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TECHNICAL NOTE

COMPUTER-AIDED ANALYSIS OF LINEAR CONTROL SYSTEM ROBUSTNESS

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Abstract—Stochastic robustness is a simple technique used to estimate the stability and performance robustness of linear, time-invariant systems. The use of high-speed graphics workstations and control system design software in stochastic robustness analysis is discussed and demonstrated.

INTRODUCTION

Stochastic robustness, a simple technique to determine the robustness of linear, time-invariant systems by Monte Carlo methods was introduced in [1] and presented in detail in [2, 3]. These references introduced the scalar probability of instability (and associated binomial confidence intervals) as a stability robustness metric. The stochastic root locus, or probability density of the closed loop eigenvalues, was shown to portray robustness properties graphically. Concepts behind stochastic stability robustness are applied to performance specifications in [4]. With the advent of fast graphics workstations and supercomputers, the stochastic robustness of a system is easily computed by Monte Carlo simulation, and results can be displayed pictorially, providing insight into otherwise hidden robustness properties. The examples described in [2–4] were performed using a combination of software developed on a Silicon Graphics Personal Iris, and commercial control system design software for a Macintosh. This paper serves primarily to document the use of present technology and software in control system robustness analysis.

GRAPHICAL ROBUSTNESS ANALYSIS

The examples of stochastic robustness analysis in [2–4] have shown that stochastic root loci, or plots of the probability densities of the closed-loop eigenvalues, give added insight into robustness properties of a system. Stochastic root loci are portrayed by graphing contours of equal root density on the two-dimensional plot, or by plotting an oblique view of the three-dimensional histogram or root density surface. Analysis is enhanced by allowing the user to rotate, rescale, shade, or contour the three-dimensional stochastic root locus. Numerical contouring using interpolation is computation intensive, but the colormap associated with graphics display can be used to produce contour-rendered or shaded images with minimal computation. Colormap contour rendering is performed by associating each z-value (root density) with a color and generating a solid surface for each bin. These tasks can be implemented quickly and efficiently using graphics libraries available on high-speed graphics workstations such as the Iris.

The plotted root loci surfaces become smoother as the number of Monte Carlo evaluations increase. Alternatively, numerical smoothing [5] can be applied to the root density (ρ) at each grid point to account for sampling effects on the plotted surface:

$$\rho_{n,m} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} w_{i,j}\rho_{n+i,m+j}. \tag{1}$$

Equation (1) is a simple smoother, where $\rho_{n,m}$ is weighted based on its unsmoothed value and the root density at the eight surrounding points. Weights $w_{i,j}$ are selected so that their sum is one; a typical weighting is $w_{n,m} = 0.2$, and all surrounding weights equal 0.1. Numerical smoothing of a stochastic root locus based on few evaluations may show sufficient qualitative information about system robustness and eigenvalue trends to give control system design insight without large numbers of Monte Carlo evaluations. The use of graphics in robustness analysis is the subject of continued research.

Consider a second-order system with uncertain damping ratio (ζ) and natural frequency (ω_n) [2]:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0. \tag{2}$$

The nominal parameter values are 0.707 and 1, respectively, and each is a Gaussian-distributed random variable with a standard deviation of 0.2. Figure 1a shows a three-dimensional stochastic root locus for 50,000 Monte Carlo evaluations. The large number of evaluations gives a relatively smooth stochastic root locus extending into the right-half plane. Figure 1b shows a "rough" root locus of the system based on 5000 evaluations, and Fig. 1c shows the 5000 evaluation root locus with the smoother described above applied. While 50,000 evaluations gives the most accurate rendering of the stochastic root locus, the smoothed root locus provides an adequate rendering.

STOCHASTIC ROBUSTNESS IMPLEMENTATION

Commercial software that includes control functions is ideal for implementation of stochastic robustness. In this section, stochastic frequency response distributions are used to demonstrate the ease of implementing stochastic robustness using MATLAB [6] and to show the insight gained from simple analysis procedures and 2-D graphics.

A Monte Carlo evaluation can be applied to determine the stochastic frequency response of one or more transfer functions of a multivariable system. Probabilistic bandwidth, gain margins, and phase margins determined from the frequency response distributions give measures of stability and performance robustness that are consistent with classical concepts. MATLAB has a control system design toolbox [6] whose "tools", or functions, are well suited to such analysis. MATLAB also has functions to generate and shape random numbers required for Monte Carlo evaluation. Frequency response analysis is applied to a fourth-order transfer function

$$\frac{x(s)}{u(s)} = \frac{K(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)}{(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)(s^2 + 2\zeta_3\omega_3 s + \omega_3^2)} \tag{3}$$

whose parameter vector $\mathbf{p} = [\zeta_1\ \omega_1\ \zeta_2\ \omega_2\ \zeta_3\ \omega_3]$ has the following mean value ($\bar{\mathbf{p}}$) and per cent Gaussian standard deviation (σ):

$$\bar{\mathbf{p}} = [1.0\ 10.0\ 0.5\ 10.0\ 0.707\ 50.0] \tag{4}$$

$$\sigma = [40\ 1\ 40\ 1\ 80\ 1]. \tag{5}$$

The numerator and denominator coefficients are written (as functions of the parameters) as MATLAB vectors for input to MATLAB function *bode* that computes magnitude and phase. This is followed by another function *margin* that computes gain margin, phase margin, and crossover frequencies. *hist* plots the histograms of the gain margins and phase margins that result from J Monte Carlo evaluations of the frequency response and margins. Figure 2 shows the stochastic frequency response for 100 Monte Carlo evaluations; the mean response is indicated by solid curves, and the darkened distribution and associated outliers indicate the stochastic responses. From this simple analysis, Fig. 2 shows that there is a range of possible gain and phase crossover frequencies, and hence a range of possible gain and phase margins. The probability densities of the gain and phase margins corresponding to Fig. 2 are given in Fig. 3. Figure 3 demonstrates that parameter uncertainty causes a (non-Gaussian) distribution in gain and phase margins around their mean values of 6.2 dB and 74 deg, respectively. The histograms in Fig. 3 define several robustness metrics. By integrating (summing) the histograms appropriately, one can determine the probability of the phase margin being less than a specified value, the probability of the gain margin being less than a specified value, and/or the probability of both gain and phase margin being less than specified values; each of these metrics can be computed with a one-line MATLAB command. The probability of gain and phase margins that are less than zero provides an alternate stability robustness metric. Histograms also can be calculated for the gain and phase crossover frequencies. The MATLAB function for this example was developed in less than 1 h, and it requires less than 3 min of computation time on a Macintosh II (or 35 sec on a Macintosh IIfx) for 100 Monte Carlo evaluations over the frequency range specified in Fig. 2.

The stochastic frequency response metrics presented above add meaning to their deterministic counterparts. A deterministic gain margin indicates how much gain reduction/increase is allowable while still maintaining a stable system, and a deterministic phase margin points to the amount of pure phase lag that a system can tolerate; it is difficult to say how much of each margin will be lost due to parameter uncertainty. Figure 3 shows that the relationship between phase margin and gain margin loss due to parameter uncertainty can be complex. The stochastic metrics (probability of gain and/or phase margins less than specified values) indicate allowable gain reduction and phase lag *after* parameter uncertainty is considered.

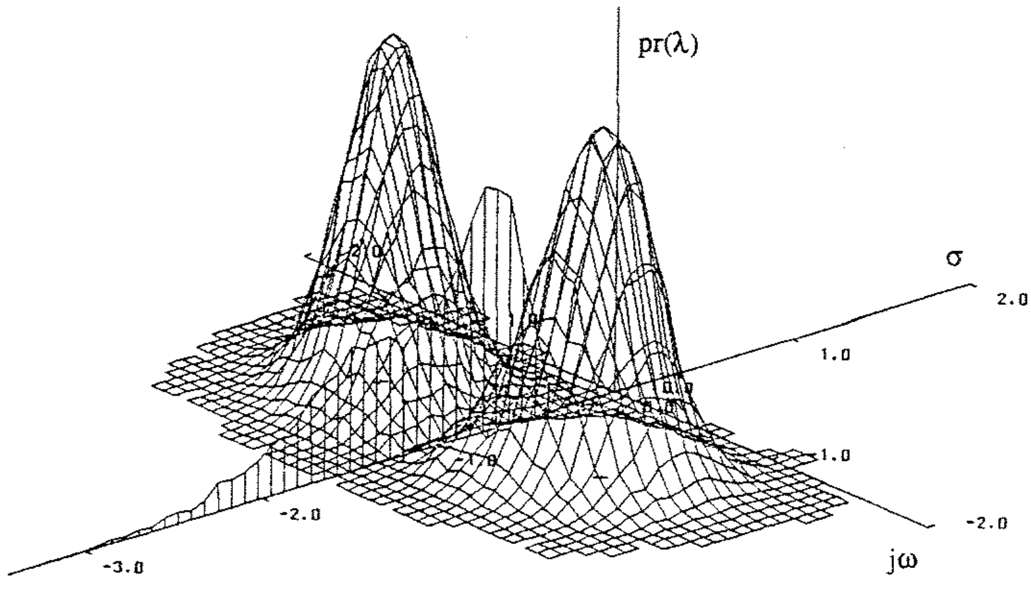

Fig. 1(a)

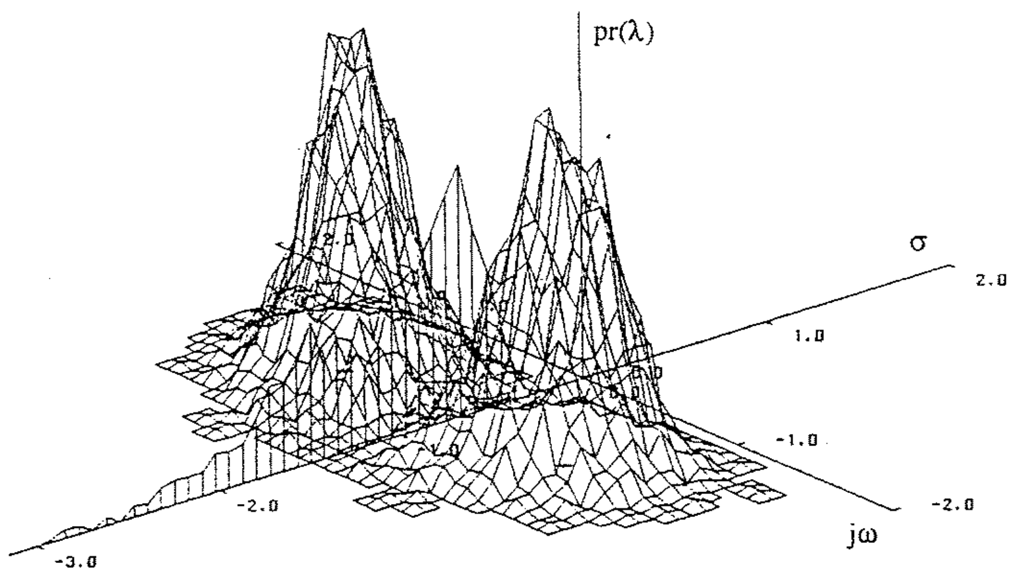

Fig. 1(b)

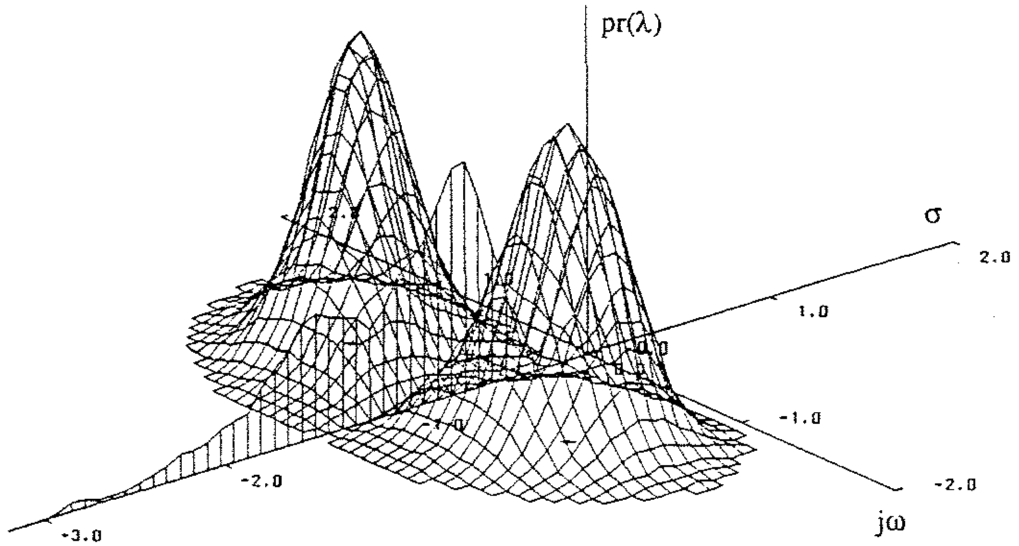

Fig. 1(c)

Fig. 1. Stochastic root loci of a second-order system with a nominal damping ratio ζ_0 0.707, and natural frequency $\omega_{n0} = 1$. Parameters are Gaussian, and each has a standard deviation of 0.2. (a) 50,000 Monte Carlo evaluations; (b) 5000 Monte Carlo evaluations; and (c) 5000 Monte Carlo evaluations with a 3-D filter applied once.

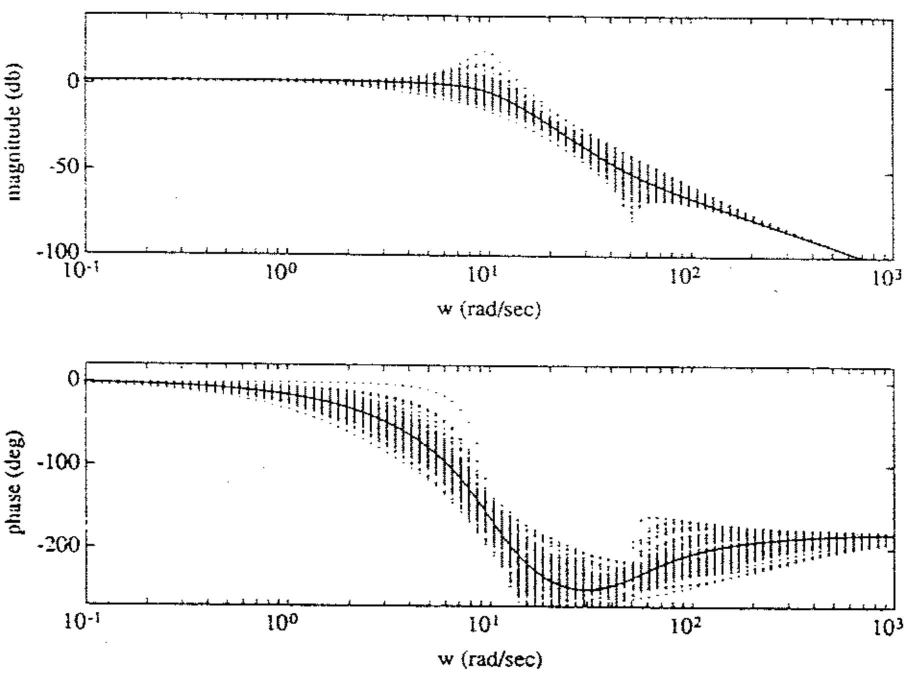

Fig. 2. Stochastic frequency response distribution for a fourth-order system with six Gaussian parameters. Based on 100 Monte Carlo evaluations.

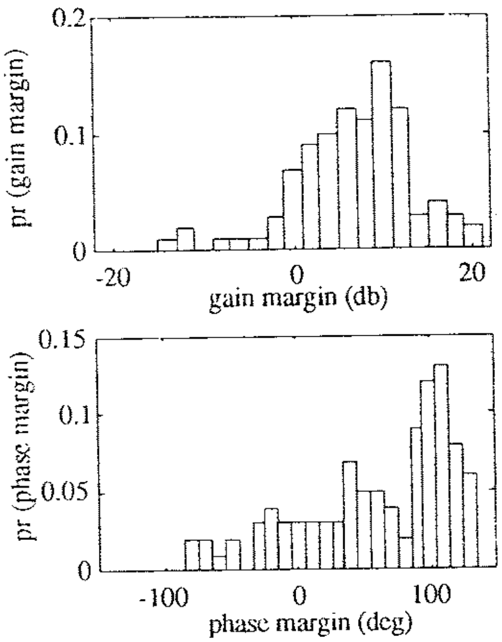

Fig. 3. Gain and phase margin histograms for a fourth-order system with six Gaussian parameters. Based on 100 Monte Carlo evaluations.

CONCLUSION

Stochastic robustness offers a rigorous yet straightforward alternative to current metrics for control system robustness that is simple to compute. It makes good use of modern computational and graphic tools, and it is easily implemented using commercial control system design and analysis software.

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