• Optimization problems and criteria
• Cost functions
• Static optimality conditions
• Examples of static optimization

Copyright 2013 by Robert Stengel. All rights reserved. For educational use only.
http://www.princeton.edu/~stengel/MAE345.html

Typical Optimization Problems

• Minimize the probable error in an estimate of the dynamic state of a system
• Maximize the probability of making a correct decision
• Minimize the time or energy required to achieve an objective
• Minimize the regulation error in a controlled system

• Estimation
• Control
Optimization Implies Choice

- Choice of best strategy
- Choice of best design parameters
- Choice of best control history
- Choice of best estimate
- Optimization provided by selection of the best control variable

Criteria for Optimization

- Names for criteria
  - Figure of merit
  - Performance index
  - Utility function
  - Value function
  - Fitness function
  - Cost function, $J$
    - Optimal cost function = $J^*$
    - Optimal control = $u^*$
- Different criteria lead to different optimal solutions
- Types of Optimality Criteria
  - Absolute
  - Regulatory
  - Feasible
Minimize Absolute Criteria

Achieve a specific objective, such as minimizing the required time, fuel, or financial cost to perform a task.

What is the control variable?

- Find feedback control gains that minimize tracking error, $\Delta x$, in presence of random disturbances.

Optimal System Regulation

Cruise Ship, Anyone?

http://www.youtube.com/watch?v=bVUF35BNKMK&list=PLCF1F89084A30FBED
Feasible Control Logic

Find feedback control structure that guarantees stability (i.e., that prevents divergence)

Desirable Characteristics of a Cost Function

- Scalar
- Clearly defined (preferably unique) maximum or minimum
  - Local
  - Global
- Preferably positive-definite (i.e., always a positive number)
Static vs. Dynamic Optimization

• **Static**
  – Optimal state, $x^*$, and control, $u^*$, are fixed, i.e., they do not change over time: $J^* = J(x^*, u^*)$
    • Functional minimization (or maximization)
    • Parameter optimization

• **Dynamic**
  – Optimal state and control vary over time: $J^* = J(x^*(t), u^*(t))$
    • Optimal trajectory
    • Optimal feedback strategy

• **Optimized cost function**, $J^*$, is a **scalar, real number** in both cases

Deterministic vs. Stochastic Optimization

• **Deterministic**
  – System model, parameters, initial conditions, and disturbances are **known without error**
  – Optimal control operates on the system with certainty
    • $J^* = J(x^*, u^*)$

• **Stochastic**
  – **Uncertainty** in system model, parameters, initial conditions, disturbances, and resulting cost function
  – Optimal control minimizes the **expected value** of the cost:
    • *Optimal cost* = $E(J[x^*, u^*])$

• **Cost function** is a **scalar, real number** in both cases
Cost Function with a Single Control Parameter

- Tradeoff between two types of cost:
  Minimum-cost cruising speed
  - Fuel cost proportional to velocity-squared
  - Cost of time inversely proportional to velocity
- Control parameter: Velocity

Tradeoff Between Time- and Fuel-Based Costs

- Nominal Tradeoff
- Fuel Cost Doubled
- Time Cost Doubled
Cost Functions with Two Control Parameters

- Minimum
- Maximum

3-D plot of equal-cost contours (iso-contours)

2-D plot of equal-cost contours (iso-contours)

Real-World Topography

Local vs. global maxima/minima
Robustness of estimates
Cost Functions with Equality Constraints

Must stay on the trail

Equality constraint may allow control dimension to be reduced

\[ c(u_1, u_2) = 0 \Rightarrow u_2 = fcn(u_1) \]

then

\[ J(u_1, u_2) = J\left[u_1, fcn(u_1)\right] = J'(u_1) \]

Cost Functions with Inequality Constraints

Must stay to the left of the trail

Must stay to the right of the trail

Must stay to the left of the trail

Must stay to the right of the trail
Necessary Condition for Static Optimality

Single control

\[ \frac{dJ}{du} \bigg|_{u=u^*} = 0 \]

i.e., the slope is zero at the optimum point

Example:

\[ J = (u - 4)^2 \]
\[ \frac{dJ}{du} = 2(u - 4) \]
\[ = 0 \quad \text{when } u^* = 4 \]

Necessary Condition for Static Optimality

Multiple controls

\[ \frac{\partial J}{\partial u} \bigg|_{u=u^*} = \left[ \frac{\partial J}{\partial u_1}, \frac{\partial J}{\partial u_2}, \ldots, \frac{\partial J}{\partial u_m} \right] = 0 \]

i.e., all slopes are concurrently zero at the optimum point

Example:

\[ J = (u_1 - 4)^2 + (u_2 - 8)^2 \]
\[ \frac{dJ}{du_1} = 2(u_1 - 4) \quad \text{when } u_1^* = 4 \]
\[ \frac{dJ}{du_2} = 2(u_2 - 8) \quad \text{when } u_2^* = 8 \]
\[ \frac{\partial J}{\partial u} \bigg|_{u=u^*} = \left[ \frac{\partial J}{\partial u_1}, \frac{\partial J}{\partial u_2} \right] \bigg|_{u=u^*} = \left[ \begin{array}{c} 4 \\ 8 \end{array} \right] = 0 \]
... But the Slope can be Zero for More than One Reason

Minimum

Maximum

Either

Inflection Point

Sufficient Condition for Static Optimum

- Single control

Minimum

Satisfy necessary condition plus

\[
\frac{d^2 J}{du^2}
\bigg|_{u = u^*} > 0
\]
i.e., curvature is positive at optimum

Example:

\[
J = (u - 4)^2
\]

\[
\frac{dJ}{du} = 2(u - 4)
\]

\[
\frac{d^2 J}{du^2} = 2 > 0
\]

Maximum

Satisfy necessary condition plus

\[
\frac{d^2 J}{du^2}
\bigg|_{u = u^*} < 0
\]
i.e., curvature is negative at optimum

Example:

\[
J = -(u - 4)^2
\]

\[
\frac{dJ}{du} = -2(u - 4)
\]

\[
\frac{d^2 J}{du^2} = -2 < 0
\]
Sufficient Condition for Static Minimum

Multiple controls

• Satisfy necessary condition
  – plus

\[
\frac{\partial^2 J}{\partial u^2} = \left[ \begin{array}{ccc}
\frac{\partial^2 J}{\partial u_1 \partial u_2} & \frac{\partial^2 J}{\partial u_1 \partial u_3} & \ldots & \frac{\partial^2 J}{\partial u_1 \partial u_m} \\
\frac{\partial^2 J}{\partial u_2 \partial u_1} & \frac{\partial^2 J}{\partial u_2 \partial u_2} & \ldots & \frac{\partial^2 J}{\partial u_2 \partial u_m} \\
\frac{\partial^2 J}{\partial u_3 \partial u_1} & \frac{\partial^2 J}{\partial u_3 \partial u_2} & \ldots & \ldots \\
\frac{\partial^2 J}{\partial u_m \partial u_1} & \frac{\partial^2 J}{\partial u_m \partial u_2} & \ldots & \frac{\partial^2 J}{\partial u_m \partial u_m}
\end{array} \right] = 0
\]

Hessian matrix

• ... what does it mean for a matrix to be “greater than zero”?

\[
\frac{\partial^2 J}{\partial u^2} \triangleq Q > 0 \quad \text{if its Quadratic Form, } x^T Q x, \quad \text{is Greater than Zero}
\]

x^T Q x \triangleq Quadratic form

Q : Defining matrix of the quadratic form

\[
[(1 \times n)(n \times n)(n \times 1)] = [(1 \times 1)]
\]

• \( \dim(Q) = n \times n \)
• \( Q \) is symmetric
• \( x^T Q x \) is a scalar
Quadratic Form of $Q$ is Positive* if $Q$ is Positive Definite

- $Q$ is positive-definite if
  - All leading principal minor determinants are positive
  - All eigenvalues are real and positive

- **3 x 3 example**

$$Q = \begin{bmatrix}
  q_{11} & q_{12} & q_{13} \\
  q_{21} & q_{22} & q_{23} \\
  q_{31} & q_{32} & q_{33}
\end{bmatrix}$$

$q_{11} > 0$, $q_{11}q_{22} > 0$, $q_{11}q_{12}q_{13} > 0$

$$\det(sI - Q) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)$$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

* except at $x = 0$

Minimized Cost Function, $J^*$

- **Gradient** is zero at the minimum
- **Hessian matrix** is positive-definite at the minimum
- Expand the cost in a *Taylor series*

$$J(u^* + \Delta u) \approx J(u^*) + \Delta J(u^*) + \Delta^2 J(u^*) + ...$$

$$\Delta J(u^*) = \Delta u^T \frac{\partial J}{\partial u}{|}_{u = u^*} = 0$$

$$\Delta^2 J(u^*) = \frac{1}{2} \Delta u^T \left[ \frac{\partial^2 J}{\partial u^2}{|}_{u = u^*} \right] \Delta u \geq 0$$

- **First variation** is zero at the minimum
- **Second variation** is positive at the minimum
How Many Maxima/Minima does the “Mexican Hat” \([z = (\sin R)/R]\) Have?

\[
\frac{\partial J}{\partial u} = \left[ \frac{\partial J}{\partial u_1} \quad \frac{\partial J}{\partial u_2} \quad \cdots \quad \frac{\partial J}{\partial u_m} \right] \bigg|_{u = u^*} = 0
\]

\[
\frac{\partial^2 J}{\partial u^2} = \left[ \begin{array}{cccc}
\frac{\partial^2 J}{\partial u_1^2} & \frac{\partial^2 J}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 J}{\partial u_1 \partial u_m} \\
\frac{\partial^2 J}{\partial u_2 \partial u_1} & \frac{\partial^2 J}{\partial u_2^2} & \cdots & \frac{\partial^2 J}{\partial u_2 \partial u_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 J}{\partial u_m \partial u_1} & \frac{\partial^2 J}{\partial u_m \partial u_2} & \cdots & \frac{\partial^2 J}{\partial u_m^2}
\end{array} \right] \bigg|_{u = u^*} \succ 0
\]

One maximum

Static Cost Functions with Equality Constraints

- Minimize \(J(u')\), subject to \(c(u') = 0\)
  - \(\dim(c) = [n \times 1]\)
  - \(\dim(u') = [(m + n) \times 1]\)
Two Approaches to Static Optimization with a Constraint

1. Use constraint to reduce control dimension
2. Augment the cost function to recognize the constraint

\[ J_A(u') = J(u') + \lambda^T c(u') \]

Example:

\[
\begin{align*}
\min_{u_1, u_2} J & \quad \text{subject to} \\
& c(u_1, u_2) = 0 \rightarrow u_2 = fcn(u_1)
\end{align*}
\]

Then

\[ J(u') = J(u_1, u_2) = J[u_1, fcn(u_1)] = J'(u_1) \]

\( \lambda \) an unknown constant
\( \lambda \) has the same dimension as the constraint
\( \text{dim}(\lambda) = \text{dim}(c) = n \times 1 \)

Solution:

First Approach

Cost function

\[ J = u_1^2 - 2u_1u_2 + 3u_2^2 - 40 \]

Constraint

\[ c = u_2 - u_1 - 2 = 0 \]
\[ \therefore u_2 = u_1 + 2 \]
Solution Example:
Reduced Control Dimension

Cost function and gradient with substitution

\begin{align*}
J &= u_1^2 - 2u_1u_2 + 3u_2^2 - 40 \\
&= u_1^2 - 2u_1(u_1 + 2) + 3(u_1 + 2)^2 - 40 \\
&= 2u_1^2 + 8u_1 - 28 \\
\frac{\partial J}{\partial u_1} &= 4u_1 + 8 = 0
\end{align*}

Optimal solution

\begin{align*}
u_1^* &= -2 \\
u_2^* &= 0 \\
J^* &= -36
\end{align*}

Solution:
Second Approach

- Partition \( u' \) into a state, \( x \), and a control, \( u \), such that
  - \( \text{dim}(x) = [n \times 1] \)
  - \( \text{dim}(u) = [m \times 1] \)
- Add constraint to the cost function, weighted by Lagrange multiplier, \( \lambda \)
- \( c \) is required to be zero when \( J_A \) is a minimum

\begin{align*}
\begin{bmatrix} x \\ u \end{bmatrix} \\
J_A(u') &= J(u') + \lambda^T c(u') \\
J_A(x,u) &= J(x,u) + \lambda^T c(x,u) \\
c(u') &= c \begin{bmatrix} x \\ u \end{bmatrix} = 0
\end{align*}
Solution: Adjoin Constraint with Lagrange Multiplier

Gradient with respect to $x$, $u$, and $\lambda$ is zero at the optimum point

\[
\frac{\partial J_A}{\partial x} = \frac{\partial J}{\partial x} + \lambda^T \frac{\partial c}{\partial x} = 0
\]

\[
\frac{\partial J_A}{\partial u} = \frac{\partial J}{\partial u} + \lambda^T \frac{\partial c}{\partial u} = 0
\]

\[
\frac{\partial J_A}{\partial \lambda} = c = 0
\]

Simultaneous Solutions for State and Control

- $(2n + m)$ values must be found $(x, \lambda, u)$
- Use first equation to find form of optimizing Lagrange multiplier $(n$ scalar equations$)$
- Second and third equations provide $(n + m)$ scalar equations that specify the state and control

\[
\lambda^* = -\frac{\partial J}{\partial x} \left( \frac{\partial c}{\partial x} \right)^{-1}
\]

\[
\frac{\partial J}{\partial u} + \lambda^* \frac{\partial c}{\partial u} = 0
\]

\[
\frac{\partial J}{\partial u} \frac{\partial \left( \frac{\partial c}{\partial x} \right)^{-1}}{\partial x} \frac{\partial c}{\partial u} = 0
\]

\[
c(x, u) = 0
\]
Solution Example: Second Approach

Cost function

\[ J = u^2 - 2xu + 3x^2 - 40 \]

Constraint

\[ c = x - u - 2 = 0 \]

Partial derivatives

\[
\begin{align*}
\frac{\partial J}{\partial x} &= -2u + 6x \\
\frac{\partial J}{\partial u} &= 2u - 2x \\
\frac{\partial c}{\partial x} &= 1 \\
\frac{\partial c}{\partial u} &= -1
\end{align*}
\]

Solution Example: Second Approach

• From first equation

\[ \lambda^* = 2u - 6x \]

• From second equation

\[
(2u - 2x) + (2u - 6x)(-1) \\
\therefore x = 0
\]

• From constraint

\[ u = -2 \]

• Optimal solution

\[
\begin{align*}
x^* &= 0 \\
u^* &= -2 \\
J^* &= -36
\end{align*}
\]
Next Time:
Numerical Optimization