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Searching for Robust Minimal-Order Compensators

A method of designing a family of robust compensators for a single-input/single-output linear system is presented. Each compensator's transfer function is found by using a genetic-algorithm search for numerator and denominator coefficients. The search minimizes the probabilities of unsatisfactory stability and performance subject to real parameter variations of the plant. As the search progresses, probabilities are estimated by Monte Carlo evaluation. The design procedure employs a sweep from the lowest feasible transfer-function order to higher order, terminating either when design goals have been achieved or when no further improvement in robustness is evident. The method provides a means for estimating the best possible compensation of a given order based on repeated searches. [DOI: 10.1115/1.1367270]

Keywords: *Stochastic Robustness, Monte Carlo Evaluation, Genetic Algorithm, Weibull Distribution.*

Introduction

Designing robust control laws for uncertain systems is a computationally complex problem. Real-valued uncertainty in system parameters leads to an infinity of possible stability and performance outcomes, and algorithms for solution may be ''NP hard,'' with nonpolynomial growth in computation as the number of uncertain parameters increases [1]. All robust control solutions are approximate: either the system model is simplified to allow analytic solution of the design equations or the solution itself is simplified to accommodate the original (possibly complex) system model [2]. Probabilistic control design follows the second approach, using algorithms with polynomial complexity to characterize system performance and to identify satisfactory controllers. Earlier studies placed explicit bounds on the minimum sample size required to estimate a probabilistic performance index [3–6], and randomized algorithms have demonstrated polynomial-time complexity [7–11]. The probabilistic approach is well-suited to parallel implementation, affording nearly linear ''speed-up'' in the number of computing nodes [12]. Increasingly efficient numerical search routines have been used, beginning with local line search [10] and proceeding to genetic algorithms [11,13,14]. The probabilistic approach is readily applied to designing robust multi-input/multi-output nonlinear controllers [15,16] as well as to linear controllers.

This paper revisits a benchmark single-input/single-output (SISO) linear control problem, with the objectives of simplifying the control design procedure and of examining trade-offs between closed-loop robustness and compensator complexity. The previous approach [10,11] used a Linear-Quadratic-Gaussian regulator (LQGR) to define the compensator structure. The current approach searches for the coefficients of the SISO compensator's transfer function. Numerator and denominator coefficients are found directly, without solving algebraic Riccati equations. The number of search parameters and the amount of computation are reduced. A family of robust compensators is designed, beginning with the lowest-order compensator that solves the problem and ending when further improvements are no longer significant, trading improved robustness against higher compensator order.

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Problem Description and Design Method

The stochastic robustness metric characterizes the probability that the closed-loop system will have unacceptable stability or performance when subject to parametric uncertainties. The probability, P, is defined as

$$P(\mathbf{d})=\int_V m[H(\mathbf{v}),C(\mathbf{d})]pr(\mathbf{v})d\mathbf{v} \tag{1}$$

H is the plant structure, $\mathbf{v}$ is a vector of varying plant parameters in space $\mathbf{V}$ with distribution $pr(\mathbf{v})$, C is an application-specific compensator, and $\mathbf{d}$ is the design parameter vector for the compensator. For each stability or performance criterion, $m[\cdot]$ is a binary indicator that equals one if $H(\mathbf{v})$ and $C(\mathbf{d})$ form an unacceptable system and is zero otherwise.

The probability in Eq. (1) can be estimated by Monte Carlo evaluation (MCE). With $pr(\mathbf{v})$ shaping the random samplings of values for $\mathbf{v}$, the estimate based on N samples is

$$\hat{P}(\mathbf{d})=\frac{1}{N}\sum_{k=1}^{N} m[H(v_k),C(\mathbf{d})] \tag{2}$$

Explicit bounds on the minimum sample size N required to estimate the probability are given in [3–6].

A. Description of the Benchmark Problem. The benchmark plant proposed in [17] is a dual-mass/single-spring system (Fig. 1). Its state-space model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & 0 & 0\\ k/m_2 & -k/m_2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1/m_1\\ 0\end{bmatrix}u+\begin{bmatrix}0\\ 0\\ 0\\ 1/m_2\end{bmatrix}w \tag{3}$$

$$z=x_2+\nu \tag{4}$$

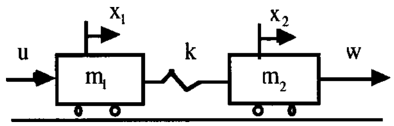

Fig. 1 The plant for the benchmark problem

where x_1 and x_2 represent the positions of the masses, x_3 and x_4 are the velocities, and u is the control force on m_1. The plant is disturbed by w on m_2, and the measurement of the second mass position is corrupted by noise v.

The nominal mass-spring values are $k_1=m_1=m_2=1$, and the limits of variation are $0.5<k<2$, $0.5<m_1<1.5$, and $0.5<m_2<1.5$. The goal is to minimize the likelihood of closed-loop instability, excess settling time, and excess control in response to a unit initial disturbance. The plant parameters have uniform probability distributions, and the stochastic robustness metrics are:

- Probability of Instability P_i : $m_i=1$ if any closed-loop eigenvalues has nonnegative real part, $m_i=0$ otherwise.
- Probability of Settling Time Exceedance P_{ts} : $m_{ts}=1$ if $|z(t)|>0.1$ for any $t>15$ s, $m_{ts}=0$ otherwise.
- Probability of Control Limit Exceedance P_u : $m_u=1$ if $|u(t)|>1$ for any t, $m_u=0$ otherwise.

The cost function is a weighted quadratic sum of the three probabilities:

$$J=\alpha P_i^2+\beta P_{ts}^2+\gamma P_u^2 \tag{5}$$

The probabilities are squared to place higher penalty on large probabilities of unsatisfactory behavior, which may be of greater concern than low probabilities of metric violation. The choice of weights has been addressed extensively in earlier papers, notably Ref. [10]. The weights are $\alpha=1$, $\beta=0.01$, and $\gamma=0.01$ to emphasize stability robustness over performance robustness.

The compensator $C(s,\mathbf{d})$ is specified in transfer-function form with numerator degree m and denominator degree n:

$$C_{(m,n)}(s,\mathbf{d})=\frac{a_0+a_1s+a_2s^2+\cdots+a_ms^m}{b_0+b_1s+b_2s^2+\cdots+b_ns^n} \tag{6}$$

The goal is to minimize J by proper choice of $\mathbf{d}=\{a_i,b_j\}$, $i=0,\cdots,m$; $j=0,\cdots,n$.

B. Searching Procedure. The genetic algorithm (GA) that computes design parameters is a randomized search method based on natural selection and adaptation [18]. In this paper, bias-roulette-wheel selection is adopted. The GA population size is chosen as 50, the crossover rate is between (0.8–1.0), and the mutation rate is in (0.001–0.01). Each candidate compensator is evaluated by 1000 MCEs. Each GA run terminates after 20 generations.

The search evaluates compensator transfer functions up to the sixth degree. With no transfer function zeros, no compensators stabilize the system for any system parameter variation. With a single zero and denominator degree greater than one, it is easy to find compensators that can stabilize the system over the designated range of parameter variation with less than unit control effort. However, the resulting compensators have slow response and cannot satisfy the settling time condition.

With second-degree numerator and denominator,

$$C_{(2,2)}(s)=\frac{a_0+a_1s+a_2s^2}{b_0+b_1s+b_2+s^2} \tag{7}$$

The design vector (or chromosome) $\mathbf{d}_{(2,2)}=\{a_0,a_1,a_2,b_0,b_1,b_2\}_{(2,2)}$ is searched. The starting search space for each parameter in $\mathbf{d}_{(2,2)}$ is $(-10,\ 10)$, as coefficient values in any range can be normalized by a common factor. Each parameter in $\mathbf{d}_{(2,2)}$ is represented by a ten-bit binary number sequence scaled to real values in $(-10,\ 10)$. Therefore, after the first GA search, the resulting parameters $(a_0',a_1',a_2',b_0',b_1',b_2')$ are accurate to 0.02. Then the search is over the range $a_i'\pm 2a_i'$, $b_j'\pm 2b_j'$, i, $j=0$, 1, 2. The search range and chromosome length can be tuned to affect the relative accuracy of results. The optimized compensator coefficients are $\mathbf{d}^*_{(2,2)}=\{a_0^*,a_1^*,a_2^*,b_0^*,b_1^*,b_2^*\}_{(2,2)}$.

Next, a compensator with two zeros and three poles is addressed:

$$C_{(2,3)}(s)=\frac{a_0+a_1s+a_2s^2}{b_0+b_1s+b_2s^2+b_3s^3} \tag{8}$$

The search for $\{a_0,a_1,a_2,b_0,b_1,b_2\}$ of $C_{(2,3)}(s)$ starts at $\mathbf{d}^*_{(2,2)}$, in the range $a_i^*\pm 2a_i^*$, $b_j^*\pm 2b_j^*$, i, $j=0$, 1, 2, and the first search of b_3 is in $(-10,\ 10)$. The search includes a chromosome with the optimal parameters of a lower-order compensator in the initial population. A family of compensators up to sixth order is obtained by increasing the number of poles and zeros in sequential fashion.

C Estimation of the Global Minimum of the Cost Function. The genetic algorithm is repeated with different random seeds, and a set of independent solutions is generated. Point estimation and confidence limits for the optimal solution are obtained in two ways. In both cases, n independent estimates are made, and they are value-ordered from highest to lowest J_i, $i=1,2,\cdots,n$. Cooke's formula [19] provides an estimate of the global minimum:

$$J^*=2J_n-\sum_{i=0}^{n-1}\left[\left(1-\frac{i}{n}\right)^n-\left(1-\frac{i+1}{n}\right)^n\right]J_{n-i} \tag{9}$$

With a large number of test points, the probability function for the results of a random search takes the form of a Weibull distribution [20]; hence, independent solutions of a randomized heuristic algorithm follow a Weibull distribution [21]. An estimate of the global minimum also can be obtained by fitting the distribution to the results of independent GA runs. The cumulative probability function for a Weibull distribution is

$$\Pr(J\le J_0)=1-\exp\left[-\left(\frac{J_0-a}{b}\right)^c\right] \tag{10}$$

where the parameter a is an estimate of the global minimum J^*, while b and c describe the spread and shape of the distribution. The confidence interval of the estimate based on N independent solutions is,

$$\Pr[J_{0,\min}-b\le J^*\le J_{0,\min}]=1-e^{-N} \tag{11}$$

where $J_{0,\min}$ is the minimum value of the N sample solutions [22].

Estimating a by Eq. (9), least-squares estimates of b and c are generated from the sample data. a is varied over an appropriate range to determine the parameter values that yield the largest correlation coefficient. Thus, J^* is estimated by Eq. (9), and the confidence interval of J^* is determined by Eq. (11).

Figure 2 shows the results of eight genetic searches for the compensator with fourth-degree denominator and third-degree numerator. By Eq. (9), $J^*=0.002243$. With a fixed at 0.002243, the Weibull distribution coefficients are estimated to be $b=0.0000353$ and $c=1.51$ (Fig. 2), with confidence interval

$$\Pr(0.0022117\le J^*\le 0.002247)=0.9997 \tag{12}$$

D. Computational Issues. Numerical search and Monte Carlo evaluation are computationally intensive processes. The design procedure uses a GA to search over a probabilistic cost space. Each point in the space is derived from 1000 Monte Carlo evaluations of a specific compensator, and each GA search uses 1000 point evaluations; therefore, there are 10^6 function evaluations. The required number of MCE for each generation could be determined dynamically [11], reducing the number of function evaluations.

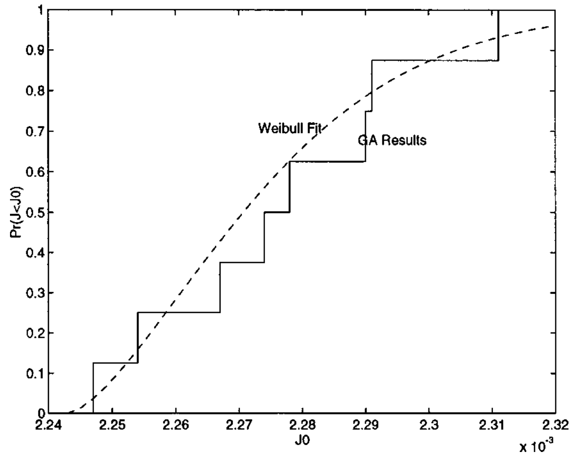

Fig. 2 Weibull curve fit to eight genetic algorithm searches

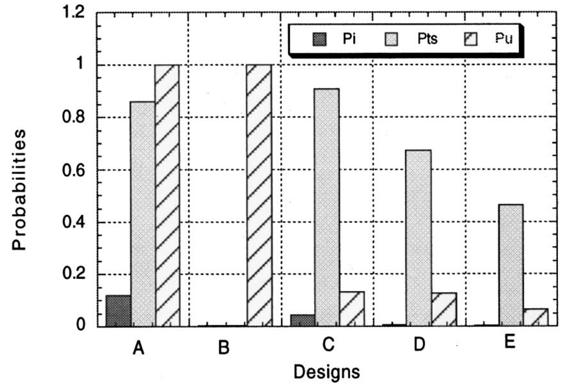

Fig. 4 Comparison of stochastic robustness of five design solutions

III Results and Discussion

A family of second- to sixth-order compensators with different numerator degrees has been designed. Each compensator is the result of a single search; hence, its design cost is subject to the Weibull statistics. Examples of the second- to fourth-order compensators are presented here:

$$C_{(2,2)}(s)=\frac{0.77+5.22s-5.76s^2}{10.12+10983s+3.98s^2} \tag{13}$$

$$C_{(2,3)}(s)=\frac{0.51+3.55s-2.92s^2}{6.46+7.68s+3.21s^2+0.67s^3} \tag{14}$$

$$C_{(3,3)}(s)=\frac{0.51+3.49s-2.89s^2+0.6s^3}{6.43+7.30s+2.87s^2+0.82s^3} \tag{15}$$

$$C_{(3,4)}(s)=\frac{0.56+3.76s-2.96s^2+0.59s^3}{6.60+7.41s+3.12s^2+0.89s^3+0.0153s^4} \tag{16}$$

$$C_{(4,4)}(s)=\frac{0.58+3.72s-2.63s^2+0.69s^3+0.79s^4}{6.76+7.43s+3.22s^2+0.297s^3+0.0397s^4} \tag{17}$$

All compensators have a zero near -0.15, which is good compensation for the plant's rigid-body mode. All have either one non-minimum-phase zero or a pair of complex ones, increasing the closed-loop damping ratio of the plant's flexible mode. The compensator transfer functions have similar zero-frequency gain (seen by setting $s=0$). For compensators from low to higher order, coefficients of common-degree terms are similar.

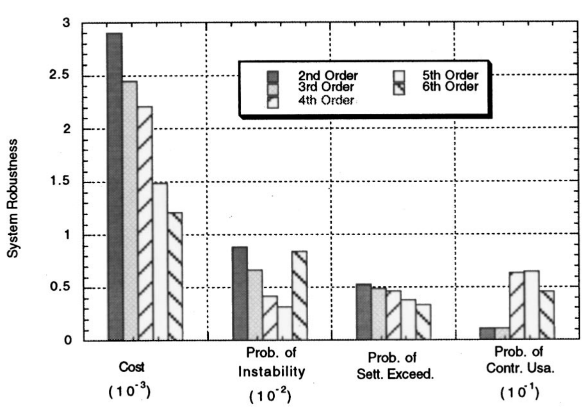

Fig. 3 System robustness for proper compensators

27,000 simulations are applied to reevaluate each of these compensators (Fig. 3). Compensators with low-pass filtering (numerator degree one less than denominator degree) perform almost as well as proper compensators, and they are less sensitive to measurement noise. Figure 3 shows that design costs are reduced as compensator order increases.

This paper's results are compared with previous designs using 27,000 Monte Carlo evaluations (Fig. 4). Design A [23] generated robust LQG regulators by adding structured covariance terms to the algebraic Riccati equations. Both Designs B [24] and C [25] used H_∞ approaches. All are fourth-order compensators, and none of the three were specifically designed to optimize the stochastic robustness metrics used here. Design D is a fifth-order LQGR compensator that is designed to minimize the same stochastic robustness cost function as that of this paper [10]. Design E is compensator $C_{(4,4)}(s)$ in this paper. Designs A and B always violate the control usage criterion, though Design B has excellent stability and settling time performance as a result. Design E has smaller probability of exceeding control limits than all of the other designs, and it has better stability and settling time performance than Designs A, C, and D.

IV Conclusions

Probabilistic evaluation combined with genetic search is an effective method for designing robust compensators. The minimized design costs follow a Weibull distribution, allowing the confidence level to be estimated. A family of robust compensators with different numerator and denominator degrees is designed, trading complexity against robustness. The system synthesis method is flexible and well tailored to satisfying practical design requirements.

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