \alpha)/m - g \cos x_2 \\ \dot{x}_2 &= [g/x_1 - x_1/(R+x_3)] \sin x_2 - L(x_1, x_3, \alpha)/mx_1 \\ \dot{x}_3 &= x_1 \cos x_2 \\ \dot{x}_4 &= x_1 \sin x_2 / (1 + x_3/R)\end{aligned}$$

D = drag
 L = lift
 M = pitch moment
 R = Earth radius

Equations of motion define the dynamic constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \alpha(t)]$$

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A Different Approach to Guidance: Optimizing a Cost Function

- Minimize a scalar function, J , of terminal and integral costs

$$J = \phi[\mathbf{x}(t_f)] + \int_{t_o}^{t_f} L[\mathbf{x}(t), \mathbf{u}(t)] dt$$

$L[\cdot]$: Lagrangian

with respect to the control, $\mathbf{u}(t)$, in (t_o, t_f) , subject to a dynamic constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad \mathbf{x}(t_o) \text{ given}$$

$\dim(\mathbf{x}) = n \times 1$
 $\dim(\mathbf{f}) = n \times 1$
 $\dim(\mathbf{u}) = m \times 1$

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Guidance Cost Function

$$\begin{aligned} & \phi[\mathbf{x}(t_f)] \\ & L[\mathbf{x}(t), \mathbf{u}(t)] \end{aligned}$$

- **Terminal cost**, e.g., in final position and velocity
- **Integral cost**, e.g., tradeoff between control usage and trajectory error

- **Minimization of cost function determines the optimal state and control, \mathbf{x}^* and \mathbf{u}^* , over the flight path duration**

$$\begin{aligned} \min_u J &= \min_u \left\{ \phi[\mathbf{x}(t_f)] + \int_{t_0}^{t_f} L[\mathbf{x}(t), \mathbf{u}(t)] dt \right\} \\ &= \phi[\mathbf{x}^*(t_f)] + \int_{t_0}^{t_f} L[\mathbf{x}^*(t), \mathbf{u}^*(t)] dt \rightarrow [\mathbf{x}^*(t), \mathbf{u}^*(t)] \end{aligned}$$

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Example of Re-Entry Flight Path Cost Function

$$J = a[V(t_f) - V_d]^2 + b[r(t_f) - r_d]^2 + \int_{t_0}^{t_f} c[\alpha(t)]^2 dt$$

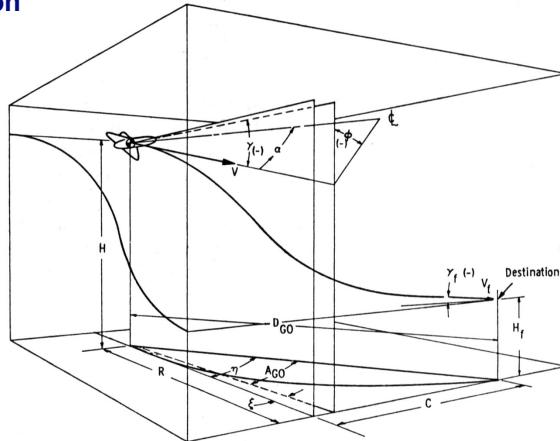
- **Cost function includes**
 - Terminal range and velocity
 - Penalty on control use
 - **a, b, and c** tradeoff importance of each factor
- **Minimization of this cost function**
 - Defines the optimal path, $\mathbf{x}^*(t)$, from t_0 to t_f
 - Defines the optimal control, $\alpha^*(t)$, from t_0 to t_f

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Extension to Three Dimensions

- Add roll angle as a control; add crossrange as a state
- For the guidance law, replace range and crossrange from the starting point by distance to go and azimuth to go to the destination

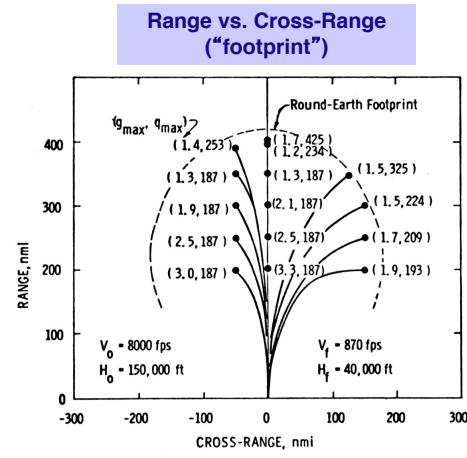
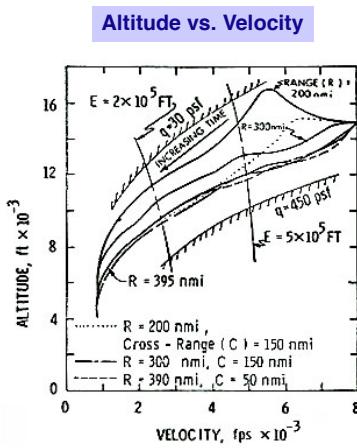


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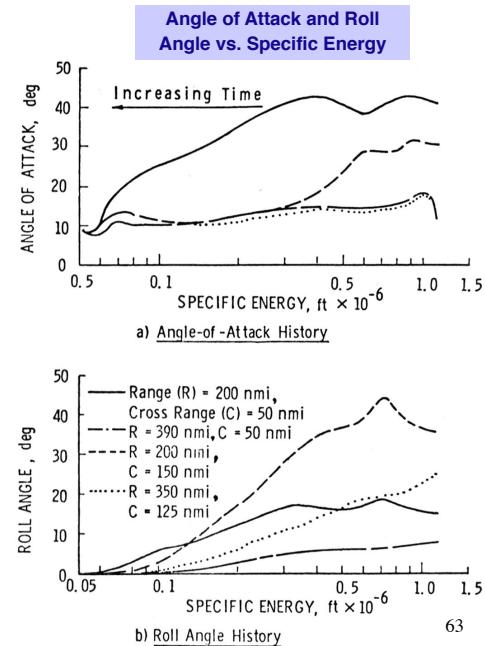
Optimal Trajectories for Space Shuttle Transition



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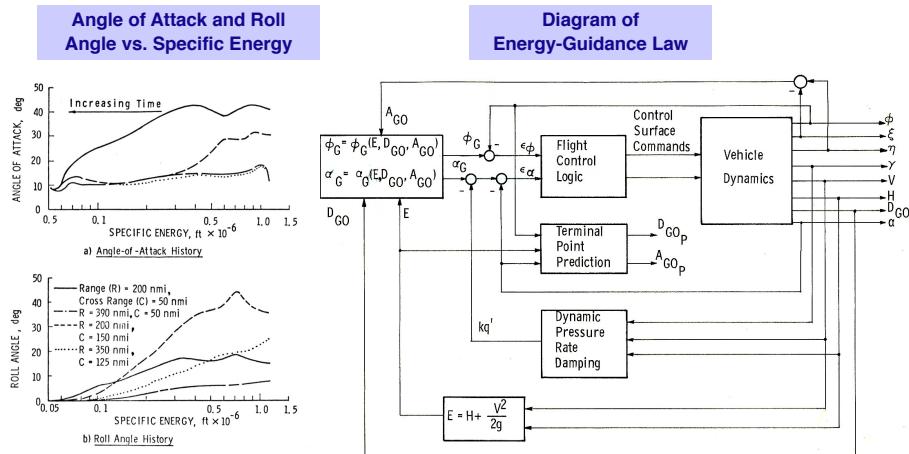
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Optimal Controls for Space Shuttle Transition



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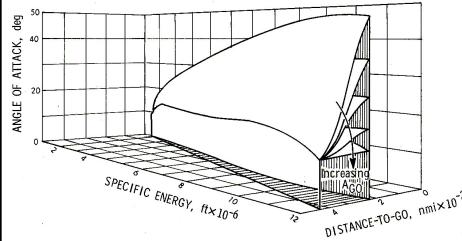
Optimal Guidance System Derived from Optimal Trajectories



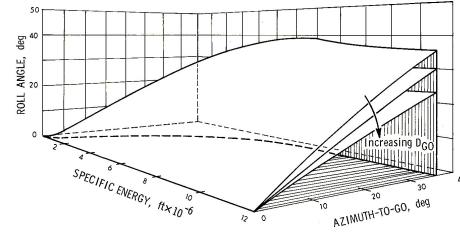
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Guidance Functions for Space Shuttle Transition

Angle of Attack Guidance Function



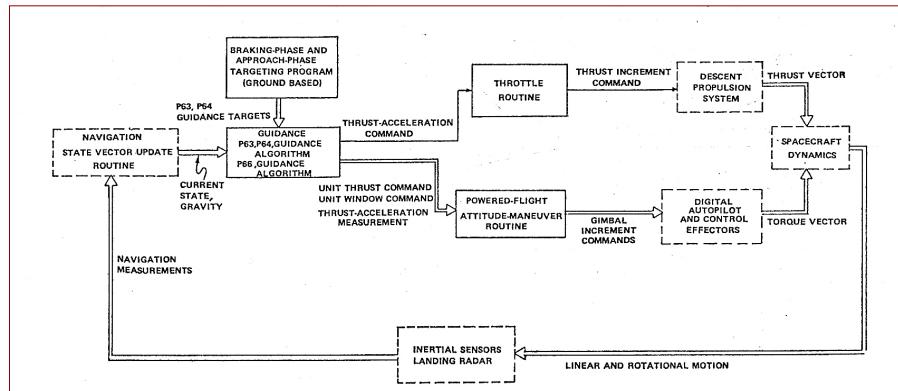
Roll Angle Guidance Function



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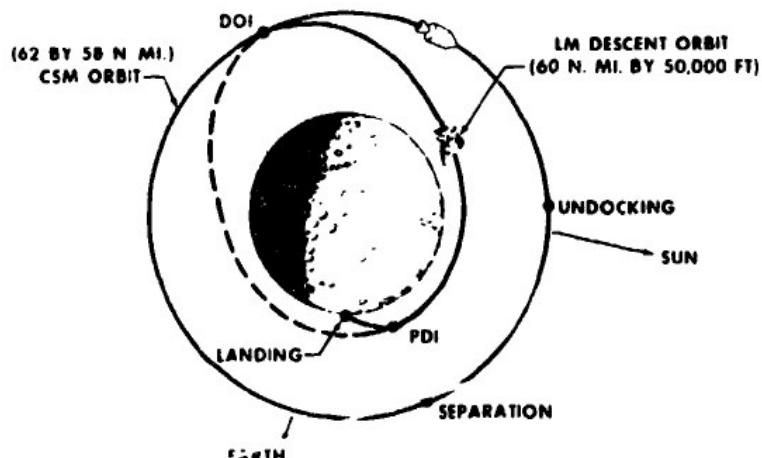
Lunar Module Navigation, Guidance, and Control Configuration



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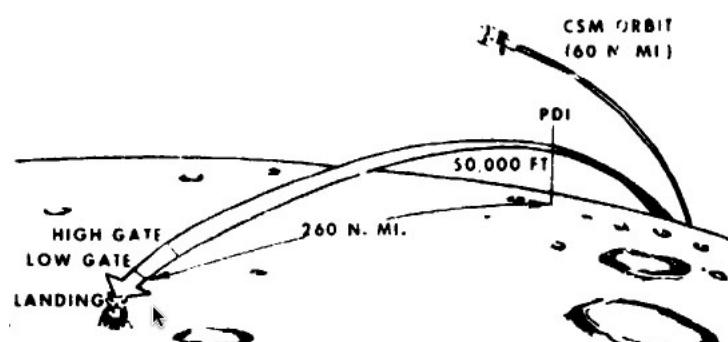
Lunar Module Transfer Ellipse to Powered Descent Initiation



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Lunar Module Powered Descent



PHASE	INITIAL EVENT	DESIGN CRITERIA
BRAKING	PDI	MINIMIZE PROPELLANT USAGE
APPROACH	HIGH GATE	CREW VISIBILITY
LANDING	LOW GATE	MANUAL CONTROL

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Lunar Module Descent Events

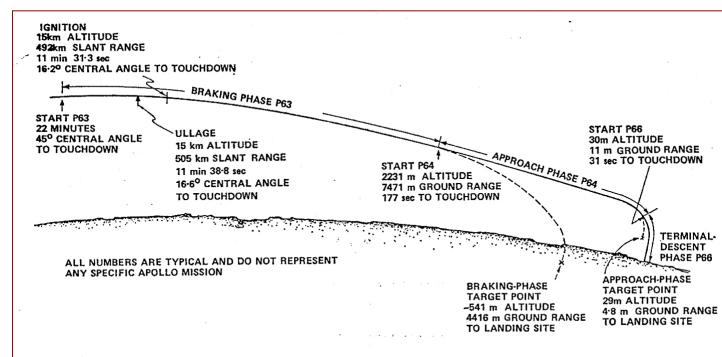
Event	TFI, min:sec	Inertial velocity, fps	Altitude rate, fps	Altitude, ft	ΔV , fps
A Ullage	-00:07				
B Power-red descent initiation	00:08	5560	-4	48 814	0
C Throttle to maximum thrust	00:26	5529	-3	48 725	31
D Rotate to windows-up position	02:56	4000	-50	44 934	1572
E LR altitude update	04:18	3065	-89	39 201	2536
F Throttle recovery	06:24	1456	-106	24 635	4239
G LR velocity update	06:42	1315	-127	22 644	4390
H High gate	08:26	506	-145	7 515	5375
I Low gate	10:06	55 (^b 68)	-16	512	6176
J Touchdown (probe contact)	11:54	-15 (^b 0)	-3	12	6775

^aTime from ignition of the DPS.^bHorizontal velocity relative to surface.

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Lunar Module Descent Targeting Sequence



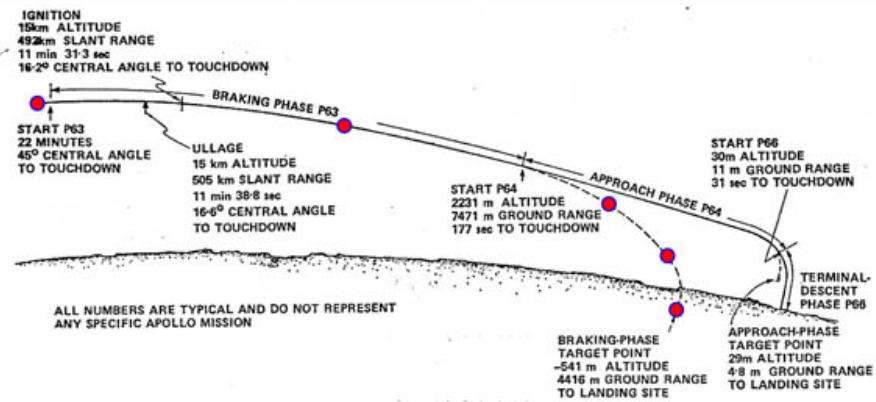
Braking Phase (P63)
Approach Phase (P64)

Terminal Descent Phase (P66)

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Characterize Braking Phase By Five Points



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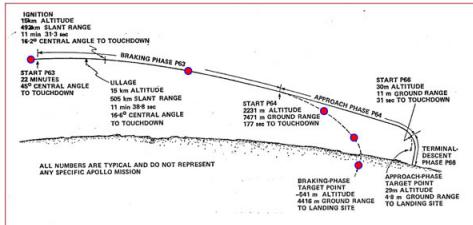
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Lunar Module Descent Guidance Logic (Klumpp, *Automatica*, 1974)

- Reference (nominal) trajectory, $\mathbf{r}_r(t)$, from target position back to starting point (Braking Phase example)
 - Three 4th-degree polynomials in time
 - 5 points needed to specify each polynomial

$$\mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24}$$

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$



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Coefficients of the Polynomials

$$\mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24}$$

- **r** = position vector
- **v** = velocity vector
- **a** = acceleration vector
- **j** = jerk vector (time derivative of acceleration)
- **s** = snap vector (time derivative of jerk)

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} \quad \mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

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Corresponding Reference Velocity and Acceleration Vectors

$$\mathbf{v}_r(t) = \mathbf{v}_t + \mathbf{a}_t t + \mathbf{j}_t \frac{t^2}{2} + \mathbf{s}_t \frac{t^3}{6}$$

$$\mathbf{a}_r(t) = \mathbf{a}_t + \mathbf{j}_t t + \mathbf{s}_t \frac{t^2}{2}$$

- **$\mathbf{a}_r(t)$** is the reference control vector
 - Descent engine thrust / mass = total acceleration
 - Vector components controlled by orienting yaw and pitch angles of the Lunar Module



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Guidance Logic Defines Desired Acceleration Vector

- If initial conditions, dynamic model, and thrust control were perfect, $\mathbf{a}_r(t)$ would produce $\mathbf{r}_r(t)$

$$\mathbf{a}_r(t) = \mathbf{a}_t + \mathbf{j}_t t + \mathbf{s}_t \frac{t^2}{2} \Rightarrow \mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24}$$

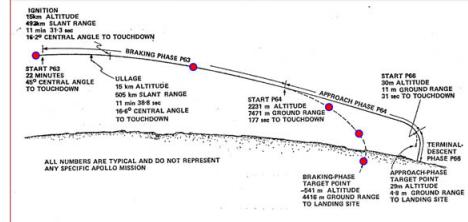
- ... but they are not
- Therefore, feedback control is required to follow the reference trajectory



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Guidance Law for the Lunar Module Descent



Linear feedback guidance law

$$\mathbf{a}_{command}(t) = \mathbf{a}_r(t) + \mathbf{K}_V [\mathbf{v}_{measured}(t) - \mathbf{v}_r(t)] + \mathbf{K}_R [\mathbf{r}_{measured}(t) - \mathbf{r}_r(t)]$$

\mathbf{K}_V : velocity error gain

\mathbf{K}_R : position error gain

Nominal acceleration profile is corrected for measured differences between actual and reference flight paths

Considerable modifications made in actual LM implementation (see Klumpp's original paper on Blackboard)

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