Linear-Quadratic-Gaussian Controllers
Robert Stengel
Optimal Control and Estimation MAE 546
Princeton University, 2015

- LTI dynamic system
- Asymptotic stability of the constant-gain LQG regulator
- Coupling due to parameter uncertainty
- Robustness (loop transfer) recovery
- Stochastic robustness analysis and design

The Problem: Control to Minimize Cost, Subject to Dynamic Constraint, Uncertain Disturbances, and Measurement Error

\[
\dot{x}(t) = Fx(t) + Gu(t) + Lw(t), \quad x(0) = x_0
\]

\[
z(t) = Hx(t) + n(t)
\]

\[
\text{min}_{u} V(t_o) = \text{min}_{u} J(t_f)
\]

\[
= \frac{1}{2} \min_{u} \left\{ E \left[ x^T(t_f) S(t_f) x(t_f) \right] + E \left[ \int_{0}^{t_f} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right] \right\}
\]

\[
J_D(t) = \left\{ \dot{x}(t), P(t), u(t) \right\}
\]
Initial Conditions and Dimensions

\[ E[x(0)] = \hat{x}_0; \quad E\left[ (x(0) - \hat{x}(0))(x(0) - \hat{x}(0))^T \right] = P(0) \]
\[ E[w(t)] = 0; \quad E[w(t)w^T(\tau)] = W\delta(t - \tau) \]
\[ E[n(t)] = 0; \quad E[n(t)n^T(\tau)] = N\delta(t - \tau) \]
\[ E[w(t)n^T(\tau)] = 0 \]

\[ \text{dim}[x(t)] = n \times 1 \]
\[ \text{dim}[u(t)] = m \times 1 \]
\[ \text{dim}[w(t)] = s \times 1 \]
\[ \text{dim}[z(t)] = \text{dim}[n(t)] = r \times 1 \]

Linear-Quadratic-Gaussian Control
The Equations (Continuous-Time Model)

System State and Measurement
\[ \dot{x}(t) = F(t)x(t) + G(t)u(t) + L(t)w(t) \]
\[ z(t) = Hx(t) + n(t) \]

Control Law
\[ u(t) = -C(t)\hat{x}(t) + C_F(t)y_c(t) \]

State Estimate
\[ \hat{x}(t) = F(t)x(t) + G(t)u(t) + K(t) \left[ z(t) - H\hat{x}(t) \right] \]
\[ = \left[ F(t) - G(t)C(t) - K(t)H \right] \hat{x}(t) + G(t)C_F(t)\dot{y}_c(t) + K(t)z(t) \]

Estimator Gain and State Covariance Estimate
\[ K(t) = P(t)H^TN^{-1}(t) \]
\[ \dot{P}(t) = F(t)P(t) + P(t)F^T(t) + L(t)W(t)L^T(t) - P(t)H^TN^{-1}(t)HP(t) \]

Control Gain and Adjoint Covariance Estimate
\[ C(t) = R^{-1}(t)G^T(t)S(t) \]
\[ S(t) = -Q(t) - F(t)^T S(t) - S(t)F(t) + S(t)G(t)R^{-1}(t)G^T(t)S(t) \]

Estimator in the Feedback Loop

Linear-Gaussian (LG) state estimator adds dynamics to the feedback signal
\[ u(t) = -C(t)\hat{x}(t) \]
\[ \hat{x}(t) = F\hat{x}(t) + Gu(t) + K(t) \left[ z(t) - H\hat{x}(t) \right] \]

Thus, state estimator can be viewed as a “compensator”

Bandwidth of the compensation is dependent on the multivariable signal/noise ratio, \( PH^T N^{-1} \)

\[ K(t) = P(t)H^TN^{-1} \]
\[ \dot{P}(t) = FP(t) + P(t)F^T \]
\[ + LWL^T - P(t)H^TN^{-1}HP(t) \]
Scalar LTI Example of Estimator Compensation

Dynamic system and measurement
\[ \dot{x} = x + w; \quad z = Hx + n \]

Estimator differential equation
\[ \dot{\hat{x}} = \hat{x} + K(z - H\hat{x}) = (1 - KH)\hat{x} + Kz \]

Laplace transform of estimator
\[ [s - (1 - KH)]\hat{x}(s) = Kz(s) \]

Estimator transfer function
Low-pass filter
\[ \hat{x}(s) = \frac{K}{[s - (1 - KH)]}z(s) \]

Steady-State Scalar Filter Gain

Signal "Power" = State Estimate Variance = \( P \)
Noise "Power" = Measurement Error Variance = \( N \)
\( H = Projection \ from \ Noise \ Space \ to \ Signal \ Space \)

Constant, scalar filter gain
\[ K = \frac{PH}{N} \]

Algebraic Riccati equation
\[ 0 = 2P + W - \frac{P^2H^2}{N}; \quad P^2 - \frac{2N}{H^2}P - \frac{WN}{H^2} = 0 \]
\[ P = \frac{N}{H^2} \pm \sqrt{\left(\frac{N}{H^2}\right)^2 + \frac{WN}{H^2}} = \frac{N}{H^2} \left[1 \pm \sqrt{1 + \frac{WH^2}{N}}\right] \]
Steady-State Filter Gain

\[ K = \left\{ \frac{N}{H^2} \left[ 1 + \sqrt{1 + \frac{WH^2}{N}} \right] \right\} H = \left\{ \frac{1}{H} \left[ 1 + \sqrt{1 + \frac{WH^2}{N}} \right] \right\} \]

\[ K \xrightarrow{W \gg N} \frac{\sqrt{W}}{N} \]

Dynamic Constraint on the Certainty-Equivalent Cost

\( P(t) \) is independent of \( u(t) \); therefore

\[ \min_u J = \min_u J_{CE} + J_S \]

\( J_{CE} \) is identical in form to the deterministic cost function

Minimized subject to dynamic constraint based on the state estimate

\[ \dot{x}(t) = F\dot{x}(t) + Gu(t) + K(t)[z(t) - H\dot{x}(t)] \]
Kalman-Bucy Filter Provides Estimate of the State Mean Value

Filter residual is a Gaussian process

\[ \hat{x}(t) = F\hat{x}(t) + Gu(t) + K(t)\left[z(t) - H\hat{x}(t)\right] \]

\[ \triangleq F\hat{x}(t) + Gu(t) + K(t)\epsilon(t) \]

Filter equation is analogous to deterministic dynamic constraint on deterministic cost function

\[ \dot{x}(t) = Fx(t) + Gu(t) + Lw(t) \]

Control That Minimizes the Certainty-Equivalent Cost

Optimizing control history is generated by a time-varying feedback control law

\[ u(t) = -C(t)\hat{x}(t) \]

The control gain is the same as the deterministic gain

\[ C(t) = R^{-1}G^{T}S(t) \]

\[ \dot{S}(t) = -Q - F^{T}S(t) - S(t)F + S(t)GR^{-1}G^{T}S(t) \]

\[ S(t_f) \text{ given} \]
Optimal Cost for the Continuous-Time LQG Controller

Certainty-equivalent cost

\[ J_{CE} = \frac{1}{2} \text{Tr} \left[ S(0)E\left[ \hat{x}(0)\hat{x}^T(0) \right] + \int_0^{t_f} S(t)K(t)NK^T(t) dt \right] \]

Total cost

\[ J = J_{CE} + J_S = \frac{1}{2} \text{Tr} \left[ S(0)E\left[ \hat{x}(0)\hat{x}^T(0) \right] + \int_0^{t_f} S(t)K(t)NK^T(t) dt \right] + \frac{1}{2} \text{Tr} \left[ S(t_f)P(t_f) + \int_0^{t_f} QP(t) dt \right] \]

Discrete-Time LQG Controller

Kalman filter produces state estimate

\[ \hat{x}_k(-) = \Phi \hat{x}_{k-1}(+) + \Gamma C_{k-1} \hat{x}_{k-1}(+) \]
\[ \hat{x}_k(+) = \hat{x}_k(-) + K_k \left[ z_k - H\hat{x}_k(-) \right] \]

Closed-loop system uses state estimate for feedback control

\[ x_{k+1} = \Phi x_k - \Gamma C_k \hat{x}_k(+) \]
Response of Discrete-Time 1st-Order System to Disturbance, and Kalman Filter Estimate from Noisy Measurement

Comparison of 1st-Order Discrete-Time LQ and LQG Control Response

Propagation of Uncertainty

Kalman Filter, Uncontrolled System

Linear-Quadratic Control with Noise-free Measurement

Linear-Quadratic-Gaussian Control with Noisy Measurement
Asymptotic Stability of the LQG Regulator

System Equations with LQG Control

With perfect knowledge of the system

\[
\begin{align*}
\dot{x}(t) &= Fx(t) + Gu(t) + Lw(t) \\
\dot{\hat{x}}(t) &= F\hat{x}(t) + Gu(t) + K(t)[z(t) - H\hat{x}(t)]
\end{align*}
\]

State estimate error

\[
\epsilon(t) = x(t) - \hat{x}(t)
\]

State estimate error dynamics

\[
\dot{\epsilon}(t) = (F - KH)\epsilon(t) + Lw(t) - Kn(t)
\]
Control-Loop and Estimator
Eigenvalues are Uncoupled

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\varepsilon}(t)
\end{bmatrix} =
\begin{bmatrix}
(F - GC) & GC \\
0 & (F - KH)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\varepsilon(t)
\end{bmatrix} +
\begin{bmatrix}
L & 0 \\
L & -K
\end{bmatrix}
\begin{bmatrix}
w(t) \\
n(t)
\end{bmatrix}
\]

Upper-block-triangular stability matrix
LQG system is stable because
\( (F - GC) \) is stable
\( (F - KH) \) is stable

Estimate error affects state response

Actual state does not affect error response
Disturbance affects both equally

Parameter Uncertainty
Introduces Coupling
Coupling Due To Parameter Uncertainty

Actual System: \( \{ F_A, G_A, H_A \} \)
Assumed System: \( \{ F, G, H \} \)

\[
\begin{align*}
\dot{x}(t) &= F_A x(t) + G_A u(t) + Lw(t) \\
\dot{\hat{x}}(t) &= F \hat{x}(t) + G u(t) + K(t) [ z(t) - H \hat{x}(t) ] \\
z(t) &= H_A x(t) + n(t) \\
u(t) &= -C(t) \hat{x}(t)
\end{align*}
\]

Closed-loop control and estimator responses are coupled

Effects of Parameter Uncertainty on Closed-Loop Stability

\[
\begin{align*}
|sI_{2n} - F_{CL}| &= \left| \begin{array}{c}
\frac{sI_n - (F_A - G_A C)}{(F_A - F) - (G_A - G) C - K(H_A - H)} \\
\frac{-G_A C}{F + (G_A - G) C - K H}
\end{array} \right| \\
&= \Delta_{CL}(s) = 0
\end{align*}
\]

- Uncertain parameters affect closed-loop eigenvalues
- Coupling can lead to instability for numerous reasons
  - Improper control gain
  - Control effect on estimator block
  - Redistribution of damping
Doyle’s Counter-Example of LQG Robustness (1978)

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1
\end{bmatrix} u
+ \begin{bmatrix}
1 \\
1
\end{bmatrix} w
\]

\[
z = \begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + n
\]

Design Matrices

\[
Q = Q \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}; \quad R = I; \quad W = W \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}; \quad N = 1
\]

Control and Estimator Gains

\[
C = \left(2 + \sqrt{4 + Q}\right) \begin{bmatrix}
1 & 1
\end{bmatrix} = \begin{bmatrix}
c & c
\end{bmatrix}
\]

\[
K = \left(2 + \sqrt{4 + W}\right) \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
k \\
k
\end{bmatrix}
\]

Uncertainty in the Control Effect

System Matrices

\[
F_A = F; \quad G_A = \begin{bmatrix}
0 \\
\mu
\end{bmatrix}; \quad H_A = H
\]

Characteristic Equation

\[
\begin{vmatrix}
(s - 1) & -1 & 0 & 0 \\
0 & (s - 1) & \mu c & \mu c \\
-k & 0 & (s - 1 + k) & -1 \\
-k & 0 & (c + k) & (s - 1 + c)
\end{vmatrix} = 0
\]

\[
s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = \Delta_{CL}(s) = 0
\]
Stability Effect of Parameter Variation

**Routh's Stability Criterion** (necessary condition)
- All coefficients of $\Delta(s)$ must be positive for stability
  - $\mu$ is nominally equal to 1
  - $\mu$ can force $a_0$ and $a_1$ to change sign
  - Result is dependent on magnitude of $ck$

\[
\begin{align*}
a_1 &= k + c - 4 + 2(\mu - 1)ck \\
a_0 &= 1 + (1 - \mu)ck
\end{align*}
\]

- Arbitrarily small uncertainty, $\mu = 1 + \varepsilon$, could cause unstability
- Not surprising: uncertainty is in the control effect

The Counter-Example Raises a Flag

**Solution**
Choose $Q$ and $W$ to be small, increasing allowable range of $\mu$

- However, .... The counter-example is irrelevant because it does not satisfy the requirements for LQ and LG stability
  - The open-loop system is unstable, so it requires feedback control to restore stability
  - To guarantee stability, $Q$ and $W$ must be positive definite, but

\[
\begin{align*}
Q &= Q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad hence, |Q| = 0 \\
W &= W \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad hence, |W| = 0
\end{align*}
\]
Robustness
(Loop Transfer Recovery)

Loop-Transfer Recovery
(Doyle and Stein, 1979)

- **Proposition:** LQG and LQ robustness would be the same if the control vector had the same effect on the state and its estimate.

\[ Cx(t) \text{ and } \hat{C}\hat{x}(t) \text{ produce same expected value of control, } E[u(t)] \]

but not the same

\[ E\left[\left(u_{LQ}(t) - u_{LQG}(t)\right)^{\prime}\left(u_{LQ}(t) - u_{LQG}(t)\right)\right] \]

as \( \hat{x}(t) \) contains measurement errors but \( x(t) \) does not.

- Therefore, restoring the correct mean value from \( z(t) \) restores closed-loop robustness.

- **Solution:** Increase the assumed “process noise” for estimator design as follows (see text for details)

\[ W = W_o + k^2GG^T \]
Stochastic Robustness Analysis and Design

Expression of Uncertainty in the System Model

System uncertainty may be expressed as
- Elements of $\mathbf{F}$
- Coefficients of $\Delta(s)$
- Eigenvalues, $\lambda$
- Frequency response/singular values/time response, $A(j\omega), \sigma(j\omega), \mathbf{x}(t)$

- **Variation may be**
  - Deterministic, e.g.,
    - Upper/lower bounds (“worst-case”)
  - Probabilistic, e.g.,
    - Gaussian distribution
- **Bounded variation is equivalent to probabilistic variation with uniform distribution**
Stochastic Root Locus: Uncertain Damping Ratio and Natural Frequency

Laplace transform of dynamic model

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]

Gaussian statistics

- \( E(\zeta) = \bar{\zeta} = 0.707 \)
- \( E(\omega_n) = \bar{\omega}_n = 1.0 \)

Uniform Statistics

- \( \zeta = [0.507, 0.907] \)
- \( \omega_n = [0.8, 1.2] \)

Gaussian Distribution of Eigenvalues

Uniform Distribution of Eigenvalues

Probability of Instability

- Nonlinear mapping from probability density functions (pdf) of uncertain parameters to pdf of roots
- Finite probability of instability with Gaussian (unbounded) distribution
- Zero probability of instability for some uniform distributions
Probabilistic Control Design

- Design constant-parameter controller (CPC) for satisfactory stability and performance in an uncertain environment
- Monte Carlo Evaluation of simulated system response with
  - competing CPC designs [Design parameters = d]
  - given statistical model of uncertainty in the plant [Uncertain plant parameters = v]
- Search for best CPC
  - Exhaustive search
  - Random search
  - Multivariate line search
  - Genetic algorithm
  - Simulated annealing

Design Outcome Follows Binomial Distribution

- Binomial distribution: Satisfactory/Unsatisfactory
- Confidence intervals of probability estimate are functions of
  - Actual probability
  - Number of trials
Example: Probability of Stable Control of an Unstable Plant

Longitudinal dynamics for a Forward-Swept-Wing Airplane

\[
F = \begin{bmatrix}
-2g f_{11} / V & \rho V^2 f_{12} / 2 & \rho V f_{13} & -g \\
-45 / V^2 & \rho V f_{22} / 2 & 1 & 0 \\
0 & \rho V^2 f_{32} / 2 & \rho V f_{33} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
g_{11} & g_{12} \\
0 & 0 \\
g_{31} & g_{32} \\
0 & 0
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
V, \text{ Airspeed} \\
\alpha, \text{ Angle of attack} \\
q, \text{ Pitch rate} \\
\theta, \text{ Pitch angle}
\end{bmatrix}
\]

Example: Probability of Stable Control of an Unstable Plant

Nominal eigenvalues (one unstable)

\[\lambda_{1-4} = -0.1 \pm 0.057 j, \quad -5.15, \quad 3.35\]

Air density and airspeed, \(\rho\) and \(V\), have uniform distributions (±30%)

10 coefficients have Gaussian distributions (\(\sigma = 30\%\))

\[
p = \begin{bmatrix}
\rho & V & f_{11} & f_{12} & f_{13} & f_{22} & f_{32} & f_{33} & g_{11} & g_{12} & g_{31} & g_{32}
\end{bmatrix}^T
\]

Environment Uncontrolled Dynamics Control Effect
LQ Regulators for the Example

Three stabilizing feedback control laws

Case a) LQR with low control weighting

\[
Q = \text{diag}(1,1,1,0); \quad R = (1,1); \quad \lambda_{1-4,\text{nominal}} = -35,-5.1,-3.3,-.02
\]

\[
C = \begin{bmatrix}
0.17 & 130 & 33 & 0.36 \\
0.98 & -11 & -3 & -1.1
\end{bmatrix}
\]

Case b) LQR with high control weighting

\[
Q = \text{diag}(1,1,1,0); \quad R = (1000,1000); \quad \lambda_{1-4,\text{nominal}} = -5.2,-3.4,-1.1,-.02
\]

\[
C = \begin{bmatrix}
0.03 & 83 & 21 & -0.06 \\
0.01 & -63 & -16 & -1.9
\end{bmatrix}
\]

Case c) Case b with gains multiplied by 5 for bandwidth (loop-transfer) recovery

\[
\lambda_{1-4,\text{nominal}} = -32,-5.2,-3.4,-0.01
\]

\[
C = \begin{bmatrix}
0.13 & 413 & 105 & -0.32 \\
0.05 & -313 & -81 & -1.1 - 9.5
\end{bmatrix}
\]

Stochastic Robustness

(Ray, Stengel, 1991)

- Distribution of closed-loop roots with
  - Gaussian uncertainty in 10 parameters
  - Uniform uncertainty in velocity and air density
- 25,000 Monte Carlo evaluations
- Probability of instability
  - a) Pr = 0.072
  - b) Pr = 0.021
  - c) Pr = 0.0076
Probabilities of Instability for the Three Cases

95% CONFIDENCE INTERVALS (with dynamic pressure effects)

- **Case a: Low LQ Control Weights**
  - Probabilities of instability with 30% uniform aerodynamic uncertainty
    - Case a: $3.4 \times 10^{-4}$
    - Case b: 0
    - Case c: 0

- **Case b: High LQ Control Weights**

- **Case c: Bandwidth Recovery**

Stochastic Root Loci for the Three Cases

with Gaussian Aerodynamic Uncertainty
ACC Benchmark Control Problem, 1991

• Parameters of 4th-order mass-spring system
  – Uniform probability density functions for
    • $0.5 < m_1, m_2 < 1.5$
    • $0.5 < k < 2$
  • Probability of Instability, $P_i$
    – $m_i = 1$ (unstable) or 0 (stable)
  • Probability of Settling Time Exceedance, $P_{ts}$
    – $m_{ts} = 1$ (exceeded) or 0 (not exceeded)
  • Probability of Control Limit Exceedance, $P_u$
    – $m_u = 1$ (exceeded) or 0 (not exceeded)
• Design Cost Function
• 10 controllers designed for the competition

\[ J = aP_i^2 + bP_{ts}^2 + cP_u^2 \]

Stochastic LQG Design for Benchmark Control Problem

• SISO Linear-Quadratic-Gaussian Regulators (Marrison)
  – Implicit model following with control-rate weighting and scalar output (5th order)
  – Kalman filter with single measurement (4th order)
  – Design parameters
    • Control cost function weights
    • Springs and masses in ideal model
    • Estimator weights
  – Search
    • Multivariate line search
    • Genetic algorithm
Comparison of Design Costs for Benchmark Control Problem

\[ J = P_i^2 + 0.01P_m^2 + 0.01P_u^2 \]
\[ J = 0.01P_i^2 + 0.01P_m^2 + P_u^2 \]
\[ J = 0.01P_i^2 + P_m^2 + 0.01P_u^2 \]

Cost Emphasizes Instability
Cost Emphasizes Excess Control
Cost Emphasizes Settling-Time Exceedance

Stochastic LQG controller more robust in 39 of 40 benchmark controller comparisons

Estimation of Minimum Design Cost Using Jackknife/Bootstrapping Evaluation
Thank you!