Dynamic Effects of Feedback Control

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- Inner, Middle, and Outer Feedback Control Loops
- Step Response of Linear, Time-Invariant (LTI) Systems
- Position and Rate Control
- Transient and Steady-State Response to Sinusoidal Inputs

Outer-to-Inner-Loop Control Hierarchy

- **Inner Loop**
  - Small Amplitude
  - Fast Response
  - High Bandwidth

- **Middle Loop**
  - Moderate Amplitude
  - Medium Response
  - Moderate Bandwidth

- **Outer Loop**
  - Large Amplitude
  - Slow Response
  - Low Bandwidth

- **Feedback**
  - Error between command and feedback signal drives next inner-most loop
Natural Feedback Control

Inner Loop
Chicken Head Control - 1
http://www.youtube.com/watch?v=_dPlkFPowCc

Middle Loop
Hovering Red-Tail Hawks
http://www.youtube.com/watch?v=-VPVZMSEvwU

Outer Loop
Osprey Diving for Fish
http://www.youtube.com/watch?v=qrgp19-N6jY

Outer-to-Inner-Loop Control Hierarchy of an Industrial Robot

- Inner Loop
  - Focus on control of individual joints

- Middle Loop
  - Focus on operation of the robot

- Outer Loop
  - Focus on goals for robot operation
Inner-Loop Feedback Control

Feedback control design must account for actuator-system-sensor dynamics

![Diagram of Inner-Loop Feedback Control](image)

Single-Input/Single-Output Example, with forward and feedback control logic ("compensation")

Thermostatic Temperature Control

- **Dynamics**
  - Delays
  - Dead Zones
  - Saturation
  - Coupling

- **Structure**
  - Layout
  - Insulation
  - Circulation
  - Multiple Spaces

- **External Effects**
  - Solar Radiation
  - Air Temperature
  - Wind
  - Rain, Humidity

... all controlled by a simple (but nonlinear) on/off switch
Thermostat Control Logic

- **Control Law** [i.e., logic that drives the control variable, \( u(t) \)]
  \[
e(t) = y_c(t) - y(t) = u_c(t) - u_b(t)
  \]
  \[
  u(t) = \begin{cases} 
  1 \text{ (on)}, & e(t) > 0 \\
  0 \text{ (off)}, & e(t) \leq 0 
  \end{cases}
  \]

- \( y_c \): Desired output variable (command)
- \( y \): Actual output
- \( u \): Control variable (forcing function)
- \( e \): Control error

...but control signal would “chatter” with slightest change of temperature

- **Solution**: Introduce *lag* to slow the switching cycle, e.g., *hysteresis*
  \[
  u(t) = \begin{cases} 
  1 \text{ (on)}, & e(t) - T > 0 \\
  0 \text{ (off)}, & e(t) + T \leq 0 
  \end{cases}
  \]
Thermostat Control Logic with Hysteresis

Hysteresis delays the response
System responds with a *limit cycle*

- Cooling control is similar with sign reversal

Speed Control of Direct-Current Motor

Linear Feedback Control Law (*c* = Control Gain)

\[ u(t) = c \, e(t) \]  
where
\[ e(t) = y_c(t) - y(t) \]

How would \( y(t) \) be measured?
Characteristics of the Model

- **Simplified Dynamic Model**
  - Rotary inertia, $J$, is the sum of motor and load inertias
  - Internal damping neglected
  - Output speed, $y(t)$, rad/s, is an integral of the control input, $u(t)$
  - Motor control torque is proportional to $u(t)$
  - Desired speed, $y_c(t)$, rad/s, is constant

Model of Dynamics and Speed Control

First-order LTI ordinary differential equation

$$\frac{dy(t)}{dt} = \frac{1}{J} u(t) = \frac{c}{J} e(t) = \frac{c}{J} [y_c(t) - y(t)], \quad y(0) \text{ given}$$

Integral of the equation, with $y(0) = 0$

$$y(t) = \frac{1}{J} \int_0^t u(t) dt = \frac{c}{J} \int_0^t e(t) dt = \frac{c}{J} \int_0^t [y_c(t) - y(t)] dt$$

- Direct integration of $y_c(t)$
- Negative feedback of $y(t)$
Step Response of Speed Controller

- Solution of the integral

\[ y(t) = y_c \left[ 1 - \exp\left(\frac{-c}{J}t\right) \right] = y_c \left[ 1 - \exp^{\lambda t} \right] = y_c \left[ 1 - \exp^{-\frac{t}{\tau}} \right] \]

- where
  - \( \lambda = \frac{-c}{J} \) = eigenvalue or root of the system (rad/sec)
  - \( \tau = \frac{J}{c} \) = time constant of the response (sec)

What does the shaft angle response look like?

Angle Control of Direct-Current Motor

- Simplified Dynamic Model
  - Rotary inertia, \( J \), is the sum of motor and load inertias
  - Output angle, \( y(t) \), is a double integral of the control, \( u(t) \)
  - Desired angle, \( y_c(t) \), is constant

Feedback Control Law

\[ u(t) = c e(t) \]

\[ e(t) = y_c(t) - y(t) \]
Model of Dynamics and Angle Control

Associated 2nd-order, linear, time-invariant ordinary differential equation

\[
\frac{d^2 y(t)}{dt^2} = \ddot{y}(t) = \frac{c}{J} \left[ y_c - y(t) \right]
\]

Model of Dynamics and Angle Control

- Corresponding set of 1st-order equations, with
  - Angle: \( x_1(t) = y(t) \)
  - Angular rate: \( x_2(t) = \frac{dy(t)}{dt} \)

\[
\dot{x}_1(t) = x_2(t)
\]

\[
\dot{x}_2(t) = \frac{u(t)}{J} = \frac{c}{J} \left[ y_c - y(t) \right] = \frac{c}{J} \left[ y_c - x_1(t) \right]
\]
State-Space Model of the DC Motor

Open-loop dynamic equation

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1/J
\end{bmatrix} u(t)
\]

Feedback control law

\[u(t) = c [y_c(t) - y_1(t)] = c [y_c(t) - x_1(t)]\]

Closed-loop dynamic equation

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-c/J & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
c/J
\end{bmatrix} y_c
\]

Step Response with Angle Feedback

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-c/J & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
c/J
\end{bmatrix} y_c
\]

Step Response of Undamped Angle Control

\[
\begin{align*}
F1 &= [0;1;1 0]; \\
G1 &= [0;1]; \quad F2 = [0;1;0.5 0]; \\
G2 &= [0;0.5]; \quad F3 = [0;1;0.25 0]; \\
G3 &= [0;0.25]; \quad Hx = [1;0;0 1]; \\
Sys1 &= \text{ss}(F1,G1,Hx,0); \\
Sys2 &= \text{ss}(F2,G2,Hx,0); \\
Sys3 &= \text{ss}(F3,G3,Hx,0); \\
\text{step}(\text{Sys1},\text{Sys2},\text{Sys3})
\end{align*}
\]

\(c/J = 1, 0.5, \text{ and } 0.25\)
What Went Wrong?

• No damping!
• Solution: Add rate feedback in the control law

\[ u(t) = c_1[y_c(t) - y_1(t)] - c_2y_2(t) \]

Closed-loop dynamic equation

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-c_1/J & -c_2/J
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
c_1/J
\end{bmatrix} y_c
\]

Alternative Implementations of Rate Feedback

\[ u(t) = c_1[y_c(t) - y_1(t)] - c_2y_2(t) = c_1[y_c(t) - y_1(t)] - c_2 \frac{dy_1(t)}{dt} \]
Step Response with Angle and Rate Feedback

\[ c_1 / J = 1 \]
\[ c_2 / J = 0, 1.414, 2.828 \]

Step Response of Damped Angle Control

\[
F_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};
\]
\[
G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};
\]
\[
F_{1a} = \begin{bmatrix} 0 & 1 \\ -1 & -1.414 \end{bmatrix};
\]
\[
F_{1b} = \begin{bmatrix} 0 & 1 \\ -1 & -2.828 \end{bmatrix};
\]
\[
H_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};
\]
\[
Sys1 = \text{ss}(F_1,G_1,H_x,0);
\]
\[
Sys2 = \text{ss}(F_{1a},G_1,H_x,0);
\]
\[
Sys3 = \text{ss}(F_{1b},G_1,H_x,0);
\]
\[
\text{step}(Sys1,Sys2,Sys3)
\]

LTI Model with Feedback Control

- Command input, \( u_c \), has dimensions of \( u \)

\[
u(t) = u_c(t) - C_y(t)
\]

\[
\dot{x}(t) = F \dot{x}(t) + Gu(t) + Lw(t)
\]
\[
y(t) = H_x x(t) + H_u u(t)
\]
Effect of Feedback Control on the LTI Model

$$\dot{x}(t) = F x(t) + G u(t) = F x(t) + G [u_c(t) - C y(t)]$$

$$= F_{\text{open loop}} x(t) + G \{u_c(t) - C [H_x x(t)]\}$$

$$= [F - GCH_x] x(t) + G u_c(t)$$

$$\triangleq F_{\text{closed loop}} x(t) + G u_c(t)$$

Feedback modifies the stability matrix of the closed-loop system

Convergence or divergence
Envelope of transient response

LTI Model with Feedback Control and Command Gain

Command input, $y_c$, is “shaped” by $C_c$

$$u(t) = u_c(t) - C y(t)$$

$$= C_c y_c(t) - C y(t)$$
Effect of Command Gain on LTI Model

\[
\dot{x}(t) = F x(t) + G u(t) = F x(t) + G \left\{ C_c y_c(t) - C y(t) \right\} \\
= F x(t) + G \left\{ C_c y_c(t) - C \left[ H_x x(t) \right] \right\} \\
= \left[ F - G C H_x \right] x(t) + G C_c y_c(t)
\]

- Steady-state response of the system \( \dot{x}(t) = 0 \)

\[
x^\star(t) = -\left[ F - G C H_x \right]^{-1} G C_c y_c^\star(t)
\]

- Command gain adjusts the steady-state response
- Has no effect on the stability of the system

Response to Sine Wave Input with Angle Feedback: No Damping

\[
y_c(t) = \sin(\omega t) = \sin(6.28 t), \text{ deg}
\]

- Why are there 2 oscillations in the output?
  - Undamped transient response to the input
  - Long-term dynamic response to the input
- System has a natural frequency of oscillation, \( \omega_n \)
- Long-term response to a sine wave is a sine wave
Response to Sine Wave Input with Rate Damping

\[ y_c(t) = \sin(\omega t) = \sin(6.28 t), \text{ deg} \]

- With damping, transient response decays
- In this case, damping has negligible effect on long-term response

System Dynamics Produces Differences in Amplitude and Phase Angle of Input and Output

Amplitude Ratio \( (AR) = \frac{y_{\text{peak}}}{y_{C_{\text{peak}}}} \)

Phase Angle \( = -360 \frac{\Delta t_{\text{peak}}}{\text{Period}}, \text{ deg} \)

- Amplitude ratio and phase angle characterize the system model
Effect of Input Frequency on Output Amplitude and Phase Angle

- With low input frequency, input and output amplitudes are about the same
- Lag of angle output oscillation, compared to input, is small
- Rate oscillation “leads” angle oscillation by ~90 deg

\[ y_c(t) = \sin(t) / 6.28 \text{, deg} \]

\[ c_1 / J = 1; c_2 / J = 1.414 \]

At Higher Frequency, Phase Angle Lag Increases

\[ c_1 / J = 1; c_2 / J = 1.414 \]

\[ y_c(t) = \sin(t) \text{, deg} \]
\[
\begin{align*}
    c_1 / J &= 1; \\
    c_2 / J &= 1.414
\end{align*}
\]

At Even Higher Frequency, Amplitude Ratio Decreases

\[y_c(t) = \sin(6.28 t), \text{deg}\]

**Frequency Response of the DC Motor with Feedback Control**

- Long-term response to sinusoidal inputs over range of frequencies
  - Determine experimentally or
  - from the transfer function
- Transfer function based on the Laplace transform of the system
- Frequency response depicted in the Bode Plot
Next Time:
Analog and Digital Control Systems

Supplemental Material
LTI Control with Forward-Loop Gain

\[ u(t) = C[y_c(t) - y(t)] \]

\[ \dot{x}(t) = Fx(t) + Gu(t) + Lw(t) \]

\[ y(t) = H_x x(t) + H_u u(t) \]

With \( C_c = C \), command input, \( y_c \), has dimensions of \( y \)