Linear Transformation of a Probability Distribution

Shape of probability distribution is unchanged

Let $y = kx$.

$$E(y) = \int_{-\infty}^{\infty} y \, pr(x) \, dx = \bar{y}$$

$$= k \int_{-\infty}^{\infty} x \, pr(x) \, dx = k \, \bar{x}$$

$$E[(y - \bar{y})^2] = \sigma_y^2 = \int_{-\infty}^{\infty} (y - \bar{y})^2 \, pr(x) \, dx$$

$$= k^2 \int_{-\infty}^{\infty} (x - \bar{x})^2 \, pr(x) \, dx = k^2 \sigma_x^2$$

$$\sigma_y = k \sigma_x$$
Linear Transformation of a Gaussian Probability Distribution

Probability distribution of $y$ is Gaussian as well

$$
pr(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\bar{x})^2}{2\sigma^2_x}}
$$

$$
pr(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(y-\bar{y})^2}{2\sigma^2_y}} = \frac{1}{\sqrt{2\pi} k\sigma_x} e^{-\frac{(y-k\bar{x})^2}{2k^2\sigma^2_x}}
$$

Skew is zero:

$$
E[(y - \bar{y})^3] = \int_{-\infty}^{\infty} (y - \bar{y})^3 \, pr(x) \, dx
= k^3 \int_{-\infty}^{\infty} (x - \bar{x})^3 \, pr(x) \, dx = 0
$$

Nonlinear Transformation of a Probability Distribution

$$
y = f(x)
$$

$$
y(x) = y(x_0) + \Delta y(\Delta x)
= f(x_0) + \frac{\partial f}{\partial x} \bigg|_{x=x_0} \Delta x + \cdots
\Rightarrow \Delta y(\Delta x) = \frac{\partial f}{\partial x} \bigg|_{x=x_0} \Delta x
$$

$$
\Pr[y(x_0), y(x_0 + \Delta x)] \Pr(x) = \Pr[x_0, x_0 + \Delta x]
$$

$$
pr[y(x_0)] \Delta y = pr[y(x_0)] \left( \frac{\partial f}{\partial x} \bigg|_{x=x_0} \Delta x \right) = pr[x_0] \Delta x
$$

$$
\Rightarrow \text{in the limit } \Delta x \to 0 \Rightarrow pr[y(x_0)] = pr[x_0] \left( \frac{\partial f}{\partial x} \bigg|_{x=x_0} \right)
$$
**Nonlinear Transformation of a Gaussian Probability Distribution**

\[
E(y) = \bar{y} = \int_{-\infty}^{\infty} y \text{pr}(x) dx
\]

\[
= \int_{-\infty}^{\infty} f(x) \text{pr}(x) dx \neq k \bar{x} \quad \text{(in general)}
\]

\[
E[(y - \bar{y})^2] = \int_{-\infty}^{\infty} (y - \bar{y})^2 \text{pr}(x) dx
\]

\[
= \int_{-\infty}^{\infty} [f(x) - \bar{y}]^2 \text{pr}(x) dx \neq k^2 (x - \bar{x})^2 \quad \text{(in general)}
\]

\[
\text{Skew} : E[(y - \bar{y})^3] = \int_{-\infty}^{\infty} (y - \bar{y})^3 \text{pr}(x) dx
\]

\[
\neq 0 \quad \text{(in general)}
\]

Probability distribution of \( y \) is not Gaussian, but it has a mean and variance.

---

**State Propagation for Nonlinear Dynamic Systems**

**Continuous-time system**

\[
\dot{x}(t) = f_c \left[ x(t), u(t), w(t), t \right], \quad x(0) \text{ given}
\]

\[
x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f_c \left[ x(t), u(t), w(t), t \right] dt
\]

\[
\triangleq x(t_k) + \Delta x(t_k, t_{k+1})
\]

**Discrete-time system**

\[
x(t_{k+1}) = f_d \left[ x(t_k), u(t_k), w(t_k), t \right], \quad x(0) \text{ given}
\]

\[
x(t_{k+1}) = x(t_k) + \Delta f_d \left[ x(t_k), u(t_k), w(t_k), t_k \right]
\]

\[
\triangleq x(t_k) + \Delta x(t_k, t_{k+1})
\]

In both cases, the state propagation can be expressed as

\[
x(t_{k+1}) \triangleq x(t_k) + \Delta x(t_k, t_{k+1})
\]
Nonlinear Dynamic Systems with Random Inputs and Measurement Error

Structure, \( f(.) \), and parameters, \( p(t) \), of the nonlinear system are known without error

Stochastic effects are additive and small

Continuous-time system with random inputs and measurement error

\[
\dot{x}(t) = f_c[x(t), u(t), w(t), t], \quad E[x(0)] \text{ given } \quad z(t) = h[x(t), u(t), n(t)]
\]

Discrete-time system with random inputs and measurement error

\[
x(t_{k+1}) = f_D[x(t_k), u(t_k), w(t_k), t], \quad E[x(0)] \text{ given } \quad z(t_k) = h[x(t_k), u(t_k), n(t_k)]
\]

Nonlinear Propagation of the Mean

Underlying deterministic model (either case)

\[
x(t_{k+1}) = x(t_k) + \Delta f[x(t_k), u(t_k), w(t_k), t_k]
\]

\[
= x(t_k) + \Delta x(t_k, t_{k+1})
\]

Propagation of the mean (continuous- or discrete-time)

\[
E[x(t_{k+1})] \triangleq \bar{x}(t_{k+1}) = E[x(t_k) + \Delta x(t_k, t_{k+1})]
\]

\[
= E[x(t_k)] + E[\Delta x(t_k, t_{k+1})] \triangleq \bar{x}(t_k) + \Delta \bar{x}(t_k, t_{k+1})
\]

“State Estimate” usually defined as estimate of the mean of associated Markov process
Probability Distribution Propagation for Nonlinear Dynamic Systems

Consequently

\[ \bar{x}(t_{k+n}) = \bar{x}(t_k) + \Delta \bar{x}(t_k) + \Delta \bar{x}(t_{k+1}) + \cdots + \Delta \bar{x}(t_{k+n-1}) \]

Central limit theorem: probability distribution of \( x(t_{k+n}) \) approaches a Gaussian distribution for large \( k+n \)

…even if \( \text{pr}[\Delta x(t_{k+1}, t_k)] \) is not Gaussian

Mean and variance are dominant measures of the probability distribution of a system’s state, \( x(t) \)

Nonlinear Propagation of the Covariance

Second central moment

\[ P(t_{k+1}) \triangleq E \left\{ \left[ x(t_{k+1}) - \bar{x}(t_{k+1}) \right] \left[ x(\tau_{k+1}) - \bar{x}(\tau_{k+1}) \right]^T \right\} \]

\[ = E \left\{ \left[ x(t_k) + \Delta x(t_k) \right] - \left[ \bar{x}(t_k) + \Delta \bar{x}(t_k) \right] \right\} \left[ \left[ x(\tau_k) + \Delta x(\tau_k) \right] - \left[ \bar{x}(\tau_k) + \Delta \bar{x}(\tau_k) \right] \right]^T \]

\[ \delta x(t_k) \triangleq x(t_k) - \bar{x}(t_k) \]

\[ P(t_{k+1}) = E \left\{ \delta x(t_k) + \delta [\Delta x(t_k)] \right\} \left\{ \delta x(\tau_k) + \delta [\Delta x(\tau_k)] \right\}^T \]

Covariance matrix

\[ = E \left[ \delta x(t_k) \delta x^T(t_k) \right] + E \left\{ \delta x(t_k) \delta [\Delta x(t_k)] \right\} + E \left\{ \delta [\Delta x(t_k)] \delta x^T(t_k) \right\} + E \left\{ \delta [\Delta x(t_k)] \delta [\Delta x(t_k)] \right\} \]

\[ \triangleq P(t_k) + M(t_k) + M^T(t_k) + \Delta P(t_k) \]
Gaussian and Non-Gaussian
Probability Distributions

- (Almost) all random variables have means and standard deviations
- Minimizing estimate error covariance tends to minimize the “spread” of the error in many (but not all) non-Gaussian cases
- Central Limit Theorem implies that estimate errors tend toward normal distribution

Neighboring-Optimal Estimator
Neighboring-Optimal Estimator

\[ \dot{x}(t) = f[x(t), u(t), w(t), t] \]
\[ z(t) = h[x(t)] + n(t) \]

- Assume
  - Nominal solution exists
  - Disturbance and measurement errors are small
  - State stays close to the nominal solution
  - Mean and variance are good approximators of probability distribution

\[ x_o(t), u_o(t), w_o(t) \text{ known in } [0,t_f] \]
\[ x_o(t) + \Delta x(t) = f[x_o(t), u_o(t), w_o(t), t] + \left[ F(t) \Delta x(t) + G(t) \Delta u(t) + L(t) \Delta w(t) \right] \]
\[ z_o(t) + \Delta z(t) = h[x_o(t)] + n_o(t) + \left[ H(t) \Delta x(t) + \Delta n(t) \right] \]

Jacobian matrices evaluated along the nominal path

\[ F(t) = \frac{\partial f}{\partial x}[x_o(t), u_o(t), w_o(t)]; \quad G(t) = \frac{\partial f}{\partial u}[x_o(t), u_o(t), w_o(t)]; \quad L(t) = \frac{\partial f}{\partial w}[x_o(t), u_o(t), w_o(t)] \]

Neighboring-Optimal Estimator

Estimate the perturbation from the nominal path

\[ \Delta \hat{x}(t) = F(t) \Delta \hat{x}(t) + G(t) \Delta u(t) + K_C(t) \left[ \Delta z(t) - H(t) \Delta \hat{x}(t) \right] \]

\[ K_C(t) \] is the LTV Kalman-Bucy gain matrix

LTV Kalman filter could be used to estimate the state at discrete instants of time

\[ \hat{x}(t) \approx x_{Nom}(t) + \Delta \hat{x}(t) \]
Neighboring-Optimal Estimator Example (from Gelb, 1974)

Radar tracking a falling body (one dimension)

\[
\begin{bmatrix}
  \dot{x}_1(t) \\
  \dot{x}_2(t) \\
  \dot{x}_3(t)
\end{bmatrix} =
\begin{bmatrix}
  x_2(t) \\
  d - g \\
  0
\end{bmatrix} , \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  \text{altitude} \\
  \text{velocity} \\
  \text{ballistic coefficient, } \beta
\end{bmatrix}
\]

\[z(t) = x_1(t) + n(t)\]

where

\[
\text{Drag} = d = \frac{\rho V^2(t)}{2 \beta(t)} = \frac{\rho x_2^2(t)}{2 x_3(t)}
\]

\[
\text{Density} = \rho = \rho_o e^{-\text{altitude}/k} = \rho_o e^{-x_3(t)/k}
\]

Estimate Error for Example (from Gelb, 1974)

- Pre-computed nominal trajectory
- Filter gains also may be pre-computed
- Position and velocity errors diverge
- Ballistic coefficient estimate error exceeds filter estimate
- Example of parameter estimation
Extended Kalman-Bucy Filter

\[ \dot{x}(t) = f\left[ x(t), u(t), w(t), \tau \right] \]
\[ z(t) = h\left[ x(t) \right] + n(t) \]

\[ F(t) \triangleq \frac{\partial f}{\partial x}[x_o(t), u_o(t), w_o(t)]; \quad G(t) \triangleq \frac{\partial f}{\partial u}[x_o(t), u_o(t), w_o(t)]; \]
\[ L(t) \triangleq \frac{\partial f}{\partial w}[x_o(t), u_o(t), w_o(t)]; \quad H(t) \triangleq \frac{\partial h}{\partial x}[x_o(t), u_o(t), w_o(t)] \]

- **Assume**
  - No nominal solution is reliable or available
  - Disturbances and measurement errors may not be small
Extended Kalman-Bucy Filter

- In the estimator
  - Replace the linear dynamic model by the nonlinear model
  - Compute the filter gain matrix using the linearized model
  - Make linear update to the state estimate propagated by the nonlinear model

\[
\hat{x}(t) = f[\hat{x}(t), u(t), t] + K_C(t) \{z(t) - h[\hat{x}(t)]\}
\]

\[
x(t_{k+1}) = \hat{x}(t_k) + \int_{t_k}^{t_{k+1}} \left\{f[\hat{x}(t), u(t), t] + K_C(t) \{z(t) - h[\hat{x}(t)]\}\right\} dt
\]

Filter Gain

\[
K_C(t) = P(t)H^T[\hat{x}(t), u(t)]R_C^{-1}(t)
\]
Extended Kalman-Bucy Filter

\[
\dot{P}(t) = F[\hat{x}(t), u(t)]P(t) + P(t)F^T[\hat{x}(t), u(t)] \\
+ L[\hat{x}(t), u(t)]Q'_C(t)L^T[\hat{x}(t), u(t)] - K_C(t)H[\hat{x}(t), u(t)]P(t)
\]

- Linear Kalman-Bucy filter
  - State estimate is affected by the covariance estimate
  - Covariance estimate is not affected by the state estimate
  - Consequently, the covariance estimate is unaffected by the output, \( z(t) \)

- Extended Kalman-Bucy filter
  - State estimate is affected by the covariance estimate
  - Covariance estimate is affected by the state estimate
  - Therefore, the covariance estimate is affected by the output, \( z(t) \)

Extended Kalman-Bucy Filter Example (from Gelb, 1974)

- Early tracking error is large
- Position, velocity, and ballistic coefficient errors converge to estimated bounds
- Filter gains must be computed on-line
Hybrid Extended Kalman Filter

Numerical integration of continuous-time propagation equations

State Estimate (−)

\[
\hat{x}[t_k(-)] = x[t_{k-1}(+)] + \int_{t_{k-1}}^{t_k} f[\hat{x}(\tau), u(\tau)] d\tau
\]

Covariance Estimate (−)

\[
P[t_k(-)] = P[t_{k-1}(+)] + \int_{t_{k-1}}^{t_k} \left[ F(\tau)P(\tau) + P(\tau)F^T(\tau) + L(\tau)Q_c(\tau)L^T(\tau) \right] d\tau
\]

Jacobian matrices must be calculated

Hybrid Extended Kalman Filter

Recursive discrete-time estimate updates

Filter Gain

\[
K(t_k) = P[t_k(-)]H^T(t_k)[H(t_k)P[t_k(-)]H^T(t_k) + R(t_k)]^{-1}
\]

State Estimate (+)

\[
\hat{x}[t_k(+)] = \hat{x}[t_k(-)] + K(t_k)\left\{z(t_k) - h[\hat{x}[t_k(-)]]\right\}
\]

Covariance Estimate (+)

\[
P[t_k(+)] = [I_n - K(t_k)H(t_k)]P[t_k(-)]
\]
Iterated Extended Kalman Filter
(Gelb, 1974)

Re-apply the update equations to the updated solution to improve the estimate before proceeding.
Re-linearize output matrix before each new update.

\[ \hat{x}_{k,i+1}(+) = \hat{x}_k(-) + K_{k,i} \{ z_k - h[\hat{x}_{k,i}(+)] - H_k[\hat{x}_{k,i}(+)][\hat{x}_k(-) - \hat{x}_{k,i}(+)] \}, \]
\[ \hat{x}_{k,0}(+) = \hat{x}_k(-) \]

Arbitrary # of iterations: \( i = 0, 1, \ldots \)

Filter Gain

\[ K_{k,i} = P_k(-)H_k^T[\hat{x}_{k,i}(+)] \{ H_k[\hat{x}_{k,i}(+)]P_k(-)H_k^T[\hat{x}_{k,i}(+)] + R_k \}^{-1} \]

Covariance Estimate (+)

\[ P_{k,i+1}(+) = \{ I_n - K_{k,i}H_k[\hat{x}_{k,i}(+)] \} P_k(-) \]

Two-Dimensional Example
(from Gelb, 1974)

Same falling sphere dynamics, with offset radar

\[ z(t_k) = \left\{ r_1^2(t_k) + \left[ x(t_k) - r_2 \right]^2 \right\}^{1/2} + n(t) \]

Figure 6.1-5 Geometry of the Two-Dimensional Tracking Problem
Comparison of Filter Results
(from Gelb, 1974)

Comparison of alternative nonlinear estimators
100-trial Monte Carlo evaluation

Figure 6.1-6  Comparative Performance of Several Tracking Algorithms (Ref. 5)

Second-order filter includes additional terms in $f[.]$ and $h[.]$

Quasilinearization
(Describing Functions)
Quasilinearization

- **True linearization**: slope of a nonlinear curve at the evaluation point
- **Quasilinearization**: amplitude-dependent slope of a nonlinear curve at the evaluation point
- **Describing function**: quasilinear function of affine form:

\[
\text{Describing Function} = \text{Bias} + \text{Scale Factor} \left( x - x_0 \right)
\]

Comparison of Clipped Sine Wave with Describing Function Approximation
Deterministic Describing Function of the Saturation Function

- Describing function depends on wave form and amplitude of the input
- Saturation function

\[ y = f(x) = \begin{cases} 
  a, & x \geq a \\
  x, & -a < x < a \\
  -a, & x \leq -a 
\end{cases} \]

- Describing function input = Nonlinear function input
- Sinusoidal input

\[ x(t) = A \sin \omega t \]

- Clipped sine wave

\[ y(t) = f[A \sin \omega t] = \begin{cases} 
  a, & x \geq a \\
  A \sin \omega t, & -a < x < a \\
  -a, & x \leq -a 
\end{cases} \]

Sinusoidal-Input Describing Function of the Saturation Function
(from Graham and McRuer, 1961)

- Approximate nonlinear function by linear function

\[ f(x) \approx d_0 + d_1(x - x_0) \]

- Most readily calculated as the first term of a Fourier series for \( y(t) \)
- For symmetric input \( (x_0 = 0) \) to symmetric nonlinearity, \( d_0 = 0 \), and

\[ d_1 = \frac{2A}{\pi} \left[ \sin^{-1} \left( \frac{a}{A} \right) + \frac{a}{A} \sqrt{1 - \left( \frac{a}{A} \right)^2} \right] \]

\( a \): Saturation limit
\( A \): Input amplitude
Sinusoidal-Input Describing Function of the Saturation Function

Describing function output

\[ y_D(t) = d \sin \omega t = \frac{2A}{\pi} \left[ \sin^{-1} \left( \frac{a}{A} \right) + \left( \frac{a}{A} \right) \sqrt{1 - \left( \frac{a}{A} \right)^2} \right] \sin \omega t \]

See “Describing Function Analysis of Nonlinear Simulink Models” in Simulink Control Design 3.1

Nonlinearity Introduces Harmonics in Output

[Graph showing cello spectrum with synthesized cello tone]
Harmonic Describing Functions of Saturation

Fourier series of symmetrically clipped sine wave includes symmetric harmonic terms

\[ y_D(t) = d_1 \sin \omega t + d_3 \sin (3\omega t + \varphi_3) + d_5 \sin (5\omega t + \varphi_5) + \cdots \]

Describing Function Derived from Expected Values of Input and Output

**Describing Function** = **Bias** + **Scale Factor** \( x - x_o \)

Approximate nonlinear function by linear function

\[ f(x) \approx d_0 + d_1 (x - x_o) \]

Statistical representation of fit error

\[ J = E \left\{ \left[ f(x) - d_0 - d_1 (x - x_o) \right]^2 \right\} \]

Minimize fit error to find \( d_0 \) and \( d_1 \)

\[ \frac{\partial J}{\partial d_0} = 0; \quad \frac{\partial J}{\partial d_1} = 0 \]
Random-Input Describing Function

\[ \tilde{x} \triangleq x - x_o \]

Bias and scale factor

\[
\begin{align*}
    d_0 &= E[f(x)] - d_1 E[\tilde{x}] \\
    d_1 &= \frac{E[\tilde{x}f(x)] - d_0 E[\tilde{x}]}{E[\tilde{x}^2]}
\end{align*}
\]

by elimination

\[
\begin{align*}
    d_0 &= \frac{E[\tilde{x}^2]E[f(x)] - E[\tilde{x}]E[\tilde{x}f(x)]}{E[\tilde{x}^2] - E^2[\tilde{x}]} \\
    d_1 &= \frac{E[\tilde{x}f(x)] - E[\tilde{x}]E[f(x)]}{E[\tilde{x}^2] - E^2[\tilde{x}]}
\end{align*}
\]

If \( f(x) \) and \( \tilde{x} \) both have zero mean

\[
\begin{align*}
    d_0 &= 0 \\
    d_1 &= \frac{E[\tilde{x}f(x)]}{E[\tilde{x}^2]} = \frac{E[xf(x)]}{E[x^2]}
\end{align*}
\]

Describing function for symmetric function

\[
\text{Describing Function} = d_0 + d_1 (x - x_o) = d_1 x
\]

\[
= \frac{E[xf(x)]}{E[x^2]} x
\]
Random-Input Describing Function for the Saturation Function
(from Graham and McRuer, 1961)

**Describing function input:** White noise with standard deviation, \( \sigma \)

\[
x(t) \sim N(0, \sigma) \sim \text{Zero-mean white noise with standard deviation, } \sigma
\]

**Describing function output**

\[
y_D(t) = \frac{E[\tilde{x}f(x)]}{E[\tilde{x}]} x(t) = d_i x(t) = \text{erf} \left( \frac{a}{\sqrt{2} \sigma} \right) x(t)
\]

where the **error function**, erf(\(z\)), is

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\lambda^2} d\lambda
\]

Comparison of Clipped White Noise with Describing Function Approximation
Random-Input and Sinusoidal Describing Functions of the Saturation Function

Describing Function, dB

General shapes are similar

Graph showing saturation function with dB values.

Multivariate Describing Functions

- Let $f(x)$ be a nonlinear vector function of a vector $x$.
- The quasilinear describing function approximation is

$$f(x) \approx b + D(x - \hat{x})$$

where $\hat{x} = E(x)$

- Cost function = Trace of the error covariance matrix

$$J = E \left[ \text{Tr} \left[ (f(x) - b - D(x - \hat{x}))^T (f(x) - b - D(x - \hat{x})) \right] \right]$$

$$= E \left[ \text{Tr} \left[ (f(x) - b - D(x - \hat{x}))^T (f(x) - b - D(x - \hat{x})) \right] \right]$$

- Quasilinear extended Kalman-Bucy filter
  - $F$, $G$, and $L$ replaced by describing function matrices
Describing Function Matrices for State Estimation

Minimize fit error to find $b$ and $D$

\[ \frac{\partial J}{\partial b} = 0; \quad \frac{\partial J}{\partial D} = 0 \]

Describing function bias (see text)

\[ b = E[f(x)] = \hat{f}(x) \]

Describing function scaling matrix is a function of the covariance inverse (see text)

\[ D = E\left\{ [f(x)\hat{x}^T] - E[f(x)]\hat{x}^T \right\}P^{-1} \]

where

\[ \hat{x} = x - \hat{x}; \quad P = E(\hat{x}\hat{x}^T) \]

Monte Carlo Comparison of Quasilinear Filter with Extended Kalman Filter and Three Others (from Gelb, 1974)

\[ \dot{x}(t) = -\sin x(t) + w(t) \]
\[ z(t) = 0.5 \sin(2x_k) + n_k \]
Next Time:
Sigma Points
(Unscented Kalman) Filters
plus Brief Introduction to Particle, Batch Least-Squares, Backward-Smoothing, Gaussian Mixture Filters

Supplemental Material
Quasilinear Filter
Propagation Equations

True linearization Jacobians replaced by quasilinear (describing function) Jacobians

\[ \hat{x}(t_k) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} f(\tau), u(\tau) \, d\tau \]

\[ P(t_k) = P(t_{k-1}) + \int_{t_{k-1}}^{t_k} \left[ D_F(\tau)P(\tau) + P(\tau)D_F^T(\tau) + D_L(\tau)Q'_C(\tau)D_L^T(\tau) \right] \, d\tau \]

\( D_F : (n \times n) \) Stability matrix of \( f(x) \) containing describing function elements

Covariance propagation is state-dependent

Quasilinear System
Stability Matrix

True linearization of nonlinear system equation

\[ \dot{x}_o(t) + \Delta x(t) \approx f[x_o(t), u_o(t), t] + F[x_o(t), u_o(t), t] \Delta x(t) + \cdots \]

\[ = f[x_o(t), u_o(t), t] + F(t) \Delta x(t) + \cdots \]

Quasilinearization of nonlinear system equation

Some or all elements of stability matrix are state-dependent

\[ \dot{x}_o(t) + \Delta x(t) \approx f[x_o(t), u_o(t), t] + F[x_o(t), u_o(t), t] \Delta x(t) + \cdots \]

\[ = f[x_o(t), u_o(t), t] + D_F\left[ E[\Delta x(t)], x_o(t), u_o(t), t \right] \Delta x(t) + \cdots \]
Quasilinear Filter Gain and Updates

Filter Gain

\[ K(t_k) = P(t_k(-))D_H^T(t_k)D_H(t_k)P(t_k(-))D_H^T(t_k) + R(t_k) \]^{-1} \]

State Estimate Update

\[ \hat{x}[t_k(+)] = \hat{x}[t_k(-)] + K(t_k)[z(t_k) - h\{\hat{x}[t_k(-)]\}] \]

Covariance Estimate Update

\[ P[t_k(+)] = \left[I_n - K(t_k)D_H(t_k)\right]P[t_k(-)] \]

\( D_H \): Output matrix of \( h(x) \) containing describing function elements