Typical Optimization Problems

- Minimize the probable error in an estimate of the dynamic state of a system
- Maximize the probability of making a correct decision
- Minimize the time or energy required to achieve an objective
- Minimize the regulation error in a controlled system
Optimization Implies Choice

- Choice of best strategy
- Choice of best design parameters
- Choice of best control history
- Choice of best estimate
- Optimization provided by selection of the best control variable

Criteria for Optimization

- Names for criteria
  - Figure of merit
  - Performance index
  - Utility function
  - Value function
  - Fitness function
  - Cost function, $J$
    - Optimal cost function = $J^*$
    - Optimal control = $u^*$
- Different criteria lead to different optimal solutions
- Types of Optimality Criteria
  - Absolute
  - Regulatory
  - Feasible
Minimize Absolute Criteria

Achieve a specific objective, such as minimizing the required time, fuel, or financial cost to perform a task

What is the control variable?

Optimal System Regulation

Design controller to minimize tracking error, $\Delta x$, in presence of random disturbances
Feasible Control Logic
Find feedback control structure that guarantees stability (i.e., that prevents divergence)

Desirable Characteristics of a Cost Function

- **Scalar**
- Clearly defined (preferably unique) maximum or minimum
  - Local
  - Global
- Preferably **positive-definite** (i.e., always a positive number)

Single Inverted Pendulum
http://www.youtube.com/watch?v=mi-tek7HvZs

Double Inverted Pendulum
http://www.youtube.com/watch?v=8HDDzKxNMEY
**Static vs. Dynamic Optimization**

- **Static**
  - Optimal state, $x^*$, and control, $u^*$, are fixed, i.e., they do not change over time: $J^* = J(x^*, u^*)$
    - Functional minimization (or maximization)
    - Parameter optimization

- **Dynamic**
  - Optimal state and control vary over time: $J^* = J(x^*(t), u^*(t))$
    - Optimal trajectory
    - Optimal feedback strategy

- **Optimized cost function** $J^*$, is a scalar, real number in both cases

**Deterministic vs. Stochastic Optimization**

- **Deterministic**
  - System model, parameters, initial conditions, and disturbances are known without error
  - Optimal control operates on the system with certainty
    - $J^* = J(x^*, u^*)$

- **Stochastic**
  - Uncertainty in system model, parameters, initial conditions, disturbances, and resulting cost function
  - Optimal control minimizes the expected value of the cost:
    - \[ \text{Optimal cost} = E\{J(x^*, u^*)} \]

- **Cost function** is a scalar, real number in both cases
Cost Function with a Single Control Parameter

- Tradeoff between two types of cost:
  - Minimum-cost cruising speed
  - Fuel cost proportional to velocity-squared
  - Cost of time inversely proportional to velocity
- Control parameter: Velocity

Tradeoff Between Time- and Fuel-Based Costs

- Nominal Tradeoff
- Fuel Cost Doubled
- Time Cost Doubled
Cost Functions with Two Control Parameters

3-D plot of equal-cost contours (iso-contours)

2-D plot of equal-cost contours (iso-contours)

Real-World Topography

Local vs. global maxima/minima
Robustness of estimates
Cost Functions with Equality Constraints

Equality constraint may allow control dimension to be reduced

Cost Functions with Inequality Constraints

Person: Stay outside the fence
Horse: Stay inside the fence
Necessary Condition for Static Optimality

Single control

\[
\frac{dJ}{du}\bigg|_{u=u^*} = 0
\]

i.e., the slope is zero at the optimum point

Example:

\[
J = (u - 4)^2
\]
\[
\frac{dJ}{du} = 2(u - 4)
\]
\[
= 0 \text{ when } u^* = 4
\]

Necessary Condition for Static Optimality

Multiple controls

\[
\frac{\partial J}{\partial u}\bigg|_{u=u^*} = \begin{bmatrix}
\frac{\partial J}{\partial u_1} & \frac{\partial J}{\partial u_2} & \cdots & \frac{\partial J}{\partial u_m}
\end{bmatrix}_{u=u^*} = 0
\]

i.e., all slopes are concurrently zero at the optimum point

Example:

\[
J = (u_1 - 4)^2 + (u_2 - 8)^2
\]
\[
dJ/du_1 = 2(u_1 - 4) = 0 \text{ when } u_1^* = 4
\]
\[
dJ/du_2 = 2(u_2 - 8) = 0 \text{ when } u_2^* = 8
\]
\[
\frac{\partial J}{\partial u}_{u=u^*} = \begin{bmatrix}
\frac{\partial J}{\partial u_1} & \frac{\partial J}{\partial u_2}
\end{bmatrix}_{u=u^*} = \begin{bmatrix}
4 \\ 8
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 \\ 0
\end{bmatrix}
\]
... But the Slope can be Zero for More than One Reason

![Graphs showing minimum, maximum, inflection point, and either case]

Sufficient Condition for Static Optimum

- **Single control**

<table>
<thead>
<tr>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satisfy necessary condition plus</td>
<td>Satisfy necessary condition plus</td>
</tr>
</tbody>
</table>

\[
\frac{d^2 J}{du^2} \bigg|_{u=u^*} > 0
\]

i.e., curvature is positive at optimum

Example:

\[
J = (u - 4)^2
\]

\[
\frac{dJ}{du} = 2(u - 4)
\]

\[
\frac{d^2 J}{du^2} = 2 > 0
\]

\[
\frac{d^2 J}{du^2} \bigg|_{u=u^*} < 0
\]

i.e., curvature is negative at optimum

Example:

\[
J = -(u - 4)^2
\]

\[
\frac{dJ}{du} = -2(u - 4)
\]

\[
\frac{d^2 J}{du^2} = -2 < 0
\]
Sufficient Condition for Static Minimum

Multiple controls

- Satisfy necessary condition
- plus

\[
\frac{\partial^2 J}{\partial \mathbf{u}^2} \bigg|_{\mathbf{u}=\mathbf{u}^*} = \begin{bmatrix}
\frac{\partial^2 J}{\partial u_1^2} & \frac{\partial^2 J}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 J}{\partial u_1 \partial u_m} \\
\frac{\partial^2 J}{\partial u_2 \partial u_1} & \frac{\partial^2 J}{\partial u_2^2} & \cdots & \frac{\partial^2 J}{\partial u_2 \partial u_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 J}{\partial u_m \partial u_1} & \frac{\partial^2 J}{\partial u_m \partial u_2} & \cdots & \frac{\partial^2 J}{\partial u_m^2}
\end{bmatrix}
\bigg|_{\mathbf{u}=\mathbf{u}^*} > 0
\]

\[
\frac{\partial^2 J}{\partial \mathbf{u}^2} \triangleq \mathbf{Q} > 0 \quad \text{if Its Quadratic Form, } \mathbf{x}^T \mathbf{Q} \mathbf{x}, \text{ is Greater than Zero}
\]

\[
\mathbf{x}^T \mathbf{Q} \mathbf{x} \triangleq \text{Quadratic form}
\]

\[
\text{Q: Defining matrix} \text{ of the quadratic form}
\]

\[
[(1 \times n)(n \times n)(n \times 1)] = [(1 \times 1)]
\]

- \(\text{dim}(\mathbf{Q}) = n \times n\)
- \(\mathbf{Q}\) is symmetric
- \(\mathbf{x}^T \mathbf{Q} \mathbf{x}\) is a scalar
Quadratic Form of Q is Positive* if Q is Positive Definite

• Q is positive-definite if
  – All leading principal minor determinants are positive
  – All eigenvalues are real and positive

 3 x 3 example

\[
Q = \begin{bmatrix}
q_{11} & q_{12} & q_{13} \\
q_{21} & q_{22} & q_{23} \\
q_{31} & q_{32} & q_{33}
\end{bmatrix}
\]

\[
q_{11} > 0, \quad q_{11} q_{12} > 0, \quad q_{11} q_{12} q_{13} > 0
\]

\[
\det(sI - Q) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)
\]

\[
\lambda_1, \lambda_2, \lambda_3 \geq 0
\]

* except at x = 0

Minimized Cost Function, J*

• Gradient is zero at the minimum
• Hessian matrix is positive-definite at the minimum
• Expand the cost in a Taylor series

\[
J(u^* + \Delta u) \approx J(u^*) + \Delta J(u^*) + \Delta^2 J(u^*) + \ldots
\]

\[
\Delta J(u^*) = \Delta u^T \frac{\partial J}{\partial u} \bigg|_{u=u^*} = 0
\]

\[
\Delta^2 J(u^*) = \frac{1}{2} \Delta u^T \frac{\partial^2 J}{\partial u^2} \bigg|_{u=u^*} \Delta u \geq 0
\]

• First variation is zero at the minimum
• Second variation is positive at the minimum
How Many Maxima/Minima does the “Mexican Hat” Have?

\[ z = \text{sinc}(R) \triangleq \frac{\sin R}{R} \]

\[ \frac{\partial J}{\partial u} |_{u^*} = \left[ \frac{\partial J}{\partial u_1} \frac{\partial J}{\partial u_2} \ldots \frac{\partial J}{\partial u_m} \right] |_{u^*} = 0 \]

\[ \frac{\partial^2 J}{\partial u^2} |_{u^*} = \left[ \begin{array}{ccc}
\frac{\partial^2 J}{\partial u_1^2} & \frac{\partial^2 J}{\partial u_1 \partial u_2} & \ldots \frac{\partial^2 J}{\partial u_1 \partial u_m} \\
\frac{\partial^2 J}{\partial u_2 \partial u_1} & \frac{\partial^2 J}{\partial u_2^2} & \ldots \frac{\partial^2 J}{\partial u_2 \partial u_m} \\
\vdots & \vdots & \ddots \\
\frac{\partial^2 J}{\partial u_m \partial u_1} & \frac{\partial^2 J}{\partial u_m \partial u_2} & \ldots \frac{\partial^2 J}{\partial u_m^2} 
\end{array} \right] |_{u^*} \]

One maximum

Wolfram Alpha

\texttt{maximize(sinc(sqrt(x^2+y^2)))}

Static Cost Functions with Equality Constraints

\begin{itemize}
  \item Minimize \( J(u') \), subject to \( c(u') = 0 \)
  \begin{itemize}
    \item \( \text{dim}(c) = [n \times 1] \)
    \item \( \text{dim}(u') = [(m + n) \times 1] \)
  \end{itemize}
\end{itemize}
Two Approaches to Static Optimization with a Constraint

1. Use constraint to reduce control dimension

2. Augment the cost function to recognize the constraint

\[ J_A(u') = J(u') + \lambda^T c(u') \]

**Example**: \( \min_{u_1, u_2} J \) subject to

\[ c(u') = c(u_1, u_2) = 0 \rightarrow u_2 = fcn(u_1) \]

\[ J(u') = J(u_1, u_2) = J[u_1, fcn(u_1)] = J'(u_1) \]

\[ \lambda \text{, an unknown constant} \]

\[ \lambda \text{ has the same dimension as the constraint} \]

\[ \dim(\lambda) = \dim(c) = n \times 1 \]

Solution: First Approach

Cost function

\[ J = u_1^2 - 2u_1u_2 + 3u_2^2 - 40 \]

Constraint

\[ c = u_2 - u_1 - 2 = 0 \]

\[ \therefore u_2 = u_1 + 2 \]
Solution Example: Reduced Control Dimension

Cost function and gradient with substitution

\[ J = u_1^2 - 2u_1u_2 + 3u_2^2 - 40 \]
\[ = u_1^2 - 2u_1(u_1 + 2) + 3(u_1 + 2)^2 - 40 \]
\[ = 2u_1^2 + 8u_1 - 28 \]
\[ \frac{\partial J}{\partial u_1} = 4u_1 + 8 = 0; \quad u_1 = -2 \]

Optimal solution

\[ u_1^* = -2 \]
\[ u_2^* = 0 \]
\[ J^* = -36 \]

Solution: Second Approach

- Partition \( u' \) into a state, \( x \), and a control, \( u \), such that
  - \( \text{dim}(x) = \text{dim}(c(x)) = [n \times 1] \)
  - \( \text{dim}(u) = [m \times 1] \)
- Add constraint to the cost function, weighted by Lagrange multiplier, \( \lambda \)
  - \( \text{dim}(\lambda) = [n \times 1] \)
- \( c \) is required to be zero when \( J_A \) is a minimum

\[ J_A(u') = J(u') + \lambda^T c(u') \]
\[ J_A(x,u) = J(x,u) + \lambda^T c(x,u) \]
\[ c(u') = \begin{pmatrix} x \\ u \end{pmatrix} = 0 \]
Solution: Adjoin Constraint with Lagrange Multiplier

Gradient with respect to \( x, u, \) and \( \lambda \) is zero at the optimum point

\[
\frac{\partial J_A}{\partial x} = \frac{\partial J}{\partial x} + \lambda^T \frac{\partial c}{\partial x} = 0
\]

\[
\frac{\partial J_A}{\partial u} = \frac{\partial J}{\partial u} + \lambda^T \frac{\partial c}{\partial u} = 0
\]

\[
\frac{\partial J_A}{\partial \lambda} = c = 0
\]

Simultaneous Solutions for State and Control

- \((2n + m)\) values must be found \((x, \lambda, u)\)
- Use first equation to find form of optimizing Lagrange multiplier \((n\) scalar equations\))
- Second and third equations provide \((n + m)\) scalar equations that specify the state and control

\[
\lambda^* = -\frac{\partial J}{\partial x} \left( \frac{\partial c}{\partial x} \right)^{-1}
\]

\[
\lambda^* = \left[ \left( \frac{\partial c}{\partial x} \right)^{-1} \right]^T \left( \frac{\partial J}{\partial x} \right)^T
\]

\[
\frac{\partial J}{\partial u} + \lambda^* \frac{\partial c}{\partial u} = 0
\]

\[
\frac{\partial J}{\partial u} - \frac{\partial J}{\partial x} \left( \frac{\partial c}{\partial x} \right)^{-1} \frac{\partial c}{\partial u} = 0
\]

\[
c(x,u) = 0
\]
Solution Example: Second Approach

Cost function
\[ J = u^2 - 2xu + 3x^2 - 40 \]

Constraint
\[ c = x - u - 2 = 0 \]

Partial derivatives
\[ \frac{\partial J}{\partial x} = -2u + 6x \]
\[ \frac{\partial J}{\partial u} = 2u - 2x \]
\[ \frac{\partial c}{\partial x} = 1 \]
\[ \frac{\partial c}{\partial u} = -1 \]

Solution Example: Second Approach

- From first equation
  \[ \lambda^* = 2u - 6x \]

- From second equation
  \[ (2u - 2x) + (2u - 6x)(-1) \]
  \[ \therefore x = 0 \]

- From constraint
  \[ u = -2 \]

- Optimal solution
  \[ x^* = 0 \]
  \[ u^* = -2 \]
  \[ J^* = -36 \]
Next Time:
Numerical Optimization