Stability of Dynamic Systems

Robert Stengel
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- Stability about an equilibrium point
- Bounds on the system norm
- Lyapunov criteria for stability
- Eigenvalues
- Transfer functions
- Continuous- and discrete-time systems

Dynamic System Stability

- Well over 100 definitions of stability
- **Common thread**: Response, $x(t)$, is bounded as $t \to \infty$
- Our principal focus: Initial-condition response of **NTI** and **LTI** dynamic systems
Vector Norms for Real Variables

- "Norm" = Measure of length or magnitude of a vector, \( x \)
- Euclidean or Quadratic Norm
  \[ L^2 \text{ norm} = \|x\|_2 = (x^T x)^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \]
- Weighted Euclidean Norm
  \[ \|y\|_2 = (y^T y)^{1/2} = (y_1^2 + y_2^2 + \cdots + y_m^2)^{1/2} \]
  \[ = (x^T D^T D x)^{1/2} = \|Dx\|_2 \]
  \[ x^T D^T D x \triangleq x^T Q x \]
  \[ Q \triangleq D^T D = \text{Defining matrix} \]

Uniform Stability

- Autonomous dynamic system
  - Time-invariant
  - No forcing input
  \[ \dot{x}(t) = f[x(t)] \]
- Uniform stability about \( x = 0 \)
  \[ \|x(t_0)\| \leq \delta, \ \delta > 0 \]
  Let \( \delta = \delta(\epsilon) \)
  If, for every \( \epsilon \geq 0 \),
  \[ \|x(t)\| \leq \epsilon, \ \epsilon \geq \delta > 0, \ \ t \geq t_0 \]
  Then the system is uniformly stable
- If system response is bounded, then the system possesses uniform stability
Local and Global Asymptotic Stability

- **Local asymptotic stability**
  - Uniform stability plus
    \[
    \lim_{t \to \infty} \|x(t)\| = 0
    \]

- **Global asymptotic stability**
  - System is asymptotically stable for any \( \varepsilon \)

- If a linear system has uniform asymptotic stability, it also is **globally stable**

Exponential Asymptotic Stability

- Uniform stability about \( x = 0 \) plus
  \[
  \|x(t)\| \leq k e^{-\alpha t} \|x(0)\|; \quad k, \alpha \geq 0
  \]

- If norm of \( x(t) \) is contained within an exponentially decaying envelope with convergence, system is **exponentially asymptotically stable (EAS)**

- Linear, time-invariant system that is asymptotically stable is **EAS**
Exponential Asymptotic Stability

\[ k \int_0^\infty e^{-\alpha t} \, dt = -\left( \frac{k}{\alpha} \right) \left. e^{-\alpha t} \right|_0^\infty = \frac{k}{\alpha} \]

Therefore, time integrals of the norm of an EAS system are bounded

\[ \int_0^\infty ||\dot{x}(t)|| \, dt = \int_0^\infty \left[ x^T(t) x(t) \right]^{1/2} \, dt \leq \left( \frac{k}{\alpha} \right) ||x(0)|| \]

and

\[ \int_0^\infty ||\dot{x}(t)||^2 \, dt \text{ is bounded} \]

Exponential Asymptotic Stability

Weighted Euclidean norm and its square are bounded if system is EAS

\[ \int_0^\infty \|\dot{x}(t)\| \, dt = \int_0^\infty \left[ x^T(t) D^T D x(t) \right]^{1/2} \, dt \text{ is bounded} \]

with \( 0 < D^T D \triangleq Q < \infty \)

\[ \int_0^\infty \left[ x^T(t) Q x(t) \right] \, dt \text{ is bounded} \]

Conversely, if the weighted Euclidean norm is bounded, the LTI system is EAS
Initial-Condition Response of an EAS Linear System

\[
x(t) = \Phi(t,0)x(0) = e^{F(t)}x(0)
\]
\[
\|x(t)\|^2 = x^T(0)\Phi^T(t,0)\Phi(t,0)x(0) \text{ is bounded}
\]

- To be shown
  - Continuous-time LTI system is stable if all of its eigenvalues have negative real parts
  - Discrete-time LTI system is stable if all of its eigenvalues lie within the unit circle

Lyapunov’s First Theorem

- A nonlinear system is asymptotically stable at the origin if its linear approximation is stable at the origin, i.e.,
  - for all trajectories that start “close enough” (in the neighborhood)
  - within a stable manifold (closed boundary)

\[
\dot{x}(t) = f[x(t)] \text{ is stable at } x_o = 0 \text{ if } \\
\Delta \dot{x}(t) = \left. \frac{\partial f[x(t)]}{\partial x} \right|_{x_o = 0} \Delta x(t) \text{ is stable}
\]

“At the origin” is a fuzzy concept
Lyapunov Function

Define a scalar Lyapunov function, a positive definite function of the state in the region of interest

\[ V[x(t)] \geq 0 \]

\[ V[0] = 0 \] for \( t \geq 0 \)

Examples

\[
V = E = \frac{mV^2}{2} + mgh; \quad \frac{E}{mg} = \frac{E}{\text{weight}} = \frac{V^2}{2g} + h
\]

\[
V = \frac{1}{2}x^T x; \quad V = \frac{1}{2}x^T P x
\]


Lyapunov’s Second Theorem

Evaluate the time derivative of the Lyapunov function

\[
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f[x(t)]
\]

\[
= \frac{\partial V}{\partial x} f[x(t)] \text{ for autonomous systems}
\]

* If \( \frac{dV}{dt} < 0 \) in the neighborhood of the origin, the origin is asymptotically stable
Quadratic Lyapunov Function for LTI System

Lyapunov function
\[ V\left[x(t)\right] = x^T(t)Px(t) \]

Linear, Time-Invariant System
\[ \dot{x}(t) = Fx(t) \]

Rate of change for quadratic Lyapunov function
\[
\frac{dV}{dt} = \frac{\partial V}{\partial x} \dot{x} = x^T(t)P\dot{x}(t) + \dot{x}^T(t)Px(t) \\
= x^T(t)(PF + F^TP)x(t) \triangleq -x^T(t)Qx(t)
\]

Lyapunov Equation

The LTI system is stable if the Lyapunov equation is satisfied with positive-definite \(P\) and \(Q\)

\[
PF + F^TP = -Q
\]
with
\[
P > 0, \quad Q > 0
\]
Lyapunov Stability:
1st-Order Example

\[ \Delta \dot{x}(t) = a \Delta x(t), \Delta x(0) \text{ given} \]

\[ F = a, P = p, Q = q \]

\[ \Delta x(t) = \int_0^t \Delta \dot{x}(t) \, dt = \int_0^t a \Delta x(t) \, dt = e^{at} \Delta x(0) \]

PF + \( F^T P = -Q \)

- with \( p > 0, a < 0 \)
- \( 2pa < 0 \) and \( q > 0 \)
- \therefore \text{system is stable}

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Lyapunov Stability and the HJB Equation

\[ V\left[ x(t) \right] = x^T(t)Px(t) \]

Lyapunov stability

\[ \frac{dV}{dt} < 0 \]

Dynamic programming optimality

\[ \frac{\partial V^*}{\partial t} = -\min_{u(t)} H \]
Lyapunov Stability for Nonautonomous (Time-Varying) Systems

\[ \dot{x}(t) = f[x(t), t]; \quad f[0, t] = 0 \text{ for } t \geq 0 \]

Time-varying Lyapunov function bounded by time-invariant Lyapunov functions

\[ V[x(t), t] \geq V_a[x(t)] > 0 \text{ in neighborhood of } x(0) = 0 \]

For asymptotic stability

\[ V[x(t), t] \leq V_b[x(t)] > 0 \text{ for large values of } x(t) \]

Time-derivative of Lyapunov function must be negative-definite

\[ \frac{dV[x(t), t]}{dt} = \frac{\partial V[x(t), t]}{\partial t} + \frac{\partial V[x(t), t]}{\partial x} f[x(t)] < 0 \text{ in neighborhood of } x = 0 \]

Laplace Transforms and Linear System Stability
**Fourier Transform of a Scalar Variable**

\[ F[x(t)] = x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt, \quad \omega = \text{frequency, rad} / \text{s} \]

- \( x(t) \): real variable
- \( x(j\omega) \): complex variable
  \[ = a(\omega) + jb(\omega) \]
  \[ = A(\omega)e^{j\phi(\omega)} \]
- \( A \): amplitude
- \( \phi \): phase angle

**Laplace Transforms of Scalar Variables**

Laplace transform of a scalar variable is a complex number

\( s \) is the Laplace operator, a complex variable

\[ L[x(t)] = x(s) = \int_{0}^{\infty} x(t)e^{-st} dt, \quad s = \sigma + j\omega, \quad (j = i = \sqrt{-1}) \]

Laplace transformation is a linear operation

**Multiplication by a constant**

\[ L[a x(t)] = a x(s) \]

**Sum of Laplace transforms**

\[ L[x_1(t) + x_2(t)] = x_1(s) + x_2(s) \]
Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

\[ L[x(t)] = x(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \end{bmatrix} \]

Laplace transform of a matrix variable

\[ L[A(t)] = A(s) = \begin{bmatrix} a_{11}(s) & a_{12}(s) & \cdots \\ a_{21}(s) & a_{22}(s) & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} \]

Laplace transform of a time-derivative

\[ L[\dot{x}(t)] = sx(s) - x(0) \]

Transformation of the LTI System Equations

Time-Domain System Equations

\[ \dot{x}(t) = Fx(t) + Gu(t) \quad \text{Dynamic Equation} \]
\[ y(t) = H_x x(t) + H_u u(t) \quad \text{Output Equation} \]

Laplace Transforms of System Equations

\[ sx(s) - x(0) = Fx(s) + Gu(s) \quad \text{Dynamic Equation} \]
\[ y(s) = H_x x(s) + H_u u(s) \quad \text{Output Equation} \]
Laplace Transform of the State Vector
Response to Initial Condition and Control

Rearrange Laplace Transform of Dynamic Equation

\[ s\mathbf{x}(s) - \mathbf{F}\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\mathbf{u}(s) \]
\[ [s\mathbf{I} - \mathbf{F}]\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\mathbf{u}(s) \]
\[ \mathbf{x}(s) = [s\mathbf{I} - \mathbf{F}]^{-1}[\mathbf{x}(0) + \mathbf{G}\mathbf{u}(s)] \]

The matrix inverse is

\[ [s\mathbf{I} - \mathbf{F}]^{-1} = \frac{\text{Adj}(s\mathbf{I} - \mathbf{F})}{|s\mathbf{I} - \mathbf{F}|} \quad (n \times n) \]

\[ \text{Adj}(s\mathbf{I} - \mathbf{F}) : \text{Adjoint matrix} \quad (n \times n) \quad \text{Transpose of matrix of cofactors} \]
\[ |s\mathbf{I} - \mathbf{F}| = \det(s\mathbf{I} - \mathbf{F}) : \text{Determinant} \quad (1 \times 1) \]

Characteristic Polynomial of a Dynamic System

Matrix Inverse

\[ [s\mathbf{I} - \mathbf{F}]^{-1} = \frac{\text{Adj}(s\mathbf{I} - \mathbf{F})}{|s\mathbf{I} - \mathbf{F}|} \quad (n \times n) \]

Characteristic matrix of the system

\[ (s\mathbf{I} - \mathbf{F}) = \begin{bmatrix}
(s - f_{11}) & -f_{12} & \ldots & -f_{1n} \\
-f_{21} & (s - f_{22}) & \ldots & -f_{2n} \\
\ldots & \ldots & \ldots & \ldots \\
-f_{n1} & -f_{n2} & \ldots & (s - f_{nn}) \\
\end{bmatrix} \quad (n \times n) \]

Characteristic polynomial of the system

\[ |s\mathbf{I} - \mathbf{F}| = \det(s\mathbf{I} - \mathbf{F}) \]
\[ \equiv \Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 \]
Eigenvalues

Eigenvalues of the LTI System

Characteristic equation of the system

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 = 0$$

$$= (s - \lambda_1)(s - \lambda_2)(\ldots)(s - \lambda_n) = 0$$

Eigenvalues, \(\lambda_i\), are solutions (roots) of the polynomial, \(\Delta(s) = 0\)

\[
\lambda_i = \sigma_i + j\omega_i \\
\lambda_i^* = \sigma_i - j\omega_i
\]
Factors of a 2nd-Degree Characteristic Equation

\[ |sI - F| = \begin{vmatrix} s - f_{11} & -f_{12} \\ -f_{21} & s - f_{22} \end{vmatrix} = \Delta(s) \]

\[ = s^2 - (f_{11} + f_{22})s + (f_{11}f_{22} + f_{12}f_{21}) \]

\[ = (s - \lambda_1)(s - \lambda_2) \quad \text{[real or complex roots]} \]

\[ = s^2 + 2\zeta\omega_n s + \omega_n^2 \quad \text{with complex-conjugate roots} \]

\[ \lambda_1 = \sigma_1, \quad \lambda_2 = \sigma_2 \]

\[ \lambda_1 = \sigma_1 + j\omega_1 \]

\[ \lambda_2 = \sigma_1 - j\omega_1 \]

\[ \omega_n : \text{natural frequency, rad/s} \]

\[ \zeta : \text{damping ratio, dimensionless} \]

* z Transforms and Discrete-Time Systems*
Application of Dirac Delta Function to Sampling Process

- Periodic sequence of numbers
  \[ \Delta x_k = \Delta x(t_k) = \Delta x(k\Delta t) \]

- Dirac delta function
  \[ \delta(t_0 - k\Delta t) = \begin{cases} \infty, & (t_0 - k\Delta t) = 0 \\ 0, & (t_0 - k\Delta t) \neq 0 \end{cases} \]
  \[ \int_{(t_0 - k\Delta t)-\epsilon}^{(t_0 - k\Delta t)+\epsilon} \delta(t_0 - k\Delta t) \, dt = 1 \]

- Periodic sequence of scaled delta functions
  \[ \Delta x(k\Delta t) \delta(t_0 - k\Delta t) \]

Laplace Transform of a Periodic Scalar Sequence

- Periodic sequence of numbers
  \[ \Delta x_k = \Delta x(t_k) = \Delta x(k\Delta t) \]

- Periodic sequence of scaled delta functions
  \[ \Delta x(k\Delta t) \delta(t - k\Delta t) \]

- Laplace transform of the delta function sequence
  \[
  L \left[ \Delta x(k\Delta t) \delta(t - k\Delta t) \right] = \Delta x(z) = \int_0^\infty \Delta x(k\Delta t) \delta(t - k\Delta t) e^{-z\Delta t} \, dt \\
  = \int_0^\infty \Delta x(k\Delta t) e^{-z\Delta t} \, dt \triangleq \sum_{k=0}^{\infty} \Delta x(k\Delta t) z^{-k}
  \]
### Z Transform of the Periodic Sequence

Z transform is the Laplace transform of the delta function sequence

\[
L[\Delta x(k\Delta t)\delta(t - k\Delta t)] = \sum_{k=0}^{\infty} \Delta x(k\Delta t)e^{-s\Delta t}e^{-sk\Delta t} = \sum_{k=0}^{\infty} \Delta x(k\Delta t)e^{-sk\Delta t}
\]

### Z Transform (time-shift) Operator

\[
Z \triangleq e^{s\Delta t} \quad [\text{advance by one sampling interval}]
\]

\[
Z^{-1} \triangleq e^{-s\Delta t} \quad [\text{delay by one sampling interval}]
\]

### Z Transform of a Discrete-Time Dynamic System

System equation in sampled time domain

\[
\Delta x_{k+1} = \Phi \Delta x_k + \Gamma \Delta u_k + \Lambda \Delta w_k
\]

Laplace transform of sampled-data system equation ("Z Transform")

\[
z\Delta x(z) - \Delta x(0) = \Phi \Delta x(z) + \Gamma \Delta u(z) + \Lambda \Delta w(z)
\]
Transform of a Discrete-Time Dynamic System

Rearrange

$$z\Delta x(z) - \Phi \Delta x(z) = \Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)$$

Collect terms

$$\left(zI - \Phi\right)\Delta x(z) = \Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)$$

Pre-multiply by inverse

$$\Delta x(z) = \left(zI - \Phi\right)^{-1}\left[\Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)\right]$$

Characteristic Matrix and Determinant of Discrete-Time System

$$\Delta x(z) = \left(zI - \Phi\right)^{-1}\left[\Delta x(0) + \Gamma \Delta u(z) + \Lambda \Delta w(z)\right]$$

Inverse matrix

$$\left(zI - \Phi\right)^{-1} = \frac{\text{Adj}(zI - \Phi)}{zI - \Phi} \quad (n \times n)$$

Characteristic polynomial of the discrete-time model

$$\left|zI - \Phi\right| = \det(zI - \Phi) \equiv \Delta(z)$$

$$= z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$$
Eigenvalues (or Roots) of the LTI Discrete-Time System

Characteristic equation of the system

\[ \Delta(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \]
\[ = (z - \lambda_1)(z - \lambda_2)(\ldots)(z - \lambda_n) = 0 \]

Eigenvalues, \( \lambda_i \), of the state transition matrix, \( \Phi \), are solutions (roots) of the polynomial, \( \Delta(z) = 0 \)

Eigenvalues are complex numbers that can be plotted in the \( z \) plane

\[ \lambda_i = \sigma_i + j\omega_i \]
\[ \lambda_i^* = \sigma_i - j\omega_i \]

Laplace Transforms of Continuous- and Discrete-Time State-Space Models

Initial condition and disturbance effect neglected

\[ \Delta x(s) = (sI - F)^{-1}G\Delta u(s) \]
\[ \Delta y(s) = H(sI - F)^{-1}G\Delta u(s) \]

Equivalent discrete-time model

\[ \Delta x(z) = (zI - \Phi)^{-1}\Gamma\Delta u(z) \]
\[ \Delta y(z) = H(zI - \Phi)^{-1}\Gamma\Delta u(z) \]
Scalar Transfer Functions of Continuous- and Discrete-Time Systems

\[
\frac{\Delta y(s)}{\Delta u(s)} = H(sI - F)^{-1}G = \frac{H \text{Adj}(sI - F)G}{|sI - F|} = Y(s)
\]

\[
\frac{\Delta y(z)}{\Delta u(z)} = H(zI - \Phi)^{-1}\Gamma = \frac{H \text{Adj}(zI - \Phi)\Gamma}{|zI - \Phi|} = Y(z)
\]

Comparison of \(s\)-Plane and \(z\)-Plane Plots of Poles and Zeros

- **\(s\)-Plane Plot of Poles and Zeros**
  - Poles in left-half-plane are stable
  - Zeros in left-half-plane are minimum phase

- **\(z\)-Plane Plot of Poles and Zeros**
  - Poles within unit circle are stable
  - Zeros within unit circle are minimum phase

Note correspondence of configurations

Increasing sampling rate moves poles and zeros toward the (1,0) point
Next Time:
Time-Invariant Linear-Quadratic Regulators

SUPPLEMENTARY MATERIAL
Second-Order Oscillator

Differential Equations for 2nd-Order System

Laplace Transforms of 2nd-Order System

Small Perturbations from Steady, Level Flight

\[ \Delta \mathbf{x}(t) = F \Delta \mathbf{x}(t) + G \Delta \mathbf{u}(t) + L \Delta \mathbf{w}(t) \]
Eigenvalues of Aircraft Longitudinal Modes of Motion

\[ |sI - F| = \det (sI - F) \equiv \Delta(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4) \]
\[ = (s - \lambda_p)(s - \lambda^*_p)(s - \lambda_{SP})(s - \lambda^*_{SP}) \]
\[ = \left( s^2 + 2\zeta_p \omega_{np} s + \omega_{np}^2 \right) \left( s^2 + 2\zeta_{SP} \omega_{nsp} s + \omega_{nsp}^2 \right) = 0 \]

Eigenvalues determine the damping and natural frequencies of the linear system’s modes of motion

\[ \left( \zeta_p, \omega_{np} \right) : \text{phugoid (long-period) mode} \]
\[ \left( \zeta_{SP}, \omega_{nSP} \right) : \text{short-period mode} \]

Initial-Condition Response of Business Jet at Two Time Scales

Same 4th-order responses viewed over different periods of time

- 0 - 100 sec
  - Reveals Long-Period Mode
- 0 - 6 sec
  - Reveals Short-Period Mode