A Monte Carlo Approach to the Analysis of Control System Robustness*

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Key Words—Robustness; multivariable control systems; control system analysis; Monte Carlo methods.

Abstract—Stochastic robustness, a simple technique used to estimate the stability and performance robustness of linear, time-invariant systems, is described. The scalar probability of instability is introduced as a measure of stability robustness. Examples are given of stochastic performance robustness measures based on classical time-domain specifications. The relationship between stochastic robustness measures and control system design parameters is discussed. The technique is demonstrated by applying an LQG/LTR system designed for a flexible robot arm. It is concluded that the analysis of stochastic robustness offers a good alternative to existing robustness metrics. It has direct bearing on engineering objectives, and it is appropriate for evaluating robust control system synthesis methods currently practiced.

Introduction

Standard linear control system design methods rely on accurate models of the system to be controlled. Because models are never perfect, robustness analysis is necessary to determine the possibility of instability or inadequate performance in the face of uncertainty. Synthesis of robust control system is predicated on a good measure of robustness. Consequently, much research activity during the past two decades has been devoted to developing adequate measures of robustness for linear, time-invariant systems that can in turn be used for robust control system synthesis. In most instances, robustness is treated deterministically, using singular-value analysis (e.g., Lehtomaki et al., 1981; Doyle, 1982) or parameter-space methods (e.g., Siljak, 1989; Vijino et al., 1990). These methods can be applied without regard to actual bounds on system parameters; hence, the relationship of the metric to uncertainties in the physical systems often is weak. Furthermore, deterministic metrics can be conservative and/or difficult to determine for systems with many uncertain parameters. Consequently, overconservative control system designs or designs that are insufficiently robust in the face of real world uncertainties are a danger.

Stochastic Robustness Analysis (SRA) uses statistical descriptions of parameter uncertainty to determine whether stability/performance robustness criteria are met. Stengel (1980) introduced Monte Carlo analysis of the scalar probability of instability, which is central to the analysis of stability robustness. SRA is described in more detail in Stengel and Ray (1991); exact confidence intervals for the scalar probability of instability are presented, and the stochastic root locus, or probability density of the closed-loop eigenvalues, is shown to portray robustness properties graphically. Ray and Stengel (1991) illustrates the use of SRA to compare control system designs for full-state feedback aircraft control systems and to analyse systems with finite-dimensional uncertain dynamics. Because it provides a statistical measure of robustness and because it uses knowledge of the statistics of parameter variations, SRA is inherently intuitive and accurate. The physical meaning behind the probability of instability is apparent, and overconservative or insufficiently robust designs can be avoided.

Concepts behind stochastic stability robustness are readily extended to provide measures of performance robustness. Design specifications such as rise time, peak overshoot, settling time, and steady-state error are normally used as indicators of adequate performance and (often) amount to the same kind of analysis as described above. SRA can be applied to classical criteria giving probabilistic bounds on scalar performance measures. Metrics resulting from stability and performance robustness can be related to controller parameters, thus providing a foundation for design tradeoffs and optimization. This paper summarizes stochastic stability and performance robustness analysis. The analysis is illustrated by studying the effectiveness of robustness recovery on a stochastic optimal control system with parameter uncertainties.

Stochastic stability robustness

Stochastic stability robustness is described in Stengel and Ray (1991) and is summarized here. Consider a linear, time-invariant system where the dynamic, control, and output matrices, $F(p), G(p)$, and $H(p)$ may be arbitrary functions of an $r$-dimensional parameter vector, $p$. The control gain matrix $C$ is designed using some nominal or "mean" value of the dynamic model, denoted $F, G$, and $H$, that represents $F(p), G(p), H(p)$ evaluated at the nominal parameter vector. The actual system has an unknown description that depends on the actual (unknown) value of the parameter vector $p$. Environment, variations in the nominal state, system failures, parameter estimation errors, wear, and manufacturing differences all can contribute to mismatch between the actual system and that used to design the controller. For SRA, $p$ is assumed to have a known or estimated probability distribution function, denoted $p_r(p)$, that expresses the parameter uncertainty statistics due to the above factors.

The eigenvalues of the matrix $[F(p) - G(p)CH(p)]$ determine closed-loop stability. To estimate the probability of instability ($P$) using Monte Carlo evaluation, the closed-loop eigenvalues are evaluated $J$ times with each element of $p$ specified by a random-number generator whose individual outputs are shaped by $p_r(p)$. For less than an infinite number of evaluations, the resulting probability is an estimate, denoted $\hat{P}$, and given by the number of evaluations where one or more eigenvalues has a positive real part divided by $J$. Because $F$ is a binomial variable (i.e., the outcome of each Monte Carlo evaluation takes one of two values: stable or unstable) exact confidence intervals for $P$ are calculated using the binomial test ($C$-norm, 1980). Confidence intervals also can be calculated for the difference $\Delta P$ between $P_1$ and $P_2$ of two different control systems (Ray

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and Stengel, 1990):

\[ \Pr[(L_1 - U_1) \leq \Delta P \leq (U_2 - L_2)] \geq 1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2, \]  

(1)

\((L_1, U_1)\) and \((L_2, U_2)\) are the binomial confidence intervals for \(P_1\) and \(P_2\), respectively, and \(\alpha_1, \alpha_2\) are the individual confidence coefficients. Confidence intervals for \(P\) are detailed in Stengel and Ray (1991).

The presentation of a probability with exact confidence intervals is precise (nonsensitive), defensible, and repeatable in the context of probability and statistics, even though it differs from classical deterministic measures. Although the accuracy of \(P\) in describing the true robustness depends on the accuracy with which the parameters are described, known or estimated parameter uncertainties are required to determine whether a system is robust for any robustness measure (deterministic or stochastic). While deterministic measures such as a stability margin (Viczio et al., 1990) can be computed without characterizing the uncertainty, the control system cannot be pronounced robust until the stability margin is evaluated with respect to the uncertainties expected in the system. Otherwise, the control system may be too robust, at the expense of performance, or nonrobust in the face of real-world uncertainties. By considering only allowed uncertainties, SRA eliminates this extra step. For deterministic measures, the problem is compounded by the fact that the metric itself can be conservative and/or difficult to compute.

Deterministic evaluation of the probability of instability. Continuously distributed parameters have a true underlying probability of instability that remains unknown for \(J \rightarrow \infty\), although given a sufficient number of evaluations, \(P\) can be bounded within a desired confidence interval (Stengel and Ray, 1991). Quantized approximations distribute continuous distributions, approaching them in the limit as the number of discrete parameter values goes to infinity. When a continuous distribution is approximated by a quantized distribution, it may be possible to perform fewer deterministic evaluations and obtain an equally good estimate \(\hat{P}\). For few parameters \((r)\) and few quantization levels \((w)\), deterministic evaluation may be a valid option. If the parameter distributions are in fact discrete, then evaluation of the \(w^r\) deterministic combinations would give \(P\), while Monte Carlo evaluation provides an estimate \(\hat{P}\) along with confidence intervals for \(P\). If \(r\) and/or \(w\) are large, then many deterministic evaluations are required, and Monte Carlo analysis may provide adequate bounds for fewer evaluations.

The tradeoff between Monte Carlo analysis and deterministic evaluation depends on the number of parameters, their distributions, the number of quantization levels, and required confidence intervals. For example, a system with two binary parameters and \(P = 0.25\) requires \(2^2\) or four deterministic evaluations to obtain \(P\), but 6238 Monte Carlo evaluations are needed to compute 95% confidence intervals with a width of 10% of \(P\). A system with 20 binary parameters and \(P = 0.25\) requires \(2^{20}\) or over \(10^{10}\) deterministic evaluations, yet its probability of instability can be bounded within 10% of \(P\) at a 95% confidence level with the same 6238 Monte Carlo evaluations. Evaluation of binary combinations is comparable to determining the stability of "corners" in parameter space. If the parameters distributions are not binary but the maximum real eigenvalue component is monotonic in the individual elements of \(p\), then the deterministic binary evaluations circumscribe results obtained for the actual continuous or quantized distributions with the same limits, and a conservative estimate \(\hat{P}\) is provided.

Extensions of stochastic stability robustness analysis

Stability robustness of systems with estimators. Stochastic stability robustness analysis is easily extended to systems incorporating dynamic state estimators. Using \(F_A, G_A\), and \(H_A\) as the actual system matrices and \(F, G,\) and \(H\) as the design system, the closed-loop system matrix for state \((x - \hat{x})\) and error dynamics \((x - \hat{x})\) is (Stengel, 1986)

\[
F_d = \left[ \begin{array}{c}
F_A - G_A C \\
(F - F_A) - (G - G_A) C + \text{KH}
\end{array} \right] - G_A C \\
F - (G - G_A) C - \text{KH}
\]  

(2)

where \(\hat{x}\) is the state estimate and \(K\) is the estimator gain matrix. The effect of parameter uncertainty on stability robustness is computed by Monte Carlo evaluation of the eigenvalues of equation (2), with \(F(p), G(p),\) and \(H(p)\) substituted for \(F_A, G_A,\) and \(H_A\). Closed-loop eigenvalue densities portrayed on the stochastic root locus show the possible interaction of dynamic and estimator state elements, and the possible robustness degradation due to the estimator.

Well-known loss of LQ stability margins when a state estimator is added (Doyke, 1978) can be quantified by the probability of instability.

Performance robustness analysis. While stability is an important element of robustness, performance robustness analysis is vital to determining whether important design specifications are met. Stochastic stability robustness is described by a single parameter, the probability of instability. Adequate performance—initial condition response, response to commanded inputs, control authority, and rejection of disturbances is difficult to describe by a single scalar. However, elements of SRA (e.g., Monte Carlo evaluation and use of the binomial confidence intervals) apply, independent of the performance criteria chosen.

Numerous criteria stemming from classical control concepts exist as measures of adequate performance. Appealing to these, one can begin a smooth transition from stability robustness analysis to performance robustness analysis simply by analysing the degree of stability or instability rather than strict stability (Stengel and Ray, 1991). One method of doing this is to shift the vertical discriminant line from zero to \(\Sigma\) less than (or greater than) zero. Histograms and cumulative distributions for degree of stability are readily given by the Monte Carlo estimate of \(\Pr(\Sigma)\), the probability that the maximum real eigenvalue component is less than \(\Sigma\), where \(-\infty < \Sigma < \infty\). Binomial confidence intervals are applicable to each point of the cumulative distribution, as there are just two values of interest, e.g. "satisfactory" or "unsatisfactory". The robustness measure resulting from the cumulative probability distribution is directly related to classical concepts of rates of decay (growth) of the closed-loop poles, time-to-half and time-to-double. Rather than a vertical discriminant line one can confine the closed-loop roots to sectors in the complex plane bounded by lines of constant damping and arcs of constant natural frequency. Systems with roots confined to sectors would be expected to display a certain transient response speed. Again, the probability of roots lying within a sector is a binomial variable, and binomial confidence interval calculations apply.

While the speed of the transient response depends on the closed-loop poles, its magnitude and overall shape depend on the coefficients of the characteristic exponential and sinusoidal terms, and time responses provide the most clear-cut means of evaluating performance. When time responses are computed, stochastic performance robustness can be portrayed as a distribution of possible trajectories around a nominal or desired trajectory. Envelopes can be defined around a nominal trajectory based on stated performance criteria, and the probability of exceeding the envelope becomes the scalar, binomial performance robustness metric (Ray and Stengel, 1990). While it is simple to conclude that a response violates such an envelope, individual responses within the envelope may not be acceptable. In such cases, the derivative of a response and envelopes around the derivative also can be used to evaluate performance. There is no unique set of criteria defining envelopes that bound an acceptable response; the envelopes may be defined by segments connecting points based on minimum/maximum dead time, rise time, time to peak
Fig. 1. Illustration of design insight revealed by SRA. Solid points indicate closed-loop eigenvalues enclosed by hypothetical "uncertainty circles". (a) Root-locus where stability robustness decreases monotonically with increased gain. (b) Root-locus where stability robustness decreases, increases, then decreases with increased gain.

overshoot, peak overshoot, settling time, and steady-state error. Segmented envelopes can be smoothed, or other scalars can be used to define points on the envelope. However, once an envelope is defined, time response distributions can be computed by Monte Carlo methods. The closed-loop time response to a command input, disturbance, initial condition, or some combination is evaluated J times, and for each evaluation, the trajectory is a binomial variable; it either stays within the envelope or violates the envelope. Although computing time responses are more computation-intensive than probability-of-instability or degree of stability estimation, such analysis is well within the capability of existing workstations.

Stochastic robustness as a control design aid. While general "rules of thumb" regarding the design of robust control systems are useful, SRA can identify non-obvious robustness behavior in particular problems. Figure 1 provides one example. Consider Fig. 1a, which shows the root-locus of a system that has a complex pair of poles and a right-half-plane zero. Hypothetical bounded "uncertainty circles" are drawn around possible closed-loop-root locations; the uncertainty circle at the pole represents possible open-loop eigenvalue locations due to uncertainty in the dynamic matrix F. As gain increases along the root-locus, the uncertainty is magnified, and uncertainty in the control effect matrix contributes to the widening circles. Stability robustness decreases, and closed-loop roots may be in the right-half plane at high enough gain. This case illustrates one where the decrease in robustness is monotonic, as indicated by plotting the probability of instability vs. the root-locus gain in Fig. 2a. Figure 1b postulates a system with a real pole and a complex pair of poles and zeros located in a "pole over zero" configuration. The complex portion of the root-locus starts near the jω axis before ending at the zero in the left-half plane. Again, uncertainty circles enlarge as gain increases. In this case, it is possible that eigenvalue distributions cross into the right-half plane, are entirely in the left-half plane as gain increases, and finally, cross back into the right-half plane at very high gain. Here, stability robustness (as measured by the probability of instability) may have local or global minima as functions of gain (Fig. 2b). For multivariable systems with many parameters, the intrinsic structure of the problem and the tradeoff between the spread in closed-loop-root uncertainty vs the magnitude of the control gains may not be immediately evident. Plots of stochastic robustness metrics vs scalar controller design parameters provide the necessary insight.

Case study: LQG/LTR system robustness analysis

It is well-known that the stability margins guaranteed for an LQ system are not retained in LQG systems (Doyle, 1978). Nevertheless, Loop Transfer Recovery (LTR) (Doyle et al., 1979) has become an established method of recovering transfer properties of the LQ system (or the linear-optimal estimator, Kwakernaak (1976)) in minimum-phase LQG systems. The condition for recovery of LQ-transfer properties is derived from the fact that if u(t) had the same effect on both x(t) and χ(t), then the LQG system would have the same transfer function properties as the LQ system. If the actual system matrices match those used to design the estimator, the transform relationships are (Doyle and Stein,

Fig. 2. Probability of instability as a function of control system design parameter, for the two cases in Fig. 1.
\[
    x(s) = \begin{bmatrix} s^2L - F \end{bmatrix}^{-1} G u(s),
\]
(3)
\[
    \bar{x}(s) = \left[ (s^2L - F) + KH \right] \begin{bmatrix} s^2L - F \end{bmatrix}^{-1} G u(s) = s x(s).
\]
(4)
When the estimator gain is chosen according to the recovery condition \( K / v = G \), equation (4) becomes,
\[
    \bar{x}(s) = A^{-1} G [H A^{-1} G]^{-1} H x(s) = A^{-1} G u(s) = x(s).
\]
(5)
where \( A = (s^2L - F) \). Asymptotic recovery occurs as the positive, scalar design parameter \( \nu \) approaches \( \infty \), and \( q \) estimator eigenvalues approach the \( q \) transmission zeros of \( H (s^2L - F)^{-1} G \). The procedure recovers the original loop only if the recovery condition, the nominal disturbance spectral density matrix, \( W_0 \), is appended as
\[
    W = W_0 + \nu^2 G G^T.
\]
(6)
The term \( \nu^2 G G^T \) represents additional process noise that introduces the optimality of the estimator with respect to actual measurement noise and disturbances; hence, performance suffers as more process noise is added, but the good transfer function properties of the \( LQ \) system are recovered asymptotically. The tradeoff between estimator performance and system robustness is made by adjusting \( \nu \).

Unstructured-singular-value analysis (USVA) typically is used to determine when \( LQ \) properties are recovered. Nevertheless, USVA does not indicate the effectiveness of LTR in systems with parameter uncertainties, as pointed out in Tahk and Speyer (1987) and Shakel and Soroika (1988). When the system description is uncertain, the actual system matrices do not match those used to design the estimator, and
\[
    \bar{x}(s) = A^{-1} G [H A^{-1} G]^{-1} H A^{-1} G A x(s).
\]
(7)
where \( A_x = (s^2L - F_x) \) and \( F, G_A \) represent the actual system matrices. Equation (7) shows that when parameter uncertainties are present, the original loop is not recovered, although partial recovery may improve robustness over that of the nominal LQG system. Estimator poles approach the transmission zeros as \( \nu \to \infty \), and finite number of transmission zeros may move to the right-hand plane due to parameter uncertainty in \( H A^{-1} G \). Estimator poles also are influenced by parameter uncertainty and can vary around the transmission zero; hence, increasing \( \nu \) indefinitely can decrease robustness in systems with one or more uncertain parameters. In such systems, SRA determines the value of \( \nu \) required to recover sufficient robustness while maintaining adequate performance, as demonstrated by the example that follows. This characteristic of SRA is not limited to analysis of \( LQG/LTR \) system but is useful with any control design approach in which one or more design parameters is arbitrary.

Single-link robot arm. A flexible one-link robot typically is used to study problems associated with controlling a compliant system when the sensor and actuator are not collocated (e.g., Cannon and Schmitz, 1984). In such systems, robustness concerns can be severe. The linear model of the single-link robot arm retains the first three flexible modes, and the tip of the link is controlled by applying a control force to the hub, or base of the link. Because the model is representative of a general flexible structure, physical parameters are easily identifiable, and robustness is a concern, it is a good candidate for SRA.

The dynamic, control effect, and output matrices are given by
\[
    F = \begin{bmatrix}
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0
    \end{bmatrix}
\]
(8)
\[
    G = \frac{1}{I_r} \begin{bmatrix}
        0 & 0 & \phi(0) & \phi(0) & \phi(0) & \phi(0) \\
        0 & 0 & \phi(0) & \phi(0) & \phi(0) & \phi(0) \\
        \phi(0) & \phi(0) & 0 & \phi(0) & \phi(0) & \phi(0) \\
        0 & 0 & \phi(0) & \phi(0) & \phi(0) & \phi(0) \\
        \phi(0) & \phi(0) & 0 & 0 & \phi(0) & \phi(0) \\
        \phi(0) & \phi(0) & 0 & 0 & \phi(0) & \phi(0)
    \end{bmatrix}
\]
(9)
\[
    H = \begin{bmatrix}
        0 & \phi(L) & \phi(L) & \phi(L) & 0 & 0 \\
        0 & \phi(L) & \phi(L) & \phi(L) & 0 & 0 \\
        0 & \phi(L) & \phi(L) & \phi(L) & 0 & 0 \\
        0 & \phi(L) & \phi(L) & \phi(L) & 0 & 0 \\
        0 & \phi(L) & \phi(L) & \phi(L) & 0 & 0 \\
        0 & \phi(L) & \phi(L) & \phi(L) & 0 & 0
    \end{bmatrix}
\]
(10)
where \( x \) is the length along the arm, \( \phi(x) \), are the normal modes, \( \phi(x) = \frac{d^2 \phi}{dx^2} \), \( L \) is the length of the arm, and \( I_r \) is the total inertia of the arm. The measurements taken through \( H \) are the tip displacement and hub rate, respectively. The flexibility of the open-loop system is apparent in the open-loop eigenvalues, which are 0, 0, -0.177 + 11.8i, -0.432 + 21.6i, and -0.968 + 48.37i. The transfer function between tip displacement and hub torque is non-minimum phase, with zeros 12.4, -12.0, 21.6 + 24.2i, -22.5 + 24.2i; hence, a non-minimum phase response can be expected for tip displacement. The system has a readily identifiable 14-element parameter vector:
\[
    p = \begin{bmatrix}
        \omega_1 & \omega_2 & \omega_3 & \omega_4 & \phi(0) & \phi(0) & \phi(0) \\
        L & \phi(L) & \phi(L) & \phi(L) & I_r
    \end{bmatrix}
\]
(11)

Details concerning the modeling and parameter identification are given in Cannon and Schmitz (1984). The linear-quadratic regulator designed in Cannon and Schmitz (1984) is used for demonstration of SRA. The performance index weights tip position and tip rate, and the LQR state-weighting, control-weighting, and control gain matrices are
\[
    Q = 0.01 F^T H^T \begin{bmatrix}
        1 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 & 0
    \end{bmatrix}
\]
(12)-(13)
\[
    R = 0.001
\]
(14)

The nominal closed-loop eigenvalues are -5.41 ± 48.8i, -6.47 ± 23.8i, -6.1 ± 2.6i, -7.7 ± 11.4i.

A uniform probability density function models the parameter uncertainties, with variations between ±2% of the nominal values for \( L \) and \( I_r \) and ±25% for the remaining parameters. The 50,000-evaluation stochastic root-locus for the full-state feedback system is given in Fig. 3. The nominal eigenvalues are marked, and the distribution is indicated by the height above the complex plane in units of roots/length along the real axis and roots/area in the complex plane. The “bin” size in Fig. 3 is 0.9 along the axis and 0.9 x 0.9 off the axis. For 50,000 evaluations, \( P \)=0, with 95% confidence intervals \((L, U) = (0.74 \times 10^{-2})\). Each of the four complex eigenvalue pairs appears in Fig. 3 as a “peak”, with a surrounding distribution due to parametric uncertainty. The peaks can be well-defined (as in the complex frequency complex pair) or broad (at the highest frequency pair) and the nominal eigenvalues are not necessarily at the distributions’ peak. Parameter uncertainty causes complex pairs to coalesce into real roots resulting in a distribution along the real axis. The closed-loop eigenvalues tend to spread into the left-hand plane, while definite boundaries are delineated on the right. For binary parameter variations of the same magnitude as the maximum uniform variations, 2^14 or 16,384 deterministic evaluations also give \( P = 0 \). These results indicate good stability robustness in the face of reasonably large uncertainties.

Moving to performance robustness analysis, Fig. 3 shows sector bounds defined by \( 4 \leq \alpha_0 \leq 65 \) and \( \varepsilon > 0.1 \). For 50,000 evaluations, the probability that closed-loop eigenvalues lie outside of these bounds is 0.0147, with 95% confidence intervals \((L, U) = (0.0136, 0.1258)\). While the shape of the time response depends on closed-loop zeros, a minimum response speed can be guaranteed by requiring that all closed-loop eigenvalues lie within the specified sector.

Figure 4(a) presents segmented step-response envelopes and 500 Monte Carlo evaluations of the response for a 4.8 cm position command input. The control history corresponding to the mean response is given in Fig. 4b.
initial response is in the wrong direction, since the transfer function is non-minimum phase; the envelopes in Fig. 4 indicate the maximum acceptable non-minimum phase response. For 500 responses, the probability of violating the time response envelope is 0.184 with 95% confidence intervals \((L, U) = (0.151, 0.221)\). Individual responses characteristic of those evaluated by Monte Carlo analysis are given in Fig. 5. While responses fill out the envelope, some of the individual responses within the envelope may not be acceptable in the face of real-world criteria governing rate of change of the response (Fig. 5c). This is a case where checking envelopes around the derivative of the response may be necessary. Similar analyses can be performed on control trajectories to make sure bandwidth and control effort limitations are not violated.

It is instructive from a design standpoint to plot robustness measures vs design parameters used to calculate feedback gains. Since there is a single control in this example, the scalar control weighting matrix \(R\) can be used as the design parameter. Two stochastic performance robustness measures are plotted vs \(R\) in Fig. 6—the probability of violating the time-response envelope and the probability of degree of instability. As control gains increase, the closed-loop roots are pushed farther into the left-half plane, but they also tend to migrate farther from their nominal values. At some value of control gain, there is a tradeoff between how far roots migrate and their location in

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**Fig. 3.** Stochastic root-locus for the single-link robot with uniform parameters, 50,000 evaluations. Nominal eigenvalues are marked by 'x'.

**Fig. 4.** Time histories associated with tip position command of 4.8 cm. (a) 500 Monte Carlo evaluations of the tip response. Envelopes are defined by scalar performance criteria. Nominal response is indicated by the solid line. (b) Nominal control input.

**Fig. 5.** Examples of individual tip responses. (a) Acceptable response within envelope. (b) Response violates envelope. (c) Response is within envelope, but criteria governing its derivative may be required.
Fig. 3. Stochastic root-locus for the single-link robot with uniform parameters, 50,000 evaluations. Nominal eigenvalues are marked by "x".

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Fig. 8. Evaluation of Loop Transfer Recovery. (a) Probability of Instability vs LOG/LTR design parameter $v$. (b) Sampled estimate of tip position and hub rate covariances vs LOG/LTR design parameter $v$.

indicates estimator performance by sampled estimates of the covariance of the output, $P = E[(v(t) - x(t))^2|x]$ where $E[]$ is the expectation operator. The output covariance (based on simulation of the LOG system) shows that performance degradation over that of the nominal LOG system is small at the minimizing value of $v$.

Conclusion

Stochastic Robustness Analysis offers a rigorous yet straightforward alternative to current metrics for control system robustness that is simple to compute and is unfettered by normally difficult problem statements, such as non-Gaussian statistics, arbitrary functions of uncertain parameters appearing as matrix elements, and structured uncertainty. Principles behind stochastic robustness can be applied to scalar performance metrics and/or time responses, making it a good candidate for overall robustness analysis. Stability and performance measures resulting from the analysis can provide details relating intrinsic robustness characteristics and control system design parameters. The analysis makes good use of the computing power of modern workstations. The example demonstrates stochastic robustness analysis applied to LOG/LTR. The analysis determines the effectiveness of Loop Transfer Recovery on uncertain systems, and the Loop Transfer Recovery design parameter that gives adequate stability robustness with minimal performance degradation is readily identified.

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