Stability of Dynamic Systems

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• Bounds on the system norm
• Lyapunov criteria for stability
• Eigenvalues
• Transfer functions
• Continuous- and discrete-time systems

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http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Vector Norms for Real Variables

• “Norm” = Measure of length or magnitude of a vector, \( x \)

• Euclidean or Quadratic Norm

\[
L^2 \text{ norm } = \| x \|_2 = (x^T x)^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}
\]

• Weighted Euclidean Norm

\[
\| y \|_2 = (y^T y)^{1/2} = (y_1^2 + y_2^2 + \cdots + y_m^2)^{1/2}
\]

\[
= (x^T D^T D x)^{1/2} = \| Dx \|_2
\]

\[
x^T D' D x \triangleq x^T Q x
\]

\[
Q \triangleq D'D = \text{Defining matrix}
\]
Uniform Stability

- Autonomous dynamic system
  - Time-invariant
  - No forcing input

- Uniform stability about \( x = 0 \)
  \[
  \|x(t_0)\| \leq \delta, \quad \delta > 0
  \]

Let \( \delta = \delta(\epsilon) \)
If, for every \( \epsilon \geq 0, \)
\[
\|x(t)\| \leq \epsilon, \quad \epsilon \geq \delta > 0, \quad t \geq t_0
\]
Then the system is uniformly stable

- If system response is bounded, then the system possesses uniform stability

Local and Global Asymptotic Stability

- **Local asymptotic stability**
  - Uniform stability plus
  \[
  \|x(t)\| \xrightarrow{t \to \infty} 0
  \]

- **Global asymptotic stability**
  System is asymptotically stable for any \( \epsilon \)

- If a linear system has uniform asymptotic stability, it also is globally stable

\[
\dot{x}(t) = F\ x(t)
\]
Exponential Asymptotic Stability

- Uniform stability about $x = 0$ plus
  \[ \|x(t)\| \leq ke^{-\alpha t} \|x(0)\|; \quad k, \alpha \geq 0 \]

- If norm of $x(t)$ is contained within an exponentially decaying envelope with convergence, system is \textit{exponentially asymptotically stable (EAS)}

- Linear system that is stable is EAS

Therefore, time integrals of the norm of an EAS system are bounded

\[ k \int_{0}^{\infty} e^{-\alpha t} \, dt = - \left( \frac{k}{\alpha} \right) e^{-\alpha t} \Bigg|_{0}^{\infty} = \frac{k}{\alpha} \]

and

\[ \int_{0}^{\infty} \|x(t)\| \, dt = \int_{0}^{\infty} \left[ x'(t)x(t) \right]^{1/2} \, dt \leq \left( \frac{k}{\alpha} \right) \|x(0)\| \]

\[ \int_{0}^{\infty} \|x(t)\|^2 \, dt \text{ is bounded} \]
Exponential Asymptotic Stability

Weighted Euclidean norm and its square are bounded if system is EAS

\[ \int_0^\infty \|Dx(t)\| dt = \int_0^\infty \left[ x^T(t)D^T Dx(t) \right]^{1/2} dt \text{ is bounded} \]

with \( Q = D^T D > 0 \)

\[ \int_0^\infty [x^T(t)Qx(t)] dt \text{ is bounded} \]

Conversely, if the weighted Euclidean norm is bounded, the system is EAS

Initial-Condition Response of an EAS Linear System

\[ x(t) = \Phi(t,0)x(0) = e^{F(t)}x(0) \]

\[ \|x(t)\|^2 = x^T(0)\Phi^T(t,0)\Phi(t,0)x(0) \text{ is bounded} \]

- To be shown
  - Continuous-time LTI system is stable if all of its eigenvalues have negative real parts
  - Discrete-time LTI system is stable if all of its eigenvalues lie within the unit circle
Lyapunov’s First Theorem

- A nonlinear system is asymptotically stable at the origin if its linear approximation is stable at the origin, i.e.,
  - for all trajectories that start “close enough”
  - within a stable manifold

\[
\dot{x}(t) = f(x(t)) \text{ is stable at } x_o = 0 \text{ if } \Delta \dot{x}(t) = \frac{\partial f[x(t)]}{\partial x} \bigg|_{x_o=0} \Delta x(t) \text{ is stable}
\]

“At the origin” is a fuzzy concept

Lyapunov’s Second Theorem*

Define a scalar Lyapunov function, a positive definite function of the state in the region of interest

\[
V(x(t)) \geq 0
\]

**Examples**

\[
V = E = \frac{mV^2}{2} + mgh; \quad \frac{E}{mg} = \frac{E}{\text{weight}} = \frac{V^2}{2g} + h
\]

\[
V = \frac{1}{2}x^T x; \quad V = \frac{1}{2}x^TPx
\]

Lyapunov’s Second Theorem

Evaluate the time derivative of the Lyapunov function

\[
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \dot{x}} \dot{x}
\]

= \frac{\partial V}{\partial \dot{x}} \dot{x} \text{ for autonomous systems}

• If \( \frac{dV}{dt} < 0 \) in the neighborhood of the origin, the origin is asymptotically stable.

Quadratic Lyapunov Function

Lyapunov function

\[
V[\mathbf{x}(t)] = \mathbf{x}^T(t) \mathbf{P} \mathbf{x}(t)
\]

Linear, Time-Invariant System

\[
\dot{\mathbf{x}}(t) = \mathbf{F} \mathbf{x}(t)
\]

Rate of change for quadratic Lyapunov function

\[
\frac{dV}{dt} = \frac{\partial V}{\partial \dot{x}} \dot{x} = \mathbf{x}^T(t) \mathbf{P} \dot{\mathbf{x}}(t) + \dot{\mathbf{x}}^T(t) \mathbf{P} \mathbf{x}(t)
\]

= \mathbf{x}^T(t)(\mathbf{PF} + \mathbf{F}^T \mathbf{P}) \mathbf{x}(t) \triangleq -\mathbf{x}^T(t) \mathbf{Q} \mathbf{x}(t)
Lyapunov Equation

The LTI system is stable if the Lyapunov equation is satisfied with positive-definite \( P \) and \( Q \)

\[
PF + F^T P = -Q
\]

with \( P > 0, \ Q > 0 \)

Lyapunov Stability:
1\textsuperscript{st}-Order Example

\[\Delta x(t) = a \Delta x(t), \ \Delta x(0) \text{ given}\]

\[F = a, \ P = p, \ Q = q\]

\[\Delta x(t) = \int_0^t \Delta x(t) \, dt = \int_0^t a \Delta x(t) \, dt = e^{at} \Delta x(0)\]

\[
PF + F^T P = -Q
\]

with \( p > 0, \ a < 0 \)

\[2pa < 0 \text{ and } q > 0\]

\[\therefore \text{ system is stable}\]

\[
PF + F^T P = -Q
\]

with \( p > 0, \ a > 0 \)

\[2pa < 0 \text{ and } q < 0\]

\[\therefore \text{ system is unstable}\]
Lyapunov Stability and the HJB Equation

\[ V[\mathbf{x}(t)] = \mathbf{x}^T(t) \mathbf{P}(t) \]

- **Lyapunov stability**
  \[ \frac{dV}{dt} < 0 \]

- **Dynamic programming optimality**
  \[ \frac{\partial V^*}{\partial t} = -\min_{u(t)} H \]

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Laplace Transforms and Linear System Stability
Fourier Transform of a Scalar Variable

\[ F[x(t)] = x(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt, \quad \omega = \text{frequency, rad} / s \]

\( x(t) \): real variable
\( x(j\omega) \): complex variable

\[ x(j\omega) = a(\omega) + jb(\omega) = A(\omega)e^{j\varphi(\omega)} \]

\( A \): amplitude
\( \varphi \): phase angle

Laplace Transforms of Scalar Variables

Laplace transform of a scalar variable is a complex number
\( s \) is the Laplace operator, a complex variable

\[ L[x(t)] = x(s) = \int_{0}^{\infty} x(t)e^{-st} \, dt, \quad s = \sigma + j\omega, \quad (j = i = \sqrt{-1}) \]

Laplace transformation is a linear operation

Multiplication by a constant
\[ L[a \, x(t)] = a \, x(s) \]

Sum of Laplace transforms
\[ L[x_1(t) + x_2(t)] = x_1(s) + x_2(s) \]
### Laplace Transforms of Vectors and Matrices

**Laplace transform of a vector variable**

\[
L[x(t)] = x(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \end{bmatrix}
\]

**Laplace transform of a matrix variable**

\[
L[A(t)] = A(s) = \begin{bmatrix} a_{11}(s) & a_{12}(s) & \cdots \\ a_{21}(s) & a_{22}(s) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}
\]

**Laplace transform of a time-derivative**

\[
L[\dot{x}(t)] = sx(s) - x(0)
\]

### Transformation of the System Equations

**Time-Domain System Equations**

\[
\begin{align*}
\dot{x}(t) &= Fx(t) + Gu(t) \\
y(t) &= H_x x(t) + H_u u(t)
\end{align*}
\]

**Laplace Transforms of System Equations**

\[
\begin{align*}
sx(s) - x(0) &= Fx(s) + Gu(s) \\
y(s) &= H_x x(s) + H_u u(s)
\end{align*}
\]
Second-Order Oscillator

Differential Equations for 2nd-Order System

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta\omega_n
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
\omega_n^2
\end{bmatrix} u(t)
\]

Dynamic Equation

\[
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} u(t) =
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
\]

Output Equation

\[
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(s) \\
x_2(s)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} u(s) =
\begin{bmatrix}
x_1(s) \\
x_2(s)
\end{bmatrix}
\]

Output Equation

Laplace Transforms of 2nd-Order System

\[
\begin{bmatrix}
sx_1(s) - x_1(0) \\
sx_2(s) - x_2(0)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta\omega_n
\end{bmatrix}
\begin{bmatrix}
x_1(s) \\
x_2(s)
\end{bmatrix} +
\begin{bmatrix}
0 \\
\omega_n^2
\end{bmatrix} u(s)
\]

Dynamic Equation

\[
\begin{bmatrix}
x_1(s) \\
x_2(s)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(s) \\
x_2(s)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} u(s) =
\begin{bmatrix}
x_1(s) \\
x_2(s)
\end{bmatrix}
\]

Output Equation

Laplace Transform of the State Vector Response to Initial Condition and Control

Rearrange Laplace Transform of Dynamic Equation

\[
sx(s) - F x(s) = x(0) + G u(s)
\]

\[
[sI - F] x(s) = x(0) + G u(s)
\]

\[
x(s) = [sI - F]^{-1} [x(0) + G u(s)]
\]

The matrix inverse is

\[
[sI - F]^{-1} = \frac{Adj(sI - F)}{|sI - F|} \quad (n \times n)
\]

Adj(sI - F): Adjoint matrix \quad (n \times n) \quad Transpose of matrix of cofactors

\[
|sI - F| = \det(sI - F): \quad \text{Determinant} \quad (1 \times 1)
\]
Characteristic Polynomial of a Dynamic System

Matrix Inverse

\[
[sI - F]^{-1} = \frac{\text{Adj} (sI - F)}{|sI - F|} \quad (n \times n)
\]

Characteristic matrix of the system

\[
(sI - F) = \begin{pmatrix}
(s - f_{11}) & -f_{12} & \ldots & -f_{1n} \\
-f_{21} & (s - f_{22}) & \ldots & -f_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-f_{n1} & -f_{n2} & \ldots & (s - f_{nn})
\end{pmatrix} \quad (n \times n)
\]

Characteristic polynomial of the system

\[
|sI - F| = \det (sI - F) \\
\equiv \Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0
\]

Eigenvalues
## Eigenvalues of the System

### Characteristic equation of the system

\[
\Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 = 0
\]

\[
= (s - \lambda_1)(s - \lambda_2)(\ldots)(s - \lambda_n) = 0
\]

Eigenvalues, \( \lambda_i \), are solutions (roots) of the polynomial, \( \Delta(s) = 0 \)

\[
\lambda_i = \sigma_i + j\omega_i
\]

\[
\lambda^*_i = \sigma_i - j\omega_i
\]

## Factors of a 2\textsuperscript{nd}-Degree Characteristic Equation

\[
|sI - F| = \begin{vmatrix}
(s - f_{11}) & -f_{12} \\
-f_{21} & (s - f_{22})
\end{vmatrix} \triangleq \Delta(s)
\]

\[
= s^2 - (f_{12} + f_{21})s + (f_{11}f_{22} + f_{12}f_{21})
\]

\[
= (s - \lambda_1)(s - \lambda_2) = 0 \quad \text{[real or complex roots]}
\]

\[
= s^2 + 2\zeta\omega ns + \omega_n^2 \quad \text{with complex-conjugate roots}
\]

\[
\lambda_1 = \sigma_1, \quad \lambda_2 = \sigma_2
\]

\[
\lambda_1 = \sigma_1 + j\omega_1
\]

\[
\lambda_2 = \sigma_1 - j\omega_1
\]

\( \omega_n \) : natural frequency, rad/s

\( \zeta \) : damping ratio, dimensionless
Application of Dirac Delta Function to Sampling Process

- Periodic sequence of numbers
  \[ \Delta x_k = \Delta x(t_k) = \Delta x(k\Delta t) \]

- Dirac delta function
  \[ \delta(t_0 - k\Delta t) = \begin{cases} \infty, & (t_0 - k\Delta t) = 0 \\ 0, & (t_0 - k\Delta t) \neq 0 \end{cases} \]
  \[ \int_{(t_0 - k\Delta t) - c}^{(t_0 - k\Delta t) + c} \delta(t_0 - k\Delta t) \, dt = 1 \]

- Periodic sequence of scaled delta functions
  \[ \Delta x(k\Delta t) \delta(t_0 - k\Delta t) \]
Laplace Transform of a Periodic Scalar Sequence

- Periodic sequence of numbers
  \[ \Delta x_z = \Delta x(t_k) = \Delta x(k \Delta t) \]

- Periodic sequence of scaled delta functions
  \[ \Delta x(k \Delta t) \delta(t - k \Delta t) \]

- Laplace transform of the delta function sequence
  \[
  L[\Delta x(k \Delta t) \delta(t - k \Delta t)] = \Delta x(z) = \sum_{k=0}^{\infty} \Delta x(k \Delta t) e^{-s \Delta t} dt
  = \sum_{k=0}^{\infty} \Delta x(k \Delta t) e^{-sk \Delta t} \overset{\Delta}{=} \sum_{k=0}^{\infty} \Delta x(k \Delta t) z^{-k}
  \]

\[ z \text { Transform of the Periodic Sequence} \]
\[ z \text { transform is the Laplace transform of the delta function sequence} \]

\[ L[\Delta x(k \Delta t) \delta(t - k \Delta t)] = \sum_{k=0}^{\infty} \Delta x(k \Delta t) e^{-sk \Delta t} \overset{\Delta}{=} \sum_{k=0}^{\infty} \Delta x(k \Delta t) z^{-k} \]

\[ z \overset{\Delta}{=} e^{s \Delta t} \quad \text{[advance by one sampling interval]} \]
\[ z^{-1} \overset{\Delta}{=} e^{-s \Delta t} \quad \text{[delay by one sampling interval]} \]
**z Transform of a Discrete-Time Dynamic System**

System equation in sampled time domain

\[
\Delta x_{k+1} = \Phi \Delta x_k + \Gamma \Delta u_k + \Lambda \Delta w_k
\]

Laplace transform of sampled-data system equation ("z Transform")

\[
z\Delta x(z) - \Delta x(0) = \Phi \Delta x(z) + \Gamma \Delta u(z) + \Lambda \Delta w(z)
\]

**Rearrange**

\[
z\Delta x(z) - \Phi \Delta x(z) = \Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)
\]

Collect terms

\[
(zI - \Phi)\Delta x(z) = \Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)
\]

Pre-multiply by inverse

\[
\Delta x(z) = (zI - \Phi)^{-1}[\Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)]
\]
Characteristic Matrix and Determinant of Discrete-Time System

\[ \Delta x(z) = (zI - \Phi)^{-1}[\Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)] \]

Inverse matrix

\[ (zI - \Phi)^{-1} = \frac{\text{Adj}(zI - \Phi)}{|zI - \Phi|} \quad (n \times n) \]

Characteristic polynomial of the discrete-time model

\[ |zI - \Phi| = \det(zI - \Phi) \equiv \Delta(z) \]
\[ = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \]

Eigenvalues (or Roots) of the Discrete-Time System

Characteristic equation of the system

\[ \Delta(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \]
\[ = (z - \lambda_1)(z - \lambda_2)(\ldots)(z - \lambda_n) = 0 \]

Eigenvalues, \( \lambda_i \), of the state transition matrix, \( \Phi \), are solutions (roots) of the polynomial, \( \Delta(z) = 0 \)

Eigenvalues are complex numbers that can be plotted in the \( z \) plane

\[ \lambda_i = \sigma_i + j\omega_i \quad \lambda_i^* = \sigma_i - j\omega_i \]
Laplace Transforms of Continuous- and Discrete-Time State-Space Models

Initial condition and disturbance effect neglected

\[
\Delta x(s) = (sI - F)^{-1}G\Delta u(s) \\
\Delta y(s) = H(sI - F)^{-1}G\Delta u(s)
\]

Equivalent discrete-time model

\[
\Delta x(z) = (zI - \Phi)^{-1}\Gamma\Delta u(z) \\
\Delta y(z) = H(zI - \Phi)^{-1}\Gamma\Delta u(z)
\]

Scalar Transfer Functions of Continuous- and Discrete-Time Systems

\[
\frac{\Delta y(s)}{\Delta u(s)} = H(sI - F)^{-1}G = \frac{H\text{Adj}(sI - F)G}{|sI - F|} = Y(s)
\]

\[
\frac{\Delta y(z)}{\Delta u(z)} = H(zI - \Phi)^{-1}\Gamma = \frac{H\text{Adj}(sI - \Phi)\Gamma}{|sI - \Phi|} = Y(z)
\]
Comparison of s-Plane and z-Plane Plots of Poles and Zeros

- **s-Plane Plot of Poles and Zeros**
  - Poles in left-half-plane are **stable**
  - Zeros in left-half-plane are **minimum phase**

- **z-Plane Plot of Poles and Zeros**
  - Poles within unit circle are **stable**
  - Zeros within unit circle are **minimum phase**

**Note correspondence of configurations**

**Increasing sampling rate moves poles and zeros toward the (1,0) point**

**Next Time:**
**Time-Invariant Linear-Quadratic Regulators**
Small Perturbations from Steady, Level Flight

\[ \dot{\Delta x}(t) = F \Delta x(t) + G \Delta u(t) + L \Delta w(t) \]

\[ \Delta x(t) = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} \Delta V \\ \Delta \gamma \\ \Delta q \\ \Delta \alpha \end{bmatrix} \]

- velocity, m/s
- flight path angle, rad
- pitch rate, rad/s
- angle of attack, rad

\[ \Delta u(t) = \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta \delta E \\ \Delta \delta T \end{bmatrix} \]

- elevator angle, rad
- throttle setting, %

\[ \Delta w(t) = \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix} = \begin{bmatrix} \Delta V_w \\ \Delta \alpha_w \end{bmatrix} \]

- ~horizontal wind, m/s
- ~vertical wind/V_{\text{nom}}, rad

Supplementary Material
Eigenvalues of Aircraft
Longitudinal Modes of Motion

\[ |sI - F| = \text{det}(sI - F) = \Delta(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) \]
\[ = (s - \lambda_p)(s - \lambda^*_{p})(s - \lambda_{SP})(s - \lambda^*_{SP}) \]
\[ = (s^2 + 2\zeta_p\omega_{np}s + \omega^2_{np})(s^2 + 2\zeta_{SP}\omega_{nsp}s + \omega^2_{nsp}) = 0 \]

Eigenvalues determine the damping and natural frequencies of the linear system’s modes of motion

\[ \left( \zeta_p, \omega_{np} \right): \text{phugoid (long-period) mode} \]
\[ \left( \zeta_{SP}, \omega_{nsp} \right): \text{short-period mode} \]

Initial-Condition Response of Business Jet at TwoTime Scales

Same 4th-order responses viewed over different periods of time

- 0 - 100 sec
  - Reveals Long-Period Mode

- 0 - 6 sec
  - Reveals Short-Period Mode