Spectral Properties of Linear-Quadratic Regulators

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- Stability margins of single-input/single-output (SISO) systems
- Characterizations of frequency response
- Loop transfer function
- Return difference function
- Kalman inequality
- Stability margins of scalar linear-quadratic regulators

Return Difference Function and Closed-Loop Roots
Single-Input/Single-Output Control Systems
SISO Transfer Function and Return Difference Function

- **Unit feedback control law**

- **Block diagram algebra**

\[ y(s) = A(s)[y_C(s) - y(s)] \]
\[ [1 + A(s)]y(s) = A(s)y_C(s) \]

\[ \frac{y(s)}{y_C(s)} = \frac{A(s)}{1 + A(s)} : \text{Closed-Loop Transfer Function} \]

- **Open-Loop Transfer Function**

\[ A(s) : \]

- **Return Difference Function**

\[ A(s) = \frac{kn(s)}{d(s)} \]

\[ 1 + A(s) = 1 + \frac{kn(s)}{d(s)} = 0 \text{ defines locus of roots} \]

\[ d(s) + kn(s) = 0 \text{ defines locus of closed-loop roots} \]
Return Difference Example

$$A(s) = \frac{kn(s)}{d(s)} = \frac{k(s - z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1.25(s + 40)}{s^2 + 2(0.3)(7)s + (7)^2}$$

$$1 + A(s) = 1 + \frac{k(s - z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} = 1 + \frac{1.25(s + 40)}{s^2 + 2(0.3)(7)s + (7)^2} = 0$$

$$\left[ s^2 + 2(0.3)(7)s + (7)^2 \right] + 1.25(s + 40) = 0$$

Closed-Loop Transfer Function Example

$$\frac{A(s)}{1 + A(s)} = \frac{1.25(s + 40)}{s^2 + 2(0.3)(7)s + (7)^2}$$

$$A(s) = \frac{1.25(s + 40)}{s^2 + 2(0.3)(7)s + (7)^2} + 1.25(s + 40)$$

$$= \frac{1.25(s + 40)}{s^2 + \left[ 2(0.3)(7) + 1.25 \right] s + \left[ (7)^2 + 1.25(40) \right]}$$

$$= \frac{kn(s)}{d(s) + kn(s)}$$
Open-Loop Frequency Response: Bode Plot

\[ A(j\omega) = \frac{K(j\omega - z)}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{1.25(j\omega + 40)}{(j\omega)^2 + 2(0.3)(7)j\omega + (7)^2} \]

**Two plots**
- \(20\log|A(j\omega)|\) vs. \(\log\omega\)
- \(\angle[A(j\omega)]\) vs. \(\log\omega\)

**Gain Margin**
- Referenced to 0 dB line
- Evaluated where phase angle = \(-180^\circ\)

**Phase Margin**
- Referenced to \(-180^\circ\)
- Evaluated where amplitude ratio = 0 dB
Open-Loop Frequency Response: Nyquist Plot

\[ \text{Re}(A(j\omega)) \text{ vs. Im}(A(j\omega)) \]

- Single plot; input frequency not shown explicitly
- Gain and Phase Margins referenced to \((-1)\) point
- GM and PM represented as length and angle

Only positive frequencies need be considered

Open-Loop Frequency Response: Nichols Chart

\[ 20\log_{10}[A(j\omega)] \text{ vs. } \angle[A(j\omega)] \]

- Single plot
- Gain and Phase Margins shown directly
Algebraic Riccati Equation in the Frequency Domain

Linear-Quadratic Control

- Quadratic cost function for infinite final time

\[
J = \frac{1}{2} \int_{t_i}^{t_f} \begin{bmatrix} \Delta x' \Delta u' \\ 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \Delta u(t) \end{bmatrix} \, dt = \frac{1}{2} \int \left[ [\Delta x'(t)Q\Delta x(t) + \Delta u(t)R\Delta u'(t)] \right] \, dt
\]

- Linear, time-invariant dynamic system

\[
\Delta \dot{x}(t) = F\Delta x(t) + G\Delta u(t)
\]

- Constant-gain optimal control law

\[
\Delta u(t) = -R^{-1}G^T P \Delta x(t) = -C \Delta x(t)
\]

Algebraic Riccati equation

\[
0 = -Q - F^T P - PF + PGR^{-1}G^T P
\]

\[
Q = -F^T P - PF + C^T RC
\]
Frequency Characteristics of the Algebraic Riccati Equation

Add and subtract $sP$ such that

$$P(-F) + (-F^T)P + C^T RC = Q$$

$$P(sI_n - F) + (-sI_n - F^T)P + C^T RC = Q$$

Pre-multiply each term by

$$G^T \left(-sI_n - F^T\right)^{-1}$$

Post-multiply each term by

$$\left(sI_n - F\right)^{-1} G$$

$$G^T \left(-sI_n - F^T\right)^{-1} PG + G^T P \left(sI_n - F\right)^{-1} G + G^T \left(-sI_n - F^T\right)^{-1} C^T RC \left(sI_n - F\right)^{-1} G$$

$$= G^T \left(-sI_n - F^T\right)^{-1} Q \left(sI_n - F\right)^{-1} G$$
Frequency Characteristics of the Algebraic Riccati Equation

Substitute with the control gain matrix

\[ C = R^{-1}G^TP \]
\[ G^TP = RC \]

\[
G^T \left( -sI_n - F^T \right)^{-1} C^T R + RC \left( sI_n - F \right)^{-1} G + G^T \left( -sI_n - F^T \right)^{-1} C^T RC \left( sI_n - F \right)^{-1} G
\]

\[ = G^T \left( -sI_n - F^T \right)^{-1} Q \left( sI_n - F \right)^{-1} G \]

Add \( R \) to both sides

\[
R + G^T \left( -sI_n - F^T \right)^{-1} C^T R + RC \left( sI_n - F \right)^{-1} G + G^T \left( -sI_n - F^T \right)^{-1} C^T RC \left( sI_n - F \right)^{-1} G
\]

\[ = R + G^T \left( -sI_n - F^T \right)^{-1} Q \left( sI_n - F \right)^{-1} G \]

The left side can be factored as*

\[
\left[ I_m + G^T \left( -sI_n - F^T \right)^{-1} C^T \right] R \left[ I_m + C \left( sI_n - F \right)^{-1} G \right]
\]

\[ = R + G^T \left( -sI_n - F^T \right)^{-1} Q \left( sI_n - F \right)^{-1} G \]

* Verify by multiplying the factored form
Modal Expression of Algebraic Riccati Equation

Define the loop transfer function matrix

$$A(s) = C(sI_n - F)^{-1} G$$

Modal Expression of Algebraic Riccati Equation

Recall the cost function transfer matrix

$$Y(s) = H(sI_n - F)^{-1} G, \quad \text{where} \ Q = H^T H$$

Laplace transform of algebraic Riccati equation becomes

$$[I_m + A(-s)]^T R[I_m + A(s)] = R + Y^T (-s) Y(s)$$
Algebraic Riccati Equation

\[
\begin{bmatrix}
I_m + A(-s)
\end{bmatrix}^T R \begin{bmatrix}
I_m + A(s)
\end{bmatrix} = R + Y^T (-s) Y(s)
\]

• **Cost function transfer matrix, \(Y(s)\)**
  - Reflects control-induced state variations in the cost function
  - Governs closed-loop modal properties as \(R\) becomes small
  - Does not depend on \(R\) or \(P\)

• **Loop transfer function matrix, \(A(s)\)**
  - Defines the modal control vector when \(s = s_i\)

\[
Y(s) = H(sI_n - F)^{-1} G
\]

\[
\Delta u_i = -C(s_i I_n - F)^{-1} G \Delta u_i
\]

\[
= -A(s_i) \Delta u_i, \quad i = 1, n
\]

LQ Regulator Portrayed as a Unit-Feedback System

\[
A(s) = C(sI_n - F)^{-1} G
\]

\[
I_m + A(s) = I_m + C(sI_n - F)^{-1} G : \text{Return Difference Matrix}
\]
Determinant of Return Difference Matrix Defines Closed-Loop Eigenvalues

\[
\begin{vmatrix}
I_m + A(s)
\end{vmatrix} = \begin{vmatrix}
I_m + C(sI_n - F)^{-1} G
\end{vmatrix}
\]

\[
= \begin{vmatrix}
I_m + \frac{C \text{Adj}(sI_n - F) G}{|sI_n - F|}
\end{vmatrix} = \begin{vmatrix}
I_m + \frac{C \text{Adj}(sI_n - F) G}{\Delta_{OL}(s)}
\end{vmatrix}
\]

**Characteristic Equation**

\[
\Delta_{OL}(s) \left| \begin{vmatrix}
I_m + A(s)
\end{vmatrix}
\right| = \Delta_{OL}(s) \left| \begin{vmatrix}
I_m + \frac{C \text{Adj}(sI_n - F) G}{\Delta_{OL}(s)}
\end{vmatrix}
\right|
\]

\[
= \left| \Delta_{OL}(s) I_m + C \text{Adj}(sI_n - F) G \right| = \Delta_{CL}(s) = 0
\]

Stability Margins and Robustness of Scalar LQ Regulators
Scalar Case

Multivariable algebraic Riccati equation

\[
\begin{bmatrix}
1 + A(-s) & \mathbb{R}
\end{bmatrix}^T 
\begin{bmatrix}
1 + A(s)
\end{bmatrix}
= \mathbb{R} + Y^T(-s)Y(s)
\]

Algebraic Riccati equation with scalar control

\[
[1 + A(-s)]r[1 + A(s)] = r + Y^T(-s)Y(s)
\]

where

\[
A(s) = C(sI_n - F)^{-1}G \quad (1 \times 1)
\]
\[
dim(C) = (1 \times n)
\]
\[
dim(F) = (n \times n)
\]
\[
dim(G) = (n \times 1)
\]

\[
Y(s) = H(sI_n - F)^{-1}G \quad (p \times 1)
\]
\[
dim(Y(s)) = (p \times 1)
\]
\[
dim(H) = (p \times n)
\]

Scalar Case

Let \( s = j\omega \)

\[
[1 + A(-j\omega)]r[1 + A(j\omega)] = r + Y^T(-j\omega)Y(j\omega)
\]

or

\[
[1 + A(-j\omega)][1 + A(j\omega)] = 1 + \frac{Y^T(-j\omega)Y(j\omega)}{r}
\]

\( A(j\omega) \) is a complex variable

\[
[1 + A(-j\omega)][1 + A(j\omega)] = \left\{ \left[ 1 + c(\omega) \right] - jd(\omega) \right\} \left\{ \left[ 1 + c(\omega) \right] + jd(\omega) \right\}
\]
\[
= \left\{ \left[ 1 + c(\omega) \right]^2 + d^2(\omega) \right\} = \left\| 1 + A(j\omega) \right\|^2 \quad \text{(absolute value)}
\]
Kalman Inequality

In frequency domain, cost transfer function becomes

\[ Y_i(j\omega) = [l_i(\omega) + jm_i(\omega)], \quad i = 1, p \]

\[
1 + \frac{Y^T(-j\omega)Y(j\omega)}{r} = 1 + \sum_{i=1}^{p} \left[ \frac{l_i^2(\omega) + m_i^2(\omega)}{r} \right]
\]

Consequently, the return difference function magnitude is greater than one

\[ |1 + A(j\omega)| \geq 1 \quad \text{Kalman Inequality} \]

Nyquist Plot Showing Consequences of Kalman Inequality
Uncertain Gain and Phase Modifications to the LQ Feedback Loop

How large an uncertainty in loop gain or phase angle can be tolerated by the LQ regulator?

LQ Gain Margin Revealed by Kalman Inequality

- Stability is preserved if
  - No encirclements of the $-1$ point, or
  - Number of counterclockwise encirclements of the $-1$ point equals the number of unstable open-loop roots
- Loop gain change expands or shrinks entire Nyquist plot
Gain Change Expands or Shrinks Entire Plot

\[ k_U \triangleq \text{Uncertain gain} \]
\[ A_{\text{optimal}}(j\omega) = C_{LQ} \left( j\omega I_n - F \right)^{-1} G \]
\[ A_{\text{non-optimal}}(j\omega) = k_U A_{\text{optimal}}(j\omega) \]
\[ = k_U C_{LQ} \left( j\omega I_n - F \right)^{-1} G \]
\[ \left| A_{\text{non-optimal}}(j\omega) \right| = k_U \left| A_{\text{optimal}}(j\omega) \right| \]

- Closed-Loop LQ system is stable until \(-1\) point is reached, and \# of encirclements changes

Scalar LQ Regulator Gain Margin

- Increased gain margin = Infinity
- Decreased gain margin = 50%
LQ Phase Margin
Revealed by Kalman Inequality

- Stability is preserved if
  - No encirclements of the –1 point
  - Number of counterclockwise encirclements of the –1 point equals the number of unstable open-loop roots

Return Difference Function, $1 + A(j\omega)$, is excluded from a unit circle centered at (–1,0)

$|A(j\omega)| = 1$ intersects a unit circle centered at the origin

Intersection of the unit circles occurs where the phase angle of $A(j\omega) = (-180^\circ \pm 60^\circ)$

Therefore, Phase Margin of LQ regulator $\geq 60^\circ$

LQ Regulator Preserves Stability with Phase Uncertainties of At Least –60°

- Phase-angle change rotates entire Nyquist plot
- Closed-Loop LQ system is stable until –1 point is reached
Reduced-Gain-/Phase-Margin Tradeoff

Reduced loop gain decreases allowable phase lag while retaining closed-loop stability

Next Time:
Singular Value Analysis of LQ Systems
Supplemental Material

Effect of Time Delay
Time Delay Example: DC Motor Control

Control command delayed by $\tau$ sec

Effect of Time Delay on Step Response

With no delay

Phase lag due to time delay reduces closed-loop stability
As input frequency increases, phase angle eventually exceeds $-180^\circ$.

Bode Plot of Pure Time Delay

\[ AR(e^{-j\tau \omega}) = 1 \]
\[ \phi(e^{-j\tau \omega}) = -\tau \omega \]

Effect of Pure Time Delay on LQ Regulator Loop Transfer Function

Laplace transform of time-delayed signal:
\[
L[u(t - \tau)] = e^{-\tau s} L[u(t)] = e^{-\tau s} u(s)
\]

\[ \tau_U \triangleq \text{Uncertain time delay, sec} \]

\[
A_{optimal}(j\omega) = C_{LQ}(j\omega I_n - F)^{-1} G
\]
\[
A_{non-optimal}(j\omega) = e^{-\tau_U j\omega} A_{optimal}(j\omega)
\]
\[
= e^{-\tau_U j\omega} C_{LQ}(j\omega I_n - F)^{-1} G
\]
Effect of Pure Time Delay on LQ Regulator Loop Transfer Function

Crossover frequency, $\omega_{cross}$, is frequency for which

$$|A_{optimal}(j\omega_{cross})| = |C_{LQ}(j\omega_{cross}I_n - F)^{-1}G| = 1$$

Time delay that produces 60° phase lag

$$\tau_U = \frac{60^\circ}{\omega_{cross}} \left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{3\omega_{cross}} \text{, sec}$$